Old Dominion University

Math 307 **Solutions Final Exam** **Spring 2000**

To receive full credit show all work in arriving at your answers.

No.1 (5%) Solve the differential equation $(2x+3y)\,dx - x\,dy = 0$

Solution 1: Here $\frac{\partial M}{\partial y} = 3$ and $\frac{\partial N}{\partial x} = -1$ and so the given equation is not exact. However, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3-(-1)}{-x} = -\frac{4}{x}$ Hence there exists an integrating factor $e^{\int -4\frac{dx}{x}} = e^{-4\ln x} = x^{-4}$ This gives the differential equation $\left(\frac{2}{x^3} + \frac{3y}{x^4}\right)dx - \frac{1}{x^3}\,dy = 0$ Where now we have $\frac{\partial M}{\partial y} = \frac{3}{x^4}$ and $\frac{\partial N}{\partial x} = \frac{3}{x^4}$ and so the equation is exact with solution $\phi(x,y) = \frac{1}{x^2} + \frac{y}{x^3} = C^*$ where C^* is a constant

Solution 2: The given equation has homogeneous functions. Hence we can make the substitution $y = vx$ with $dy = vdx + xdv$ to obtain $(2x+3vx)dx - x(vdx+xdv) = 0$ Separate the variables to obtain $\frac{2xdx}{x^2} = \frac{dv}{1+v}$ Now integrate to get
$\ln x^2 = \ln(1+v) + \ln C$ or $x^2 = C(1+v)$ Substitute $v = y/x$ to obtain $x^2 = C(1+\frac{y}{x})$ or $C^* = \frac{1}{C} = \frac{1}{x^2} + \frac{y}{x^3}$

No.2 (5%) Solve the initial value problem $\dfrac{dy}{dt} = 2y+1, \quad y(0) = 3$

Solution 1: Separate the variables and integrate $\dfrac{dy}{2y+1} = dt$ so that
$\ln(2y+1) = 2t + C^*$ or $2y+1 = e^{2t+C^*} = e^{2t}e^{C^*} = Ce^{2t}$ or $y = -\frac{1}{2} + \frac{C}{2}e^{2t}$ The initial condition requires that $y(0) = 3 = -1/2 + C/2$ or $C = 7$. This gives the final solution $y = -\frac{1}{2} + \frac{7}{2}e^{2t}$

Solution 2: First solve the homogeneous equation $\frac{dy}{dt} - 2y = 0$ This gives the complimentary solution $y_c = C_1e^{2t}$ By the method of undetermined coefficients assume $y_p = A$ we find $A = -1/2$. This gives the final solution $y = y_c + y_p = -\frac{1}{2} + C_1e^{2t}$ where C_1 is a constant. The initial condition requires $y(0) = 3 = -1/2 + C_1$ This gives $C_1 = 7/2$ and final solution $y = -\frac{1}{2} + \frac{7}{2}e^{2t}$.

No.3 (5%) Find the differential equation associated with the family of curves $y^2 = cx$ where c is a constant.

Solution 1: Write the family of curves in the form $\phi(x,y) = c$ or $\phi = \frac{y^2}{x} = c$ The exact differential equation is therefore $\dfrac{\partial\phi}{\partial x}\,dx + \dfrac{\partial\phi}{\partial y}\,dy = 0$ Here we have $\frac{\partial\phi}{\partial x} = -\frac{y^2}{x^2}$ and $\frac{\partial\phi}{\partial y} = 2\frac{y}{x}$ This gives the differential equation $-\frac{y^2}{x^2}\,dx + 2\frac{y}{x}\,dy = 0$ or $-y\,dx + 2x\,dy = 0$ or $\frac{dy}{dx} = \frac{y}{2x}$.

Solution 2: Differentiate $y^2 = cx$ implicitly to get $2y\frac{dy}{dx} = c$ Now eliminate the constant c from these two equations. We can solve for c from the first equation $c = y^2/x$ and substitute into the second equation $2y\frac{dy}{dx} = c = y^2/x$. Simplify to get $\frac{dy}{dx} = \frac{y}{2x}$.

No.4 (5%) Show that $\mu = x$ is an integrating factor of the differential equation $(2y+3x)\,dx + x\,dy = 0$ and then solve the differential equation.

Solution : Multiply by x to get $(2xy+3x^2)\,dx + x^2\,dy = 0$ Then $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$ so the equation is now exact. This means the solution $\phi(x,y) = c$ is such that $\frac{\partial\phi}{\partial x} = 2xy + 3x^2$ and $\frac{\partial\phi}{\partial y} = x^2$ Integrate the first equation $\int d\phi = \int 2xy\,\partial x + 3x^2\,\partial x$ or $\phi = x^2y + x^3 + f(y)$. This solution must satisfy $\frac{\partial\phi}{\partial y} = x^2 + \frac{df(y)}{dy} = x^2$ This gives $f(y) = 0$ and so the solution can be represented $\phi(x,y) = x^2y + x^3 = C$.

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No.5 (5%) Solve the differential equation $(y-x)\,dx + (x+y)\,dy = 0$

Solution: The given equation is exact because $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$. Hence the solution $\phi(x,y) = C$ must satisfy $\frac{\partial \phi}{\partial x} = y - x$ and $\frac{\partial \phi}{\partial y} = x + y$. Integrate the first equation to get $\phi = yx - \frac{x^2}{2} + f(y)$. This solution must satisfy $\frac{\partial \phi}{\partial y} = x + \frac{df(y)}{dy} = x + y$ or $\frac{df(y)}{dy} = y$ or $f(y) = \frac{y^2}{2}$. This gives the solution $\phi(x,y) = yx - \frac{x^2}{2} + \frac{y^2}{2} = C$.

No.6 (5%) Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = 1$

Solution: This is a first order linear ordinary differential equation with integrating factor $e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$. This gives $x\frac{dy}{dx} + y = x$ or $\frac{d}{dx}(xy) = x$ Multiply by dx and integrate to obtain $\int d(xy) = \int x\,dx$ or $xy = \frac{x^2}{2} + C$ This gives the solution $y = \frac{x}{2} + \frac{C}{x}$.

No.7 (5%) A ball is thrown vertically upward with an initial velocity of 64ft/sec. Use $g = 32\text{ft/sec}^2$ as the acceleration of gravity and find equations for the velocity and position of the ball as a function of time. (Assume $x = 0$ and time $t = 0$.)

Solution We have mass times acceleration $= m\frac{dV}{dt} =$ Sum of forces. The only force is the weight $W = mg$ acting downward. This gives the differential equation $m\frac{dV}{dt} = -mg$ or $\frac{dV}{dt} = -g$ Integrate to obtain $V = -gt + V_0$ where V_0 is a constant of integration. The initial condition requires that $V_0 = 64$ and so $V = -gt + 64$. Since $V = \frac{dx}{dt} = -gt + 64$ we have upon integrating that $x = -g\frac{t^2}{2} + 64t + x_0$ where x_0 is a constant of integration. The intial condition requires that $x_0 = 0$ and so $x = -g\frac{t^2}{2} + 64t$.

No.8 (5%) Find and describe the family of orthogonal trajectories associated with the parabolas $y = cx^2$.

Solution: We have $\dfrac{dy}{dx} = 2cx$. Now eliminate the constant c to obtain the differential equation of the family of parabolas $\dfrac{dy}{dx} = 2\dfrac{y}{x^2}x = \dfrac{2y}{x} = m$. The differential equation of the family of orthogonaly trajectories is $\dfrac{dy}{dx} = \dfrac{-1}{m} = -\dfrac{x}{2y}$ Now separate the variables $2y\,dy = -x\,dx$ and integrate to obtain the family of ellipses $y^2 + \frac{x^2}{2} = c^2$ where c^2 is the constant of integration.

No.9 (5%) Solve the differential equation $y'' + 5y' + 6y = 0$

Solution: Assume a solution $y = e^{\lambda t}$ and get the characteristic equation $\lambda^2 + 5\lambda + 6 = 0$ with roots $\lambda = -2$ and $\lambda = -3$ This gives the fundamental set $\{e^{-2x}, e^{-3x}\}$ and the general solution $y = c_1e^{-2x} + c_2e^{-3x}$

No.10 (5%) Solve the differential equation $y'' + 6y' + 9y = 0$

Solution: Assume a solution $y = e^{\lambda t}$ and get the characteristic equation $\lambda^2+6\lambda+9 = 0$ with roots $\lambda = -3$ and $\lambda = -3$ This gives the fundamental set $\{e^{-3x}, xe^{-3x}\}$ and the general solution $y = c_1e^{-3x} + c_2xe^{-3x}$

No.11 (5%) Solve the differential equation $y'' + 6y' + 13y = 0$

Solution: Assume a solution $y = e^{\lambda t}$ and get the characteristic equation $\lambda^2 + 6\lambda + 13 = 0$ with roots $\lambda = -3 + 2i$ and $\lambda = -3 - 2i$. This gives the fundamental set $\{e^{-3x}\cos 2x, e^{-3x}\sin 2x\}$ and the general solution $y = c_1e^{-3x}\cos 2x + c_2e^{-3x}\sin 2x$

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No.12 (5%) Use method of undetermined coefficients to solve $\frac{d^2y}{dt^2} + 4y = 12t + 4\cos 2t$

Solution: First solve the homogeneous equation $\frac{d^2y}{dt^2} + 4y = 0$ to get $y_c = c_1 \cos 2t + c_2 \sin 2t$ Then assume $y_p = A + Bt + Ct\cos 2t + Dt\sin 2t$ Differentiate and substitute into the given nonhomogenous differential equation to get

$$4At + 4B - 4C\sin 2t + 4D\cos 2t = 12t + 4\cos 2t$$

Now equate like terms to obtain $\begin{aligned} 4A &= 12 & 4B &= 0 & -4C &= 0 & 4D &= 4 \\ A &= 3 & B &= 0 & C &= 0 & D &= 1 \end{aligned}$ which gives the particular solution $y_p = 3t + t\sin 2t$. The general solution is then $y = y_c + y_p = c_1 \cos 2t + c_2 \sin 2t + 3t + t\sin 2t$.

No.13 (5%) The rate of disintegration (decay) of radium is proportional to the amount present. If 0.5% of the radium disappears in 12 years, what is the half-life of radium? Hint: If A_0 is the initial amount, then $A = 0.995A_0$ is left when $t = 12$ years.

Solution: Here $\frac{dA}{dt} = -kA$ has the solution $A = A_0 e^{-kt}$ where A_0 is the initial amount. After 12 years we have $A = 0.995A_0 = A_0 e^{-k12}$ which gives $0.995 = e^{-k12}$ or $(0.995)^{1/12} = e^{-k}$. Hence we can write $A = A_0 e^{-kt} = A_0 (e^{-k})^t = A_0 (0.995)^{t/12}$ When $A = \frac{1}{2}A_0 = A_0(0.995)^{t/12}$ then t must satisfy $-\ln 2 = \frac{t}{12}\ln(0.995)$ This gives $t = \frac{-12\ln 2}{\ln(0.995)} = 1660$ years.

Another way is from the equation $0.995 = e^{-k12}$ we solve for k and get $\ln(0.995) = -k(12)$ or $k = -\frac{1}{12}\ln(0.995)$. Now from $A = A_0 e^{-kt}$ we let $A = \frac{A_0}{2}$ and solve for t. This gives $\frac{A_0}{2} = A_0 e^{-kt}$ or $-\ln 2 = -kt$ or $t = \frac{\ln 2}{k} = \frac{\ln 2}{-\frac{1}{12}\ln(0.995)} = 1660$ years.

No.14 (10%) Use the method of variation of parameters to solve $\dfrac{d^2y}{dx^2} + y = \tan x$

Solution: First solve the homogeneous equation $\frac{d^2y}{dx^2} + y = 0$ to obtain the complimentary solution $y_c = c_1 \cos x + c_2 \sin x$ Assume a particular solution $y_p = u\cos x + v\sin x$ where $u = u(x)$ and $v = v(x)$ are functions to be determined from the system of differential equations

$$\begin{aligned} u'\cos x + v'\sin x &= 0 \\ u'(-\sin x) + v'\cos x &= \tan x \end{aligned}$$

Solving for $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$ we find

$$\begin{aligned} u' &= -\sin x\tan x = -\sin x\frac{\sin x}{\cos x} \\ u' &= -\frac{\sin^2 x}{\cos x} = \frac{-1+\cos^2}{\cos x} \\ u' = \frac{du}{dx} &= -\sec x + \cos x \\ u &= -\ln|\sec x + \tan x| + \sin x \end{aligned}$$

$$\begin{aligned} v' &= \cos x\tan x = \cos x\frac{\sin x}{\cos x} \\ v' = \frac{dv}{dx} &= \sin x \\ v &= -\cos x \end{aligned}$$

This gives the solution

$y = y_c + y_p = c_1\cos x + c_2\sin x + \cos x(-\ln|\sec x + \tan x| + \sin x) + \sin x(-\cos x)$

which simplifies to $y = c_1\cos x + c_2\sin x - \cos x\ln|\sec x + \tan x|$.

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No.15 (5%) A 6 lb weight stretches a linear spring 6 inches. The weight is then pulled down 4 inches below its equilibrium position and released from rest. Assume the weight is acted upon by a damping force equal to $2\frac{dx}{dt}$ in pounds where x is the displacement measured from the equilibrium position. Set up the differential equation and initial conditions which describes the motion of this spring-mass system.

Solution: The sum of forces about the equilibrium position gives

$$m\frac{d^2x}{dt^2} = \text{Sum of forces} = W - k(x+s) - 2\frac{dx}{dt}$$

subject to the initial condition $x(0) = 1/3ft$ and $x'(0) = 0$ Since $W = ks$ the above differential equation reduces to $m\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + kx = 0$ Here $W = mg$ or $m = W/g = 6/32 = 3/16$ and also $W = ks$ becomes $6 = k(1/2)$ or $k = 12$.

No. 16 (5%) Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$ if $F(s) = \dfrac{5s+23}{s^2+9s+20}$

Solution: By partial fractions $\dfrac{5s+23}{s^2+9s+20} = \dfrac{5s+23}{(s+4)(s+5)} = \dfrac{A}{s+4} + \dfrac{B}{s+5}$ we find that $A = 3$ and $B = 2$ this gives

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \mathcal{L}^{-1}\{\frac{3}{s+4}\} + \mathcal{L}^{-1}\{\frac{2}{s+5}\} = 3e^{-4t} + 2e^{-5t}$$

No.17 (5%) Use Laplace transforms to solve $\frac{dy}{dt} + y = h(t) \quad y(0) = 1$ where $h(t) = \begin{cases} 2 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

Solution: Take the Laplace transform of the given equation

$$\begin{aligned} \mathcal{L}\{\frac{dy}{dt}\} + \mathcal{L}\{y\} =& \mathcal{L}\{h(t)\} = \mathcal{L}\{2 - 2u_1(t)\} \\ sY(s) - y(0) + Y(s) =& \frac{2}{s} - \frac{2}{s}e^{-s} \\ sY(s) - 1 + Y(s) =& \frac{2}{s} - \frac{2}{s}e^{-s} \\ (s+1)Y(s) =& 1 + \frac{2}{s} - \frac{2}{s}e^{-s} \\ Y(s) =& \frac{1}{s+1} + \frac{2}{s(s+1)} - \frac{2e^{-s}}{s(s+1)} \end{aligned}$$

Using partial fractions $\dfrac{2}{s(s+1)} = \dfrac{2}{s} - \dfrac{2}{s+1}$ hence

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{1}{s+1}\} =& e^{-t} \\ \mathcal{L}^{-1}\{\frac{2}{s(s+1)}\} =& 2 - 2e^{-t} \\ \mathcal{L}^{-1}\{\frac{2e^{-s}}{s(s+1)}\} =& [2 - 2e^{-(t-1)}]u_1(t) \end{aligned}$$

This gives the solution $y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-t} + 2 - 2e^{-t} - [2 - 2e^{-(t-1)}]u_1(t)$

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No.18 (5%) Use Laplace transforms to solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 26y = u_4(t) \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$ and take the Laplace transform of the given differential equation to obtain

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 26Y(s) = \frac{e^{-4s}}{s}$$

Substitute $y(0) = 1$ and $y'(0) = 0$ and solve for $Y(s)$ to obtain

$$Y(s) = \frac{s+2}{s^2+2s+26} + \frac{e^{-4s}}{s(x^2+2s+26)}$$

Complete the square in the denominator and write

$$Y(s) = \frac{s+2}{(s+1)^2+5^2} + \frac{e^{-4s}}{s((s+1)^2+5^2)} = \frac{s+1}{(s+1)^2+5^2} + \frac{1}{(s+1)^2+5^2} + \frac{e^{-4s}}{s[(s+1)^2+5^2]}$$

We know that

$$\mathcal{L}^{-1}\{\frac{s}{s^2+5^2}\} = \cos 5t \quad \text{hence} \quad \mathcal{L}^{-1}\{\frac{(s+1)}{(s+1)^2+5^2}\} = e^{-t}\cos 5t \quad \text{first shift theorem}$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2+5^2}\} = \frac{1}{5}\sin 5t \quad \text{hence} \quad \mathcal{L}^{-1}\{\frac{1}{(s+1)^2+5^2}\} = \frac{1}{5}e^{-t}\sin 5t$$

By the integration property

$$\mathcal{L}^{-1}\{\frac{1}{s[(s+1)^2+5^2]}\} = \int \frac{1}{5}e^{-t}\sin 5t\,dt = \frac{1}{5}\left[\frac{5}{26} - \frac{1}{26}e^{-t}(5\cos 5t + \sin 5t)\right]$$

By the second shift theorem we have

$$\mathcal{L}^{-1}\{\frac{e^{-4s}}{s[(s+1)^2+5^2]}\} = \frac{1}{5}\left[\frac{5}{26} - \frac{1}{26}e^{-(t-4)}(5\cos 5(t-4) + \sin 5(t-4))\right]u_4(t)$$

This gives the solution

$$\begin{aligned} y(t) =& \mathcal{L}^{-1}\{Y(s)\} \\ =& e^{-t}\cos 5t + \frac{1}{5}e^{-t}\sin 5t + \frac{1}{5}\left[\frac{5}{26} - \frac{1}{26}e^{-(t-4)}(5\cos 5(t-4) + \sin 5(t-4))\right]u_4(t) \end{aligned}$$

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No.19 (5%) Use the convolution property to find $h(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$ where $F(s) = \dfrac{1}{s+2}$ and $G(s) = \dfrac{1}{s+3}$.

Solution: $h(t) = \mathcal{L}^{-1}\{F(s)G(s)\} = f(t)^{*}g(t)$ where

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s+2}\} = e^{-2t} \quad \text{and} \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{1}{s+3}\} = e^{-3t}$$

Hence

$$h(t) = f(t)^{*}g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t e^{-2\tau}e^{-3(t-\tau)}\,d\tau$$

$$h(t) = e^{-3t}\int_0^t e^{\tau}\,d\tau$$

$$h(t) = e^{-3t}\left[e^{\tau}\right]_0^t$$

$$h(t) = e^{-3t}\left[e^{t}-1\right]$$

$$h(t) = e^{-2t} - e^{-3t}$$

No.20 (5%) Assume a power series solution to the differential equation $\frac{dy}{dx} + xy = 0$ and find the resulting recurrence formula.

Solution: Assume a solution

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} c_n n x^{n-1}$$

Substitute into the differential equation to obtain

$$\sum_{n=0}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\text{sum on lower term} \quad c_1x^0 + \sum_{n=2}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\text{Shift summation index} \quad c_1x^0 + \sum_{n=0}^{\infty}(c_{n+2}(n+2) + c_n)x^{n+1} = 0$$

Now set equal to zero the coefficients of the various powers of x. This gives $c_1 = 0$ and $c_{n+2}(n+2)+c_n = 0$ or $c_{n+2} = \dfrac{-c_n}{n+2}$ This is the recurrence formula. We find

$$c_1 = c_3 = c_5 = ... = 0$$

i.e. all odd coefficients are zero

$$\text{The even coefficients are:} \quad c_2 = \frac{-c_0}{2}$$

$$c_4 = \frac{c_0}{2^2\,2!}$$

$$c_6 = \frac{-c_0}{2^3\,3!}$$

$$c_8 = \frac{c_0}{2^4\,4!} \quad \text{etc.}$$

This gives the solution

$$y = c_0[1 - \frac{x^2}{2} + \frac{x^4}{2^2\,2!} - \frac{x^6}{2^3\,3!} + \frac{x^8}{2^4\,4!} + ...]$$

End of Exam