# Chapter 8
# Green's Functions for ODE's

To introduce the concept of Green's functions we begin with Green's identity associated with linear second order differential operators. Every second order linear differential operator

$$L_x(y) = a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y \tag{8.1}$$

has associated with it an adjoint operator

$$\begin{aligned} L_x^*(y) &= \frac{d^2}{dx^2}[a_0(x)y] - \frac{d}{dx}[a_1(x)y] + a_2(x)y \\ L_x^*(y) &= a_0(x)\frac{d^2y}{dx^2} + (2a_0'(x) - a_1(x))\frac{dy}{dx} + (a_0''(x) - a_1'(x) + a_2(x))\,y. \end{aligned} \tag{8.2}$$

Let $u = u(x)$ and $v = v(x)$ denote two arbitrary continuous functions with first and second derivatives, then one can use the above operators to verify the Lagrange identity

$$vL_x(u) - uL_x^*(v) = \frac{d}{dx}\left[P(u,v)\right], \tag{8.3}$$

where

$$P(u,v) = \left[a_0(x)\frac{du}{dx} + a_1(x)u\right]v - \left[a_0(x)\frac{dv}{dx} + a_0'(x)v\right]u \tag{8.4}$$

is called the bilinear concomitant. The Lagrange identity (8.3) holds for all continuous differentiable functions $u(x)$ and $v(x)$ defined over some solution domain $I = \{x|\ a \le x \le b\}$. The functions $u$ and $v$ must be differentiable in order that $L_x(u)$ and $L_x^*(v)$ exist. This is the only restriction we place upon these functions. The integral of the Lagrange identity (8.3) produces the Green's identity

$$\int_a^b \left[vL_x(u) - uL_x^*(v)\right]\,dx = \big[P(u,v)\big]_a^b \tag{8.5}$$

where

$$\begin{aligned} \big[P(u,v)\big]_a^b =& \big[a_0(b)u'(b) + a_1(b)u(b)\big]v(b) - \big[a_0(b)v'(b) + a_0'(b)v(b)\big]u(b) \\ &- \big[a_0(a)u'(a) + a_1(a)u(a)\big]v(a) + \big[a_0(a)v'(a) + a_0'(a)v(a)\big]u(a) \end{aligned} \tag{8.6}$$

is an important right-hand side of boundary terms that must be analyzed for each differential operator $L_x(\ )$. An alternative derivation of the Green's identity is to define the inner product

$$(v, L_x(u)) = \int_a^b vL_x(u)\,dx$$

and then integrate this inner product by parts to obtain

$$(v, L_x(u)) = (u, L_x^*(v)) + \text{Boundary terms.}$$

In the sections that follow we shall find out how to use the Green's identity and appropriately choose the functions $u$ and $v$ in order to construct solutions to the two point boundary value problem

$$\begin{aligned} L_x(y) &= f(x), \quad a \le x \le b \\ B_1[y] = g_1 \quad &\text{and} \quad B_2[y] = g_2, \end{aligned} \tag{8.7}$$

where $L_x$ is the linear operator given by equation (8.1), $g_1, g_2$ are given constants, and $B_1$, $B_2$ are linear boundary operators of the Robin form

$$\begin{aligned} B_1[y] &= \left[\alpha_1 \frac{dy(x)}{dx} + \beta_1 y(x)\right]_{x=a} \\ B_2[y] &= \left[\alpha_2 \frac{dy(x)}{dx} + \beta_2 y(x)\right]_{x=b} \end{aligned} \tag{8.8}$$

with $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ constants, not all zero, and $f(x)$ is an arbitrary, but known, function. In the special case that $\alpha_1 = \alpha_2 = 0$ the boundary operators dictate that $y$ be specified on the boundary of the interval $I$. In this case the boundary conditions are said to be of the Dirichlet type. In the special case that $\beta_1 = \beta_2 = 0$ the boundary conditions are of the Neumann type. In the special case that $g_1 = g_2 = 0$ the boundary conditions are homogeneous otherwise they are termed nonhomogeneous.

In the following discussions of Green's functions, note that we shall be changing variables in the middle of a problem. In order to avoid confusion as to what variables are being used in the operators $L_x()$ and $L_x^*()$ just check the dummy variable used in the integration of the Green's identity as well as checking the subscript on the differential operators.

As an example of how one can use the Green's identity to solve boundary value problems, consider the Dirichlet type boundary value problem to solve

$$L_x(y) = f(x), \quad a \le x \le b, \tag{8.9}$$

where the solution is subject to the Dirichlet homogeneous boundary conditions

$$B_1[y] = y(a) = 0 \quad \text{and} \quad B_2[y] = y(b) = 0. \tag{8.10}$$

In order to solve this boundary value problem we change the variable $x$ in equations (8.1) and (8.2) to some new variable, $\xi$ and write the Green's identity as

$$\int_a^b [v(\xi)L_\xi(u) - u(\xi)L_\xi^*(v)]\,d\xi = \big[P(u(\xi), v(\xi))\big]_a^b. \tag{8.11}$$

Notice in the equation (8.11) the variable $\xi$ is used as a dummy variable of integration and so the operators $L_\xi$ and $L_\xi^*$ involve derivatives with respect to this variable. We now choose the functions $u$ and $v$ and use Green's identity to solve the boundary value problem given by equations (8.9), and (8.10). We let $u(\xi) = y(\xi)$ be the solution of the boundary value problem (8.9) (with $x$ replaced by $\xi$ ) and replace $u$ by $y$ in the above Green's identity. That is, in the Green's identity, given by equation (8.11), we let $L_\xi(y) = f(\xi)$ to obtain

$$\int_a^b v(\xi)f(\xi)\,d\xi - \int_a^b y(\xi)L_\xi^*(v)\,d\xi = \big[P(y(\xi), v(\xi))\big]_a^b, \tag{8.12}$$

where

$$\begin{aligned}\big[P(y,v)\big]_a^b = &\,[a_0(b)y'(b) + a_1(b)y(b)]\,v(b) - [a_0(b)v'(b) + a_0'(b)v(b)]\,y(b)\\ &- [a_0(a)y'(a) + a_1(a)y(a)]\,v(a) + [a_0(a)v'(a) + a_0'(a)v(a)]\,y(a).\end{aligned} \tag{8.13}$$

We now choose $v(\xi) = G^*(\xi; x)$ as the Green's function which satisfies

$$L_\xi^*(G^*(\xi; x)) = \delta(\xi - x), \quad a \le \xi \le b \tag{8.14}$$

involving the adjoint equation with the derivatives in $L_\xi^*$ with respect to $\xi$ and where $\delta(\xi - x)$ is the Dirac delta function introduced earlier which has the sifting property

$$\int_{\xi=x-\epsilon}^{\xi=x+\epsilon} y(\xi)\delta(\xi - x)\,d\xi = y(x). \tag{8.15}$$

This substitution for $v$ reduces Green's identity (8.12) to the form

$$\int_a^b G^*(\xi; x)f(\xi)\,d\xi - y(x) = P(y(b), G^*(b; x)) - P(y(a), G^*(a; x)). \tag{8.16}$$

Thus, the solution $y(x)$ to our boundary value problem can be obtained from the equation (8.16) provided that the remaining terms in equation (8.16) can all be evaluated. Let us examine the terms in the equation (8.16). In the first term $f(\xi)$ is given or known from the boundary value problem. The function $G^*$ is the Green's function which must be obtained by solving another differential equation (8.14) involving the adjoint operator $L_\xi^*()$. In the second term we have $y(x)$ as the solution we desire and this solution can be obtained provided we can solve for the Green's function and also that we can evaluate the right hand side of equation (8.16). The right hand side of equation (8.16) can be obtained from equation (8.13) by replacing $v(\xi)$ by $G^*(\xi;x)$ and by setting $B_1[y] = y(a) = 0$ and $B_2[y] = y(b) = 0$ as these are the boundary conditions required by equation (8.10). This gives for the right hand side of equation (8.16) the expression

$$P(y(\xi), G^*(\xi;x))\big|_a^b = [a_0(b)y'(b)]G^*(b;x) - [a_0(a)y'(a)]G^*(a;x). \tag{8.17}$$

The terms inside the brackets involve derivatives of the unknown solution $y$ and these terms are not known. However, we can remove these unknown terms by requiring that the Green's function satisfy the boundary conditions

$$B_1^*[G^*] = G^*(\xi;x)\big|_{\xi=b} = 0 \quad \text{and} \quad B_2^*[G^*] = G^*(\xi;x)\big|_{\xi=a} = 0. \tag{8.18}$$

These boundary conditions are called adjoint boundary conditions. In general, the adjoint boundary conditions are determined when we set the bilinear concomitant to zero and require $P(y, G^*)\big|_a^b = 0$. The above boundary conditions reduce the Green's identity given by equation (8.16) to

$$y(x) = \int_a^b G^*(\xi;x) f(\xi)\, d\xi, \tag{8.19}$$

where $G^*$ is the Green's function satisfying

$$L_\xi^*(G^*(\xi;x)) = \delta(\xi - x), \qquad a \le \xi \le b$$

with the boundary conditions

$$B_1^*[G^*] = G^*(a;x) = 0 \qquad B_2^*[G^*] = G^*(b;x) = 0. \tag{8.20}$$

The function $y(x)$ given by equation (8.19) is a solution of the boundary value problem (8.9) which satisfies the boundary conditions given by equation (8.10). Before we can actually prove it is a solution we must find out more about Green's functions.

**Green's Functions and Adjoint Green's Functions**

Define two different Green's functions $G$ and $G^*$ associated with the operators $L_x$ and $L_x^*$ given by the equations (8.1) and (8.2). We require

$$G = G(x;\xi) \quad \text{satisfy} \quad L_x(G(x;\xi)) = \delta(x-\xi), \quad a \le x \le b \tag{8.21}$$

with boundary conditions $B_1[G] = 0$ and $B_2[G] = 0$. The adjoint Green's function $G^* = G^*(x;\xi)$ is required to satisfy

$$L_x^*(G^*(x;\xi)) = \delta(x-\xi), \quad a \le x \le b \tag{8.22}$$

with adjoint boundary conditions $B_1^*[G^*] = 0$ and $B_2^*[G^*] = 0$ which are determined by an analysis of the bilinear concomitant in the Green's identity. In the above equations, all differentiation is with respect to x.

The operators $(L_x, B_1, B_2)$ and $(L_x^*, B_1^*, B_2^*)$ are called adjoint operators. The adjoint boundary conditions are chosen such that $P(G,G^*)\big|_a^b = 0$. Under these circumstances the Green's functions given by equations (8.21) and (8.22) will satisfy the relation

$$G^*(x;\xi) = G(\xi;x) \tag{8.23}$$

of symmetry with respect to the $\xi$ and $x$ variables. To prove this result we first multiply equation (8.21) by $G^*(x;t)$ and then change the variable $\xi$ in equation (8.22) to $t$. We can then multiply equation (8.22) by $G(x;\xi)$ to obtain

$$\begin{aligned} &G^*(x;t)L_x(G(x;\xi)) = G^*(x;t)\delta(x-\xi) \qquad \text{and} \\ &G(x;\xi)L_x^*(G^*(x;t)) = G(x;\xi)\delta(x-t). \end{aligned} \tag{8.24}$$

We subtract these two equations and then integrate from $x = a$ to $x = b$. We thus produce the Green's identity

$$\begin{aligned} P(G,G^*)\big|_a^b &= \int_a^b [G^*(x;t)L_x(G(x;\xi)) - G(x;\xi)L_x^*(G^*(x;t))]\,dx \\ &= \int_a^b [G^*(x,t)\delta(x-\xi) - G(x;\xi)\delta(x-t)]\,dx \\ &= G^*(\xi;t) - G(t;\xi). \end{aligned} \tag{8.25}$$

Observe that the bilinear concomitant $P(G,G^*)\big|_a^b$ equals zero because of our choice of the adjoint boundary conditions and consequently, the relation given by equation (8.23) follows. The symmetry of the Green's function allows one to express the solution to the previous Dirichlet problem given by equation (8.19) in either of the forms

$$y(x) = \int_a^b G^*(\xi; x) f(\xi)\, d\xi = \int_a^b G(x; \xi) f(\xi)\, d\xi. \tag{8.26}$$

**Green's Function by Laplace Transform**

Despite the fact that some Green's functions are defined by differential equations over finite intervals, one can still use the Laplace transform to solve for the Green's function. The following example will illustrate how this is done.

**Example 8-1. (Laplace Transforms)**
Using the Laplace transform find the Green's function for the operator

$$L_x(y) = \frac{d^2 y}{dx^2}, \qquad a \le x \le b \quad \text{with boundary conditions } y(a) = y(b) = 0.$$

**Solution:** The Green's function associated with the given differential operator is obtain as a solution of the differential equation

$$\frac{d^2 G}{dx^2} = \delta(x - \xi), \qquad a \le x \le b \quad G(a) = G(b) = 0 \tag{8.27}$$

We can still treat this differential equation over the interval $0 < x < \infty$, however, we will only be interested in the solution over the interval $a < x < b$. We let

$$\mathcal{L}\{G\} = \int_0^\infty G e^{-sx}\, dx = \overline{G}(s)$$

denote the Laplace transform of $G$. Upon taking the Laplace transform of the equation (8.27) there is obtained

$$\begin{aligned} \mathcal{L}\{G''\} = \mathcal{L}\{\delta(x - \xi)\} \qquad & \text{or} \\ s^2 \overline{G} - sG(0) - G'(0) = e^{-s\xi}. & \end{aligned} \tag{8.28}$$

Now $G(0)$ and $G'(0)$ are unknowns and so we replace these values with constants $A$ and $B$ to be determined at a later time. Solving the equation (8.28) for $\overline{G}$ gives

$$\overline{G} = \frac{A}{s} + \frac{B}{s^2} + \frac{e^{-s\xi}}{s^2}.$$

Taking the inverse Laplace transform of this equation produces

$$G = G(x; \xi) = A + Bx + (x - \xi) H(x - \xi), \tag{8.29}$$

where $H(x-\xi)$ is the Heaviside unit step function and $A$ and $B$ are constants to be determined. This equation implies that for $a \le x < \xi$ we have $G = A + Bx$ and the condition $G(a) = 0$ requires that

$$A + Ba = 0. \tag{8.30}$$

Similarly, for $\xi < x \le b$, the equation (8.29) becomes $G = A + Bx + x - \xi$ and the condition $G(b) = 0$ requires that

$$A + Bb + b - \xi = 0. \tag{8.31}$$

Solving the simultaneous equations (8.30) and (8.31) for the constants $A$ and $B$ we find the Green's function can be represented

$$G = G(x;\xi) = \frac{(\xi - b)(x-a)}{b-a} + (x-\xi)H(x-\xi). \tag{8.32}$$

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**Example 8-2. (Greens function.)**
Consider the Green's function problem to solve

$$L_x(G) = \frac{d^2G}{dx^2} + 3\frac{dG}{dx} + 2G = \delta(x-\xi), \qquad 0 < x < \ell \tag{8.33}$$

subject to the homogeneous boundary conditions $G(0) = 0$ and $G(\ell) = 0$. This equation can be solved by the method of Laplace transforms to produce the solution

$$G = G(x;\xi) = \left(\frac{e^{-2(\ell-\xi)} - e^{-(\ell-\xi)}}{e^{-\ell} - e^{-2\ell}}\right)\left(e^{-x} - e^{-2x}\right) + \left(e^{-(x-\xi)} - e^{-2(x-\xi)}\right) H(x-\xi) \tag{8.34}$$

which can also be expressed in the form

$$G = G(x;\xi) = \begin{cases} G_1(x;\xi) = e^{-2x+\xi}\left[\frac{e^x - 1}{e^\ell - 1}\right](e^\xi - e^\ell), & x < \xi \\ G_2(x;\xi) = e^{-2x+\xi}\left[\frac{e^\xi - 1}{e^\ell - 1}\right](e^x - e^\ell), & \xi < x \end{cases} \tag{8.35}$$

Consider also the adjoint Green's function problem to solve

$$L_x^*(G^*) = \frac{d^2G^*}{dx^2} - 3\frac{dG^*}{dx} + 2G^* = \delta(x-\xi), \qquad 0 < x < \ell \tag{8.36}$$

subject to the homogeneous boundary conditions $G^*(0) = 0$ and $G^*(\ell) = 0$. One can use the method of Laplace transforms and readily verify that the above boundary value problem has the solution

$$G^* = G^*(x;\xi) = \left(\frac{e^{\ell-\xi} - e^{2(\ell-\xi)}}{e^{2\ell} - e^{\ell}}\right)\left(e^{2x} - e^{x}\right) + \left[e^{2(x-\xi)} - e^{(x-\xi)}\right] H(x-\xi). \tag{8.37}$$

which can also be expressed in the form

$$G^* = G^*(x;\xi) = \begin{cases} G_1^*(x;\xi) = e^{x-2\xi}\left[\frac{e^{x}-1}{e^{\ell}-1}\right](e^{\xi} - e^{\ell}), & x < \xi \\ G_2^*(x;\xi) = e^{x-2\xi}\left[\frac{e^{\xi}-1}{e^{\ell}-1}\right](e^{x} - e^{\ell}), & \xi < x \end{cases} \tag{8.38}$$

Observe that if the variables $x$ and $\xi$ are interchanged in the above equation (8.35), then the inequalities $x < \xi$ and $\xi < x$ must also be interchanged. Note that by interchanging $x$ and $\xi$ in $G_1$ we obtain $G_2^*$. Similarly, if we interchange $x$ and $\xi$ in $G_2$ we obtain $G_1^*$. In this sense we can write $G(x;\xi) = G^*(\xi;x)$.

We purposely confuse the issue the solving the equations

$$L_\xi\left(G(\xi;x)\right) = \frac{d^2G}{d\xi^2} + 3\frac{dG}{d\xi} + 2G = \delta(\xi - x), \quad 0 < \xi < \ell \tag{8.39}$$

subject to the boundary conditions $G(0;x) = 0$ and $G(\ell;x) = 0$ to obtain the solution

$$G(\xi;x) = \begin{cases} e^{-2\xi+x}\left[\frac{e^{\xi}-1}{e^{\ell}-1}\right](e^{x} - e^{\ell}), & \xi < x \\ e^{-2\xi+x}\left[\frac{e^{x}-1}{e^{\ell}-1}\right](e^{\xi} - e^{\ell}), & x < \xi \end{cases} \tag{8.40}$$

Now note that $G(\xi;x) = G^*(x;\xi)$. Similarly, the equation

$$L_\xi^*\left(G^*(\xi;x)\right) = \frac{d^2G^*}{d\xi^2} - 3\frac{dG^*}{d\xi} + 2G^* = \delta(\xi - x), \quad 0 < x < \ell \tag{8.41}$$

subject to the boundary conditions $G^*(0;x) = 0$ and $G^*(\ell;x) = 0$ has the solution

$$G^*(\xi;x) = \begin{cases} e^{\xi-2x}\left[\frac{e^{\xi}-1}{e^{\ell}-1}\right](e^{x} - e^{\ell}), & \xi < x \\ e^{\xi-2x}\left[\frac{e^{x}-1}{e^{\ell}-1}\right](e^{\xi} - e^{\ell}), & x < \xi \end{cases} \tag{8.42}$$

Now note that $G^*(\xi;x) = G(x;\xi)$.

In summary, we observe that

$$\begin{aligned} G = G(x;\xi) &\quad \text{satisfies } L_x(G) = \delta(x-\xi) \text{ in the } x\text{-variable} \\ G^* = G^*(\xi;x) = G(x;\xi) &\quad \text{satisfies } L_\xi^*(G) = \delta(\xi-x) \text{ in the } \xi\text{-variable} \\ G^* = G^*(x;\xi) &\quad \text{satisfies } L_x^*(G^*) = \delta(x-\xi) \text{ in the } x\text{-variable} \\ G = G(\xi;x) = G^*(x;\xi) &\quad \text{satisfies } L_\xi(G^*) = \delta(\xi-x) \text{ in the } \xi\text{-variable.} \end{aligned}$$

One can therefore consider either of the equations

$$L_x(G) = \delta(x-\xi) \quad \text{or} \quad L^*_\xi(G) = \delta(\xi - x)$$

as defining the Green's function $G(x;\xi)$.

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The solution to equation (8.21) is called a direct Green's function and the solution to equation (8.22) is call the adjoint Green's function. You can interpret these Green's functions as follows: $G(x;\xi)$ satisfies the direct problem given by equation (8.21) in the $x$ variable and it satisfies the adjoint Green's function problem in the $\xi$ variable. We then obtain the equations

$$\begin{aligned} L_x(G(x;\xi)) &= \delta(x-\xi), \qquad a \le x \le b \\ L^*_\xi(G(x;\xi)) &= \delta(\xi - x), \qquad a \le \xi \le b. \end{aligned}$$

One may use either of these equations to calculate the Green's functions.

In the special case $L_x$ is a linear second order self-adjoint differential operator, we have $L_x = L^*_x$ and therefore $G = G^*$. In this case the Green's function is symmetric in $\xi$ and $x$. That is, since the equations are self-adjoint, the condition $G^*(x;\xi) = G(\xi;x)$ reduces to $G(x;\xi) = G(\xi;x)$. This is particularly useful as all second order linear differential operators can be written in self-adjoint form.

**Example 8-3. (Greens function.)**
In the case of self-adjoint operators the Green's function representation becomes simpler since $G = G^*$. We have shown that every second order differential equation

$$L_{1x}[y] = a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \quad a_0(x) \ne 0 \tag{8.43}$$

can be written in the self-adjoint form

$$L_x[y] = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0. \tag{8.44}$$

This is accomplished by multiplying the equation (8.43) by the function

$$\mu(x) = \frac{1}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)}\,dx}, \quad a_0(x) \ne 0. \tag{8.45}$$

For example, consider the differential equation

$$L_{1x}[y] = \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0, \quad 0 < x < \ell \tag{8.46}$$

with homogeneous boundary conditions $y(0) = 0$ and $y(\ell) = 0$. We multiply the equation (8.46) by $\mu(x) = e^{3x}$ to obtain the self-adjoint form

$$L_x[y] = \frac{d}{dx}\left(e^{3x}\frac{dy}{dx}\right) + 2e^{3x}y = 0. \tag{8.47}$$

The Green's function associated with this operator satisfies

$$L_x[G] = \frac{d}{dx}\left(e^{3x}\frac{dG}{dx}\right) + 2e^{3x}G = \delta(x-\xi). \tag{8.48}$$

Observe that if we make the change of variable $u = e^{3x}G$, the differential equation (8.48) becomes

$$\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = \delta(x-\xi) \tag{8.49}$$

which is the same differential equation solved in the previous example. We find the solution

$$u = e^{3x}G = \begin{cases} e^{x-2\xi}\left[\frac{e^x-1}{e^\ell-1}\right](e^\xi - e^\ell), & x < \xi \\ e^{x-2\xi}\left[\frac{e^\xi-1}{e^\ell-1}\right](e^x - e^\ell), & \xi < x \end{cases}$$

and consequently we obtain the Green's function

$$G = G(x;\xi) = \begin{cases} G_1(x;\xi) = e^{-2x-2\xi}\left[\frac{e^x-1}{e^\ell-1}\right](e^\xi - e^\ell), & x < \xi \\ G_2(x;\xi) = e^{-2x-2\xi}\left[\frac{e^\xi-1}{e^\ell-1}\right](e^x - e^\ell), & \xi < x \end{cases} \tag{8.50}$$

Observe that when the operator $L_x$ is self-adjoint we have the symmetry conditions $G_2(x;\xi) = G_1(\xi;x)$ and similarly $G_1(x;\xi) = G_2(\xi;x)$. This is an example of a self-adjoint operator where the associated Green's function is symmetric and satisfies $G(x;\xi) = G(\xi;x)$. The symmetry relation $G(x;\xi) = G(\xi;x)$ is known as Maxwell's reciprocity principle which can be interpreted as the response at point $x$ due to an impulse at the point $\xi$ is the same as the response at $\xi$ due to an impulse at the point $x$. For example, $G(\frac{1}{3},\frac{2}{3}) = G(\frac{2}{3},\frac{1}{3})$. To visualize this we plot the two functions $y_1 = G(x;\frac{2}{3})$ and $y_2 = G(x;\frac{1}{3})$ in the figure 8-1 for the special case where $\ell = 1$.