6.6 The Rank

→ **DEF** p. 283

Let \( A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \).

- The rows of \( A \):

\[
\vec{v}_1 = (a_{11}, a_{12}, \ldots, a_{1n}) \\
\vdots \\
\vec{v}_m = (a_{m1}, a_{m2}, \ldots, a_{mn})
\]

span a subspace of \( \mathbb{R}^n \), called the **row space** of \( A \).
• The columns of $A$:

$$
\overrightarrow{w_1} = \begin{pmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{m1}
\end{pmatrix}, \ldots, \overrightarrow{w_n} = \begin{pmatrix}
a_{1n} \\
a_{2n} \\
\vdots \\
a_{mn}
\end{pmatrix}
$$

span a subspace of $\mathbb{R}^m$, called the **column space** of $A$.

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**TH 6.10** → p. 284

If the matrices $A$ and $B$ are row equivalent then their row spaces are equal.
EXAMPLE 1 Find the basis for the row space of

\[
A = \begin{pmatrix}
1 & 3 & 1 & 3 \\
0 & 1 & 2 & 0 \\
-3 & 0 & 7 & -1 \\
3 & 4 & 1 & 1 \\
2 & 0 & -2 & -2
\end{pmatrix}.
\]

A is row equivalent to

\[
B = \begin{pmatrix}
1 & 3 & 1 & 3 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The row space of both B and A is

\[\text{span}\{(1, 3, 1, 3), (0, 1, 2, 0), (0, 0, 1, -1)\}.\] (Keep nonzero rows of B)
→ Procedure (p.285)

DEF → p.285

- **row rank** of $A = \dim(\text{row space of } A)$
- **column rank** of $A = \dim(\text{column space of } A)$
The row rank of $A$ of **Example 1** is 3. To find the column rank of $A$, use the Procedure on p.285:

$$
\begin{pmatrix}
1 & 3 & 1 & 3 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 3 & 1 & 3 \\
0 & 1 & 2 & 0 \\
-3 & 0 & 7 & -1 \\
3 & 4 & 1 & 1 \\
2 & 0 & -2 & -2 \\
\end{pmatrix}
$$

The first three columns of $A$ span the column space of $A$. Therefore, the column rank of $A$ is 3.
Question: Are the row and column ranks of a matrix always the same?

Answer: YES

**TH 6.11** → p. 287
row rank of $A = \text{column rank of } A$

**DEF** → p. 249
\[
\text{rank } A = \text{row rank of } A = \text{column rank of } A
\]

**TH 6.12** → p. 288
If $A$ is an $m \times n$ matrix then
\[
\text{rank } A + \text{nullity } A = n
\]
Equivalent conditions (∆ p.291)

For any \( n \times n \) matrix \( A \), the following conditions are equivalent:

1. \( A \) is nonsingular.
2. \( A \vec{x} = \vec{0} \) has only the trivial solution.
3. \( A \) is row equivalent to \( I_n \).
4. For every \( n \times 1 \) matrix \( \vec{b} \), the system \( A \vec{x} = \vec{b} \) has a unique solution.
5. \( \det(A) \neq 0 \).
6. \( \text{rank } A = n \).
7. \( \text{nullity } A = 0 \).
8. The rows of \( A \) are linearly independent.
9. The columns of \( A \) are linearly independent.