
6 Applications of Integration

In this chapter you will find several activities on various applications of integration. While Sections 6.1 and 6.3 (on area and arc length) follow the same format as previous chapters, and make use of Mathcad only, Sections 6.2 and 6.4 make use of the Maple V system. The reason for using Maple is that it offers three-dimensional graphing facilities superior to those of Mathcad.

In general, while Maple is more powerful than Mathcad, it is also more difficult to use. For this reason you will be provided with Maple documents to serve as templates. This will make it easier to obtain the desired output (typically, 3-D graphs) without requiring any knowledge of Maple syntax or programming.

In Calculus II, where numerous topics deal with 3-D objects (curves in space, surfaces, etc.), Maple will be used more.

6.1 Activity: Area of a Region Between Two Curves

Prerequisites: Read Section 6.1 LHE.

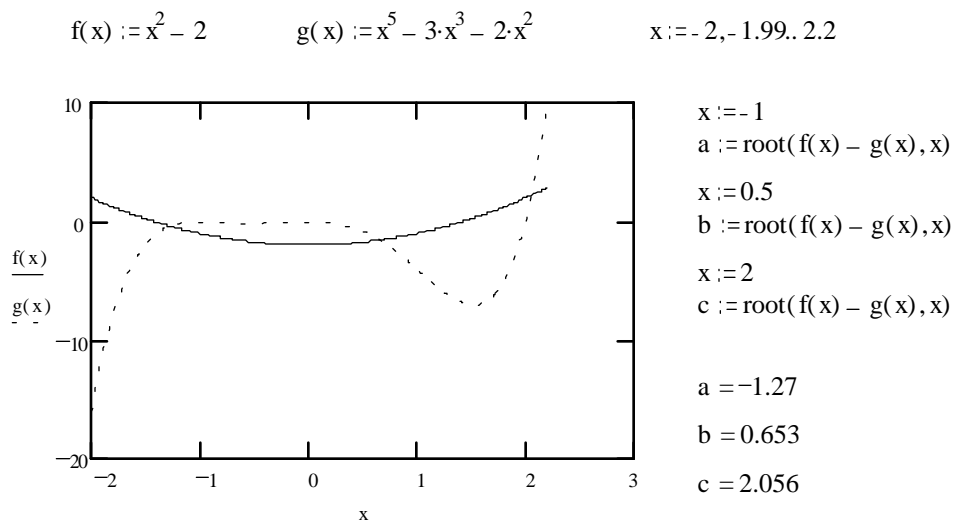
Instructions

After studying the worked example, open a blank Mathcad document and create your report there. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

Example

Use Mathcad's graphing facilities, numerical root finding (root function) and numerical integration to approximate the area between the graphs of $f(x) = x^2 - 2$ and $g(x) = x^5 - 3x^3 - 2x^2$.

Solution



$$\text{Area1} := \int_a^b (g(x) - f(x)) dx$$

$$\text{Area1} = 2.647$$

$$\text{Area2} := \int_b^c (f(x) - g(x)) dx$$

$$\text{Area2} = 6.296$$

$$\text{TOTAL AREA:} \quad \text{Area1} + \text{Area2} = 8.943$$

Problems

1. Repeat the Example for the area between the graphs of $f(x) = \sin(2x + 2)$ and $g(x) = x^4 + 2x^2 - 1$.
2. Repeat for the area of Exercise 33 p.409 LHE.
3. Repeat for the area of Exercise 34 p.409 LHE.
4. Let us suppose you got a job with Calculus City, and your tremendous mathematical skills made your boss give you the following assignment:

Determine the area of the park under development (see the map in Figure 5), so that the correct amount of some very expensive grass can be purchased for it.

Of course, generally this wouldn't be an easy problem. This time, however, luck is on your side. In the city records you find the following equations (where miles are the units) of the curves that correspond to the streets bounding the park:

$$f(x) = x \quad g(x) = \frac{\cos(4x)}{3} \quad h(x) = -(4x + 3)^3 - 0.7$$

Now you are ready to *solve* this problem, using Mathcad to sketch the functions, find their intersections (use the crosshair and then the root function), and numerically evaluate the integral(s) that you properly set up.

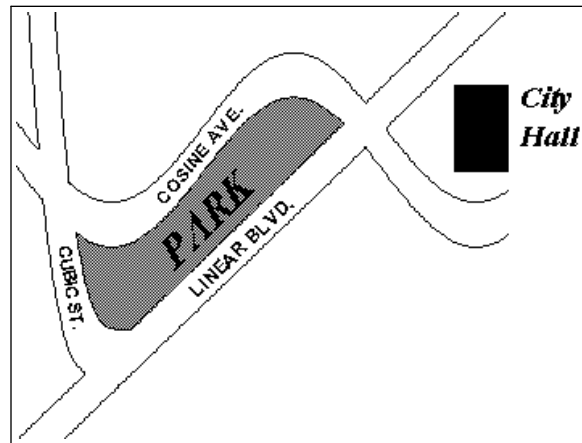


Figure 5: The Calculus City Park map.

6.2 Activity: Volume of a Solid of Revolution

Prerequisites: Read Sections 6.2 and 6.3 LHE.

The objective of this activity is to visually explore the disk (washer) and shell methods for finding volumes of solids of revolution.

Instructions

To begin this activity, start Mathcad and create a blank document. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

From the **File** menu choose **Insert**, and select the file `DISKMET.MCD`. The document you have just inserted begins with a brief description - read it. (You will see a few definitions of Mathcad functions and variables; there are quite a few more hidden from your view). The graph in Figure 6 illustrates the disk (washer) method applied in this particular example. Mathcad's 3-D visualization facilities are limited - to get a better view of the same objects you will now use the MAPLE V system.

1. Save your Mathcad document, but do not exit from Mathcad.
2. Hold the `<Alt>` key down, then press the `<Tab>` key (perhaps more than once) until the rectangle in the center of the screen says "Program Manager", then release both keys. You should now see the Program Manager window.
3. Double-click on the "Maple V" icon.
4. Once Maple starts, select **Open** from its **File** menu. Load the file `DISKMET.MS`. Your screen should now resemble Figure 7.
5. Read the instructions included in the Maple document and ignore the executable statements.



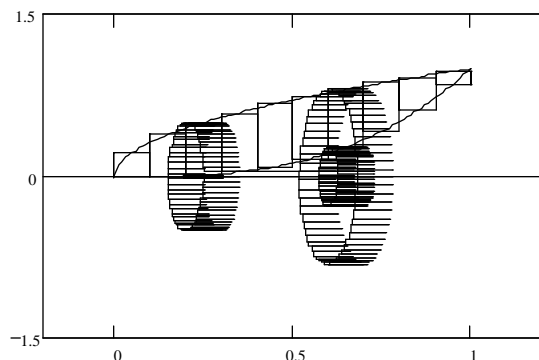


Figure 6: This picture, from the DISKMET.MCD document, illustrates the disk (washer) method.

When you are ready to execute the file, select **Execute Worksheet** from the **Format** menu. After a while, a plot window shown in Figure 8 should appear.

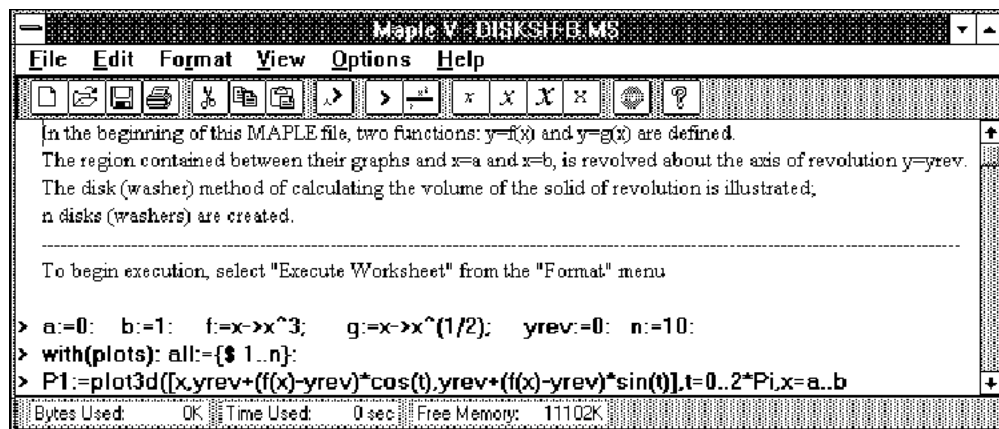


Figure 7: The Maple V main window after loading the template file.

Look at the plot. If you want to get a different view of it, drag your mouse (holding the left button) inside the plot window, then hit **<Enter>**. Experiment with it a few times. Once you get a view you like, select **Copy** from the **Edit** menu to copy the image to the clipboard.

Go back to Mathcad, using **<Alt>-<Tab>** key combination. Paste the image into your Mathcad document

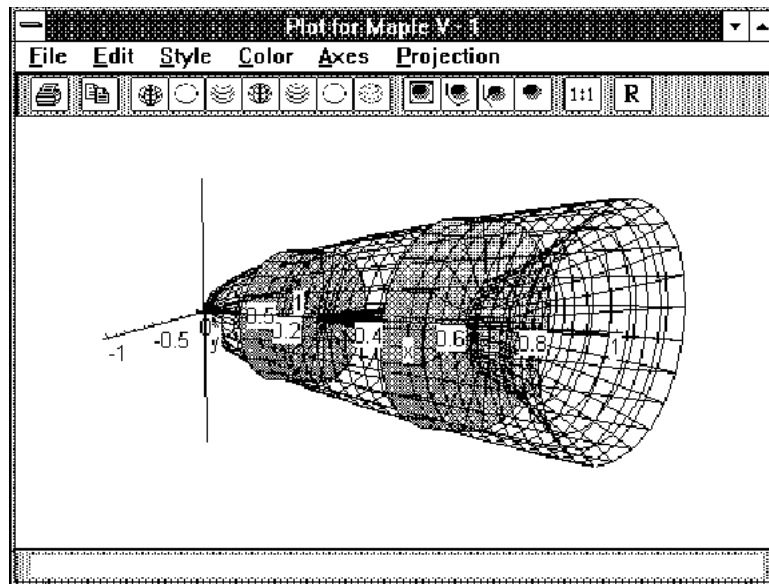


Figure 8: The Maple V plot window with an illustration of the disk (washer) method.

To see an illustration of computing the volume of the SAME solid by the shell method, insert the SHELLMET.MCD file into your Mathcad document (make sure you first position the red crosshair where you want the document inserted, typically BELOW your previous work).

Now switch to Maple using the $\langle \text{Alt} \rangle - \langle \text{Tab} \rangle$ key combination, and load the SHELLMET.MS. Then execute it, rotate the plot to get a view you like, and copy and paste the image back into Mathcad as before.

The Mathcad files and the corresponding Maple files, can be modified to change:

- disks/washers/shells to be plotted
- total number of disks/washers/shells
- axis of revolution (parallel to the x-axis)
- functions and interval determining the region being revolved

Your instructor may give you further directions.

6.3 Activity: Arc Length

Prerequisites: Read Section 6.4 LHE (pages 432-435)



The objective of this activity is to derive and explore the arc length formula. The meaning of rectifiability of curves will be addressed.

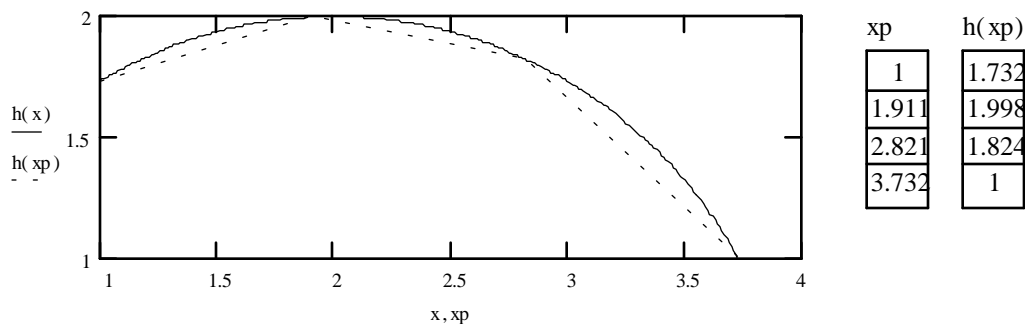


Figure 9: A snapshot from the "Arc Length" activity template document.

Instructions

A "template" document has been created for this assignment. Before proceeding, you should read the comments below. Unless your professor instructs you otherwise, you should load the document `ARCLLEN.MCD`. Follow the instructions on the screen to begin creating your lab report. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.



6.4 Activity: Area of a Surface of Revolution

Prerequisites: Read Section 6.4 LHE.

Instructions

The "template" document has been created for this assignment. Unless your professor instructs you otherwise, you should load the document `SURFREV.MCD`. Follow the instructions on the screen to begin creating your lab report. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.



Comments

Maple V templates, which are invoked in the course of this assignment, can be accessed by following these steps:



- Hold the `<Alt>` key down, then press the `<Tab>` key (perhaps more than once) until the

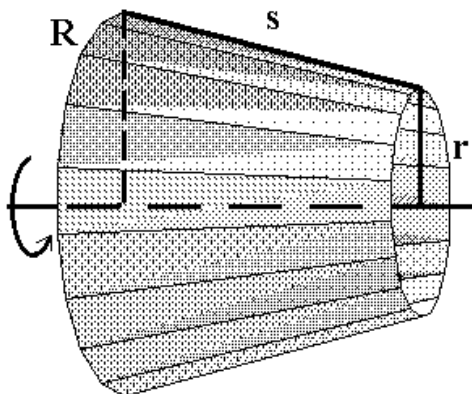


Figure 10: A frustum of a cone used in deriving the formula for the area of a surface of revolution.

rectangle in the center of the screen says "Program Manager", then release both keys. You should now see the Program Manager window.

- Double-click on the "Maple V" icon.
- Once Maple starts, select **Open** from its **File** menu. Load the appropriate template file.
- Read the instructions included in the Maple document and ignore the executable statements. When you are ready to execute the file, select **Execute Worksheet** from the **Format** menu. After a while, a plot window should appear.

6.5 Homework Help

- Solution of Exercises 13-32, p.409 LHE can be supplemented with a Mathcad illustration as in the example of Section 6.1.
- Mathcad and Maple templates discussed in Section 6.2 can be used to visualize the volumes of revolution obtained in solving most of the exercises of Sections 6.2 and 6.3 LHE.
Surfaces of revolution can also be visualized (see Section 6.4) when working on exercises in Section 6.4 LHE.
- Nearly all exercises in this chapter involve integration, thus you may want to consider using Mathcad in a way similar to Problem 1 on page 53, Problem 1 on page 61, or Problem 9 on page 63.

7 Integration Techniques and Improper Integrals

In Sections 5.2 and 5.3, you experienced some of the power and limitations of Mathcad's symbolic processor. In particular, when using Mathcad's symbolic processor to evaluate indefinite and definite integrals, you should remember the following facts.

1. Mathcad's symbolic processor can antidifferentiate the vast majority of functions encountered in calculus. However, Mathcad may obtain an expression which does not look like the antiderivative you found by pencil and paper. To see if they're equivalent (up to an arbitrary additive constant), have Mathcad simplify the difference of the two expressions.
2. (a) If your limits of integration are variable names or numbers without decimal points, Mathcad's symbolic processor will try to find an exact symbolic or numerical value for the integral.
 - (b) If you use decimals in the integrand or limits of integration, Mathcad's symbolic processor will return a 20-place numerical value if it can evaluate the integral. For example,

$$\int_0^3 \sin(x) \, dx \quad \text{yields} \quad -\cos(3) + 1$$

whereas

$$\int_0^3. \sin(x) \, dx \quad \text{yields} \quad 1.9899924966004454573$$

3. Mathcad's symbolic processor may incorrectly evaluate a definite integral. For example,

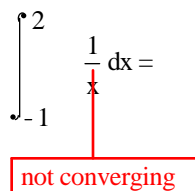
$$\int_{-1}^1 \sqrt{x^2} \, dx \quad \text{yields} \quad 0$$

since Mathcad incorrectly finds that $\sqrt{x^2}$ simplifies to x rather than $|x|$. Numerical integration gives the approximate answer

$$\int_{-1}^1 \sqrt{x^2} \, dx = 0.999999998647831$$

Of course, the exact answer is 1.

-
4. Mathcad incorrectly finds that $\int \frac{1}{x} dx$ yields $\ln x$ rather than $\ln |x|$. Similarly, Mathcad incorrectly finds that $\int \tan x dx$ yields $-\ln \cos x$ rather than $-\ln |\cos x|$.
 5. If Mathcad can't symbolically evaluate an integral, you will see the message "No closed form found for integral" or Mathcad will simply return the same integral. Expressing the integral in an equivalent form - by using a trigonometric identity or a u -substitution - may result in successful integration by Mathcad. Even if a closed form solution in terms of elementary functions doesn't exist for an integral, Mathcad may still return the solution in terms of certain special integral functions (for examples, see Section 5.3).
 6. According to Theorem 4.4 of LHE, if a function $f(x)$ is not continuous on the closed interval $[a, b]$, then the definite integral of $f(x)$ may not be defined on $[a, b]$. For example, the function $1/x$ has an infinite discontinuity at $x = 0$ and, in fact, the definite integral of $1/x$ on $[-1, 2]$ doesn't exist (it is a divergent improper integral). Therefore, we shouldn't expect Mathcad to obtain a finite value for this integral. Indeed, if we ask Mathcad to symbolically integrate $1/x$ on $[-1, 2]$, it returns the message "No closed form found for integral." If we ask Mathcad to numerically evaluate the integral, it returns

$$\int_{-1}^2 \frac{1}{x} dx =$$


not converging

Similarly, if we ask Mathcad to symbolically integrate $1/x$ on $[0, 2]$, it returns

$$\int_0^2 \frac{1}{x} dx \quad \text{yields} \quad \infty$$

On the other hand, Mathcad correctly evaluates the following convergent improper integral:

$$\int_0^2 \frac{1}{\sqrt{x}} dx \quad \text{yields} \quad 2\sqrt{2}$$

7. In order for Mathcad's symbolic processor to recognize the inverse trigonometric functions, they must be typed in Mathcad form as follows:

<i>function</i>	<i>Mathcad form</i>
$\arcsin(x)$	$\text{asin}(x)$
$\arccos(x)$	$\text{acos}(x)$
$\arctan(x)$	$\text{atan}(x)$
$\text{arccot}(x)$	$\pi - \text{atan}(x)$
$\text{arcsec}(x)$	$\text{acos}\left(\frac{1}{x}\right)$
$\text{arccsc}(x)$	$\text{asin}\left(\frac{1}{x}\right)$

Note that Mathcad does not recognize $\text{acot}(x)$, $\text{asec}(x)$ and $\text{acsc}(x)$ as the built-in functions for $\text{arccot}(x)$, $\text{arcsec}(x)$ and $\text{arccsc}(x)$; instead, the equivalent expressions listed above must be used.

7.1 Activity: Integration Techniques

Prerequisites: Read Sections 7.2-7.5 LHE

In this activity, you will practice your integration skills with the help of Mathcad's symbolic processor.

Instructions

Open a blank Mathcad document and create your report there. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

Problems

From the **Symbolic** menu, choose **Load Symbolic Processor**. Once the loading is complete, from the same menu, choose **Derivation Format**. In the Derivation Format dialog

- switch "Show derivation comments" to ON (make sure a check appears in the box);
- select "Show derivation steps" horizontally.

A closed form solution in terms of elementary functions exists for each integral listed below. However, you may need to use a trigonometric identity or u -substitution before Mathcad can sym-

bolically evaluate the integral. For each integral listed below (or the integrals specified by your instructor), you should do the following:

- (a) Ask Mathcad to symbolically evaluate the integral. If Mathcad can't evaluate the integral, state "no integral found" in your document; then (with pencil and paper) use a trigonometric identity or u -substitution to express the integral in an equivalent form, and have Mathcad evaluate it. In your Mathcad document, state the trigonometric identity or u -substitution that you used. If you used a u -substitution in an indefinite integral, then invoke the Substitute for Variable command to express Mathcad's answer in terms of the original variable of integration. Verify the answer by having Mathcad differentiate it symbolically - and simplify the result, if needed.
- (b) Evaluate the integral using pencil and paper. (Your instructor may request that you hand this in for grading.) If your answer looks different from the one obtained by Mathcad, then use Mathcad to verify that they're equivalent. As you evaluate the integral by hand, you may want to have Mathcad verify your u -substitution (as suggested by Problem 9 on page 63), or verify your partial fraction expansion (by using the **Convert to Partial Fraction** command from the **Symbolic** menu).

$$1. \int \frac{1}{x^{3/2}\sqrt{1-x}} dx$$

$$2. \int \frac{e^{3x}}{(1+e^x)^3} dx$$

$$3. \int \frac{\sec^2 x}{\tan x (\tan x - 1)} dx$$

$$4. \int \frac{\sqrt{\sqrt{x}}}{1+\sqrt{x}} dx$$

$$5. \int \frac{\sqrt{x^2-9}}{x} dx$$

$$6. \int 2x\sqrt{2x-3} dx$$

$$7. \int \frac{\sqrt{4-x^2}}{2x} dx$$

$$8. \int \sqrt{\sin x} \cos^3 x dx$$

$$9. \int \frac{\csc \sqrt{2x}}{\sqrt{x}} dx$$

$$10. \int \sqrt{1+\sqrt{x}} dx$$

$$11. \int \frac{4x-2}{3(x-1)^2} dx$$

$$12. \int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx$$

$$13. \int (\cot x) (\ln |\sin x|) dx$$

$$14. \int \frac{1}{\sec x - 1} dx$$

$$15. \int \frac{\sec^2 x}{4 + \tan^2 x} dx$$

$$16. \int \frac{\sqrt{1+x^4}}{x^3} dx$$

17. $\int_1^b \frac{\sqrt{1+x^4}}{x^3} dx$

18. $\int \sec^4 x \sqrt{\tan x} dx$

19. $\int \operatorname{arcsec} \sqrt{x} dx$

20. $\int e^{ax} \sin(bx) dx$

7.2 Activity: Improper Integrals

This activity will improve your understanding of what it means for an improper integral to converge or diverge.

Instructions

A "template" document IMPROPER.MCD is available for this assignment - your professor will inform you whether you are to use it. Answer all the problems identified by your professor. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.



Problems

From the **Symbolic** menu, choose **Load Symbolic Processor**. Once the loading is complete, from the same menu, choose **Derivation Format**. In the Derivation Format dialog

- switch "Show derivation comments" to ON (make sure a check appears in the box);
- select "Show derivation steps" horizontally.

- Use Mathcad's symbolic processor to evaluate $\int_1^b \frac{1}{x^p} dx$ ($b > 1$). Notice that the answer cannot be valid when $p = 1$. (Why?) Now, use Mathcad to show that when $p = 1$, $\int_1^b \frac{1}{x} dx = \ln b$. Of course, the above integrals can all be done by hand. You should work them out with pencil and paper just to make sure you can do it.
 - By considering the limit as $b \rightarrow \infty$ of your answers in part (a), explain why the improper integral $\int_1^\infty \frac{1}{x^p} dx$ converges when $p > 1$ and diverges when $p \leq 1$. It follows that the area under the curve $\frac{1}{x^p}$ on the interval $[1, \infty)$ is finite if and only if $p > 1$. What is the value of this area?

On the same plot, graph $\frac{1}{x^p}$ for $p = 2, 1.5, 1, 0.5, 0, -0.5$ over the interval $[1, 20]$. Which curve approaches the x -axis the fastest?

There is an infinite area under the curve $\frac{1}{x}$ on the interval $[1, \infty)$. It is interesting to observe that only a small portion of this infinite area is found on the finite interval $[1, b]$, even if b is very large. For example, calculate the area for $b = 10^{100}$ (which is larger than the number of atoms in the universe). There still must be an infinite area under $\frac{1}{x}$ on the interval $[10^{100}, \infty)$ even though the curve is within 10^{-100} of the x -axis.

Note: In numerical calculations, Mathcad treats ∞ to be the large but finite number

10^{307} . This is close to the largest number that can be numerically processed by Mathcad. To see this, numerically evaluate $\ln b$ at $b = 10^{307}$, 10^{308} and ∞ .

- (c) Use Mathcad to show that the area under the curve $\frac{1}{x^p}$ ($p > 1$) on the interval $[b, \infty)$ is $\frac{b^{1-p}}{p-1}$. Hint: Observe that $\int_b^\infty \frac{1}{x^p} dx = \int_1^\infty \frac{1}{x^p} dx - \int_1^b \frac{1}{x^p} dx$ and use the results obtained in parts (a) and (b). In particular, calculate the area under $\frac{1}{x^2}$ on $[10^{100}, \infty)$. This illustrates that there is only a tiny amount of area under the curve $\frac{1}{x^2}$ to the right of any very large number.
- (d) According to part (b), no matter how close p gets to 1, the area under the curve $\frac{1}{x^p}$ is finite provided $p > 1$. However, the area under the curve $\frac{1}{x}$ is infinite. Find the area under $\frac{1}{x^{1.001}}$ on the interval $[1, 10^{100}]$ and on the interval $[10^{100}, \infty)$. Thus, the area under the curve $\frac{1}{x^{1.001}}$ for $x > 10^{100}$ is finite but the area under the curve $\frac{1}{x}$ for $x > 10^{100}$ is infinite.
2. (a) Use Mathcad to evaluate $\int_2^b \frac{1}{x \ln x} dx$. Also work out this integral by hand (let $u = \ln x$). You should have obtained $\ln(\ln b) - \ln(\ln 2)$, which clearly approaches infinity as $b \rightarrow \infty$. Thus, the area under the curve $\frac{1}{x \ln x}$ over $[2, \infty)$ is infinite and the improper integral $\int_2^\infty \frac{1}{x \ln x} dx$ diverges.
- (b) On the same plot, graph the two functions $\frac{1}{x \ln x}$ and $\frac{1}{x^{1.001}}$ over the interval $[2, 10]$. Based on what you know about the areas under these curves, explain why $\frac{1}{x \ln x}$ cannot remain below $\frac{1}{x^{1.001}}$ as $x \rightarrow \infty$. Eventually, the curve $\frac{1}{x \ln x}$ must intersect and remain above the curve $\frac{1}{x^{1.001}}$. In fact, the point of intersection is in the range 10^{3955} to 10^{3965} , a number too large for Mathcad to process!
- (c) Use Mathcad to show that the area under $\frac{1}{x \ln x}$ on the large interval $[2, 10^{100}]$ is a relatively small number. Remember that the area under this curve from 10^{100} to ∞ is infinite. Therefore, the integral $\int_2^b \frac{1}{x \ln x} dx$ monotonically increases to infinity as $b \rightarrow \infty$, but does so *very* slowly.
3. Suppose the curve $\frac{1}{x}$ is rotated about the x -axis over the interval $[1, \infty)$ to obtain a 3-dimensional object that looks like a horn. The surface area of the horn is given by

$$\lim_{b \rightarrow \infty} \int_1^b 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}} dx.$$

The above integral gives the horn's surface area over the finite interval $[1, b]$. You should be able to derive it. If you forgot the formula for surface area, look it up and check that the above is correct.

- (a) Mathcad cannot evaluate this integral symbolically. Try it. What error message does Mathcad give?

Now evaluate this integral numerically for $b = 1000$. Explain why this integral must diverge as $b \rightarrow \infty$. Therefore, the surface area of the horn is infinite. Hint: Compare the integrand with $2\pi \left(\frac{1}{x}\right)$.

- (b) In part (a), you saw Mathcad could not evaluate $\int_1^b 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}} dx$. Use pencil and paper to verify that the substitution $u = x^2$ results in the equivalent integral $\int_1^{b^2} \pi \frac{\sqrt{u^2+1}}{u^2} du$ and then show that Mathcad can evaluate this integral for arbitrary b . In particular, evaluate this integral when $b = 10^{100}$ (simplify Mathcad's answer) and when " $b = \infty$ ".
- (c) Use Mathcad to show that the improper integral which gives the volume inside the horn over $[1, \infty)$ is equal to π . Thus, the horn has a finite volume and an infinite surface area!!
- (d) In part (b), you used Mathcad to evaluate the integral $\int_1^{b^2} \pi \frac{\sqrt{u^2+1}}{u^2} du$. In fact, you should be able to evaluate this integral by using the integration tables in Appendix C LHE, or by using an appropriate substitution for u . What substitution should you make for u in order to carry out the integration? (For homework, use pencil and paper to carry out the integration by substitution and obtain the answer that Mathcad found in part (b). Your instructor may request that you hand this in for grading.) Which integration formula in Appendix C LHE could you use to evaluate this integral?
- (e) The expression obtained by Mathcad for the integral $\int_1^{b^2} \pi \frac{\sqrt{u^2+1}}{u^2} du$ in part (b) is a function of the parameter b . Briefly explain why the limit of this expression must be infinite as b tends to infinity.

7.3 Homework Help

- In the numerous integration exercises of Sections 7.2-7.5 LHE, you may want to verify the correctness of your solution, as suggested in Section 7.1 of this manual.
- Sections: 7.7 and 7.8 LHE deal with limits, thus, some of the techniques discussed in Chapter 1 of this manual may be useful.