Chapter 1
Introduction

In this introduction we review some topics from ordinary differential equations (ODE’s) because many partial differential equations (PDE’s) can be reduced to a study of ordinary differential equations. We then introduce some important definitions and terminology associated with first and second order partial differential equations. This is followed by an introduction to some solution techniques associated with easy to solve partial differential equations.

Ordinary differential equations.

We begin with a review of some solution techniques for ordinary differential equations. The differential equations considered are linear differential equations of the form

\[ L(y) = a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y = F(x), \]  

where the coefficients \( a_0(x), a_1(x), \ldots, a_n(x) \) are continuous functions over some interval \( a \leq x \leq b \) and \( a_0(x) \) is not identically zero over this interval. The general procedure for solving such equations is to first solve the \( n \)th-order homogeneous equation \( L(y) = 0 \) and obtain a fundamental set of solutions \( \{y_1(x), y_2(x), \ldots, y_n(x)\} \). The general solution of the homogeneous equation is a linear combination of these \( n \)-independent solutions. It is called a complementary solution and written as

\[ y_c = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \]  

where \( c_1, c_2, \ldots, c_n \) are arbitrary constants. Next find any particular solution \( y_p \) of the nonhomogeneous differential equation \( L(y) = F(x) \). One usually tries to use the method of undetermined coefficients or the method of variation of parameters to construct a particular solution. The general solution of equation (1.1) is then written \( y = y_c + y_p \).

In applied problems the solutions to differential equations of the form given by equation (1.1) are required to satisfy certain auxiliary conditions. The number of these auxiliary conditions is in most applications equal to the order of the differential equation. For example, when the second order differential equation

\[ a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = 0 \quad a \leq x \leq b \]  

(1.3)
is subjected to the auxiliary conditions at a single point \( x = a \) of the form

\[
y(a) = \alpha \quad y'(a) = \beta
\]

where \( \alpha \) and \( \beta \) are given constants, the differential equation plus auxiliary conditions is referred to as an initial value problem (IVP). Initial value problems usually have unique solutions. Whenever the differential equation (1.3) is subjected to auxiliary conditions at two different points, say \( x = a \) and \( x = b \), having the form

\[
c_{11}y(a) + c_{12}y'(a) = \alpha \quad c_{11}^2 + c_{12}^2 \neq 0 \\
c_{21}y(b) + c_{22}y'(b) = \beta \quad c_{21}^2 + c_{22}^2 \neq 0
\]  

(1.4)

where \( c_{11}, c_{12}, c_{21}, c_{22}, \alpha, \beta \) are given constants, then the auxiliary conditions are called boundary conditions (BC). In this case the differential equation (1.3) together with the boundary conditions (1.4) is called a boundary value problem (BVP). Solutions to boundary value problems depend upon the boundary conditions. If not properly formulated the boundary value problem might not have a solution. In other formulations the boundary value problem might have an infinite number of solutions. In still other circumstances there will exist a unique solution and so boundary value problems must be carefully analyzed to see (i) if a solution exists and (ii) how many solutions exist. Boundary conditions of the form

\[
c_{11}y(a) + c_{12}y'(a) + c_{13}y(b) + c_{14}y'(b) = \alpha \\
c_{21}y(b) + c_{22}y'(b) + c_{23}y(a) + c_{24}y'(a) = \beta,
\]

where the \( c_{ij}, \ i = 1, 2, \ j = 1, 2, 3, 4 \), and \( \alpha, \beta \) are constants, are called mixed boundary conditions, while the boundary conditions given by equations (1.4) are called unmixed boundary conditions. Mixed boundary value problems are usually much harder to solve.

**First Order Equations.**

Consider the first order linear equation

\[
L(y) = \frac{dy}{dx} + p(x)y = q(x).
\]  

(1.5)

To solve this differential equation we first solve the homogeneous equation

\[
\frac{dy}{dx} + p(x)y = 0
\]  

(1.6)
to obtain the complementary solution $y_c$. One can separate the variables in equation (1.6) to obtain

$$\frac{dy}{y} = -p(x)\,dx.$$  \hspace{1cm} (1.7)

Define the function

$$\mathcal{P}(x) = \int_0^x p(\xi)\,d\xi$$

with

$$\frac{d\mathcal{P}}{dx} = p(x),$$

then the equation (1.7) has the integral $\ln y = -\mathcal{P}(x) + C$ where $C$ is a constant of integration. We solve for $y$

$$e^{\ln y} = e^{-\mathcal{P}(x)+C} = e^{-\mathcal{P}(x)}e^C \quad \text{or} \quad y = C_1 e^{-\mathcal{P}(x)},$$

where $C_1 = e^C$ is some new constant. This produces the complementary solution

$$y_c = C_1 e^{-\mathcal{P}(x)}.$$  

We use the method of variation of parameters and assume a particular solution of the form $y_p = u(x)e^{-\mathcal{P}(x)}$ where $C_1$ in the complementary solution has been replaced by an unknown function $u(x)$. This assumed solution has the derivative

$$\frac{dy_p}{dx} = u(x)e^{-\mathcal{P}(x)}(-p(x)) + \frac{du}{dx}e^{-\mathcal{P}(x)}.$$  

Substitute $y_p$ and $\frac{dy_p}{dx}$ into the nonhomogeneous differential equation (1.5) and simplify to obtain the differential equation which defines the function $u = u(x)$. This differential equation is

$$\frac{du}{dx} e^{-\mathcal{P}(x)} = q(x).$$

Separate the variables in this equation and solve for $u$ to obtain

$$u = u(x) = \int_0^x q(\xi)e^{\mathcal{P}(\xi)}\,d\xi.$$  

This produces the particular solution

$$y_p = e^{-\mathcal{P}(x)}\int_0^x q(\xi)e^{\mathcal{P}(\xi)}\,d\xi.$$  

The general solution to the equation (1.5) is then

$$y = y_c + y_p = e^{-\mathcal{P}(x)} \left[ C_1 + \int_0^x q(\xi)e^{\mathcal{P}(\xi)}\,d\xi \right].$$  \hspace{1cm} (1.9)

An alternate method for solving the first order linear equation (1.5) is to multiply the equation by the integrating factor $e^{\mathcal{P}(x)}$, where $\mathcal{P}(x) = \int_0^x p(\xi)\,d\xi$, to obtain the exact differential equation

$$\frac{d}{dx} \left( e^{\mathcal{P}(x)}y \right) = e^{\mathcal{P}(x)}q(x).$$

This equation is easily integrated to obtain the solution given by equation (1.9).
Second Order Equations

Consider a second order linear homogeneous differential equation having the form of equation (1.3). We assume \( \{y_1(x), y_2(x)\} \) is a fundamental set of solutions to the homogeneous second order differential equation

\[
L(y) = a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = 0.
\]

This produces the complementary solution

\[
y_c = c_1 y_1(x) + c_2 y_2(x)
\]  

(1.10)

where \( c_1, c_2 \) are arbitrary constants. The method of variation of parameters requires one to assume a particular solution to the nonhomogeneous differential equation

\[
L(y) = a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = F(x)
\]  

(1.11)

of the form

\[
y_p = u(x)y_1(x) + v(x)y_2(x)
\]  

(1.12)

where \( u(x) \) and \( v(x) \) are functions replacing the constants \( c_1, c_2 \) in equation (1.10). The functions \( u, v \) are selected to satisfy the equations

\[
\begin{align*}
    u'(x)y_1(x) + v'(x)y_2(x) &= 0 \\
    u'(x)y_1'(x) + v'(x)y_2'(x) &= \frac{F(x)}{a_0(x)} \\
\end{align*}
\]  

(1.13)

from which \( u'(x) = \frac{dv}{dx} \) and \( v'(x) = \frac{du}{dx} \) can be determined. Using Cramer’s† rule we solve for \( u' \) and \( v' \) and find

\[
\begin{align*}
    u'(x) &= \frac{du}{dx} = \frac{\begin{vmatrix} 0 & y_2(x) \\ F(x)/a_0(x) & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} \\
    v'(x) &= \frac{dv}{dx} = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & F(x)/a_0(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}}
\end{align*}
\]

which simplifies to

\[
\begin{align*}
    \frac{du}{dx} &= -\frac{y_2(x)F(x)}{a_0(x)W(x)} \\
    \frac{dv}{dx} &= \frac{y_1(x)F(x)}{a_0(x)W(x)}
\end{align*}
\]  

(1.14)

where \( W = W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) \neq 0 \) is the nonzero Wronskian† associated with the fundamental set of solutions. These equations are then integrated

† See Appendix C
to obtain the desired functions \( u = u(x) \) and \( v = v(x) \). A simple integration gives the solution to the equations (1.14) as

\[
-u = u(x) = -\int_{\alpha}^{x} \frac{y_1(\xi)F(\xi)}{a_0(\xi)W(\xi)} \, d\xi, \quad v = v(x) = \int_{\alpha}^{x} \frac{y_1(\xi)F(\xi)}{a_0(\xi)W(\xi)} \, d\xi,
\]

where \( \alpha \) is some constant initial value which is usually selected away from any singularities associated with the integrals. The resulting solution of the nonhomogeneous differential equation is the particular solution

\[
y_p = y_p(x) = y_1(x) \int_{\alpha}^{x} \frac{-y_2(\xi)F(\xi)}{p(\xi)W(\xi)} \, d\xi + y_2(x) \int_{\alpha}^{x} \frac{y_1(\xi)F(\xi)}{p(\xi)W(\xi)} \, d\xi
\]

which can also be written in the form

\[
y_p = y_p(x) = \int_{\alpha}^{x} \frac{[y_2(x)y_1(\xi) - y_1(x)y_2(\xi)]F(\xi)}{p(\xi)W(\xi)} \, d\xi.
\]

The general solution of equation (1.11) is then written as \( y = y_c + y_p \) or

\[
y = c_1y_1(x) + c_2y_2(x) + \int_{\alpha}^{x} \frac{[y_2(x)y_1(\xi) - y_1(x)y_2(\xi)]F(\xi)}{p(\xi)W(\xi)} \, d\xi.
\]

**Harmonic Motion**

One of the many easy to solve second order differential equations is the equation

\[
\frac{d^2F}{dx^2} + \lambda F = 0 \quad \text{with } \lambda \text{ constant.}
\]

This equation will arise in the study of many partial differential equations in Cartesian coordinates. The differential equation contains a parameter \( \lambda \) and so we consider the following three cases of negative, zero and positive values which can be assigned to the constant \( \lambda \).

**Case 1:** \( \lambda = -\omega^2 \) \((\omega > 0)\). The differential equation \( \frac{d^2F}{dx^2} - \omega^2 F = 0 \) is a differential equation with constant coefficients and so one can assume an exponential solution \( F = e^{mx} \). This gives the characteristic equation \( m^2 - \omega^2 = 0 \) with the characteristic roots \( m = \omega \) and \( m = -\omega \). A fundamental set of solutions is given by \( \{e^{\omega x}, e^{-\omega x}\} \). Once we know a fundamental set of solutions we can then form linear combinations of these functions and generate an infinite set of other solutions. Recall a linear combination is nothing more than multiplying the given functions by arbitrary constants (which may or may not be complex constants)
and adding the results. From the above fundamental set we can form the linear combinations
\[
\sinh \omega x = \frac{e^{\omega x} - e^{-\omega x}}{2}, \quad \cosh \omega x = \frac{e^{\omega x} + e^{-\omega x}}{2}.
\]
Many other linear combinations can be formed. One more example is,
\[
\sinh (x - x_0) = \frac{e^{-\omega x_0} e^{\omega x} - e^{\omega x_0} e^{-\omega x}}{2}, \quad \cosh (x - x_0) = \frac{e^{-\omega x_0} e^{\omega x} + e^{\omega x_0} e^{-\omega x}}{2}
\]
with \(x_0\) constant. These are additional forms for solutions. We summarize these results with the following.

<table>
<thead>
<tr>
<th>Solutions to the differential equation ( \frac{d^2 F}{dx^2} - \omega^2 F = 0 ) can be written in many forms. Four possible forms are</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F = F(x) = c_1 e^{\omega x} + c_2 e^{-\omega x})</td>
</tr>
<tr>
<td>(F = F(x) = c_1 e^{\omega(x-x_0)} + c_2 e^{-\omega(x-x_0)})</td>
</tr>
<tr>
<td>(F = F(x) = c_1 \sinh \omega x + c_2 \cosh \omega x)</td>
</tr>
<tr>
<td>(F = F(x) = c_1 \sinh \omega(x-x_0) + c_2 \cosh \omega(x-x_0))</td>
</tr>
</tbody>
</table>

where \(x_0, c_1\) and \(c_2\) are arbitrary constants.

The particular final form selected for the general solution depends upon initial or boundary conditions assigned to the problem. The final form selected is usually a form which simplifies any additional algebra associated with boundary or initial conditions.

**Case 2:** \(\lambda = 0\). The differential equation \( \frac{d^2 F}{dx^2} = 0 \) is easily solved by integrating twice to obtain \(F = F(x) = c_1 + c_2 x\). Another form for this solution is the shifted form \(F = F(x) = K_1 + K_2 (x - x_0)\) where \(x_0\) is constant. One can also assume an exponential solution \(F = e^{mx}\) and obtain the characteristic equation \(m^2 = 0\) with characteristic roots \(\{0, 0\}\). This gives the fundamental set \(\{1, x\}\) and so the general solution is \(y = c_1(1) + c_2 x\). The particular final form selected is a matter of choice. When the shifted form is expanded we have \(c_2 = K_2\) and the combination \(K_1 - K_2 x_0\) is treated as a new constant and labeled as \(c_1\).

**Case 3:** \(\lambda = \omega^2\) \((\omega > 0)\). The differential equation \( \frac{d^2 F}{dx^2} + \omega^2 F = 0 \) is a differential equation with constant coefficients. We assume an exponential solution
$F = e^{mx}$ which gives the characteristic equation $m^2 + \omega^2 = 0$ with characteristic roots $m = i\omega, m = -i\omega,$ where $i^2 = -1.$ These roots produce the fundamental set \( \{e^{i\omega x}, e^{-i\omega x}\}. \) Linear combinations of functions from the fundamental set will generate additional solution forms. For example, using the Euler identity

\[
e^{i\theta} = \cos \theta + i \sin \theta	ag{1.19}\]

one can generate the solutions

\[
\sin \omega x = \frac{e^{i\omega x} - e^{-i\omega x}}{2i}, \quad \cos \omega x = \frac{e^{i\omega x} + e^{-i\omega x}}{2}.	ag{1.20}\]

The particular linear combinations

\[
\sin \omega (x - x_0) = \frac{e^{-i\omega x_0} e^{i\omega x} - e^{i\omega x_0} e^{-i\omega x}}{2i}, \quad \cos \omega (x - x_0) = \frac{e^{-i\omega x_0} e^{i\omega x} + e^{i\omega x_0} e^{-i\omega x}}{2}
\]

with $x_0$ constant, gives the shifted solutions for the sine and cosine solutions. These results are summarized for later use.

\begin{center}
Solutions to the differential equation
\[
\frac{d^2F}{dx^2} + \omega^2 F = 0
\]
can be written in many forms. Four possible forms are

\[
F = F(x) = c_1 e^{i\omega x} + c_2 e^{-i\omega x} \\
F = F(x) = c_1 e^{i\omega (x - x_0)} + c_2 e^{-i\omega (x - x_0)} \\
F = F(x) = c_1 \sin \omega x + c_2 \cos \omega x \\
F = F(x) = c_1 \sin \omega (x - x_0) + c_2 \cos \omega (x - x_0)
\]

where $x_0, c_1$ and $c_2$ are arbitrary constants.
\end{center}

\textbf{Cauchy-Euler equation}

The Cauchy\dagger -Euler\dagger equation

\[
a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = 0
\]

\dagger See Appendix C
where \( a_0, a_1, a_2 \) are constants, is related to a differential equation with constant coefficients by way of the transformation \( t = \ln x \). Using the derivatives

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \left( \frac{1}{x} \right)
\]

and

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{-1}{x^2} \right) + \frac{d^2y}{dt^2} \left( \frac{1}{x^2} \right)
\]

the equation (1.22) becomes

\[
a_0 \frac{d^2y}{dt^2} + (a_1 - a_0) \frac{dy}{dt} + a_2 y = 0. \tag{1.23}
\]

Assuming an exponential solution of \( y = e^{mt} \) to equation (1.23) gives the characteristic equation

\[
a_0 m^2 + (a_1 - a_0) m + a_2 = 0. \tag{1.24}
\]

This characteristic equation is derivable directly from equation (1.22) by assuming a solution \( y = x^m \) since \( y = e^{mt} = e^{m \ln x} = x^m \). The roots of the characteristic equation determine the types of solutions that can exist for the Cauchy-Euler equation. We examine the following cases.

Case 1: If the characteristic roots are real and distinct \( m = \alpha, m = \beta \), then the differential equation (1.22) has the fundamental set of solutions \( \{x^\alpha, x^\beta\} \) and so the general solution can be written as the linear combination

\[
y = c_1 x^\alpha + c_2 x^\beta
\]

where \( c_1, c_2 \) are arbitrary constants.

Case 2: If the characteristic roots are repeated roots \( m = \alpha, m = \alpha \) then a fundamental set of solutions for the transformed equation (1.23) is given by \( \{e^{\alpha t}, te^{\alpha t}\} \). The transformation \( t = \ln x \) gives the fundamental set of solutions \( \{x^\alpha, x^\alpha \ln x\} \) to the original Cauchy-Euler equation (1.22). The general solution of equation (1.22) can then be written as \( y = c_1 x^\alpha + c_2 x^\alpha \ln x \) where \( c_1, c_2 \) are arbitrary constants.

Case 3: If the characteristic roots of equation (1.24) are imaginary roots of the form \( m = \alpha + i\beta, \ m = \alpha - i\beta \), with \( i^2 = -1 \), a fundamental set of solutions to the transformed differential equation (1.23) is given by \( \{e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t\} \) which by way of the transformation \( t = \ln x \), gives the fundamental set of solutions to the Cauchy-Euler equation as \( \{x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)\} \). The general solution to the Cauchy-Euler differential equation (1.22) can then be written as the linear combination

\[
y = c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x)
\]

where \( c_1, c_2 \) are arbitrary constants.

**Bessel functions**  The differential equation

\[
t^2 \frac{d^2z}{dt^2} + t \frac{dz}{dt} + (t^2 - \nu^2)z = 0 \tag{1.25}
\]
is known as Bessel’s† differential equation. This equation arises in many applied problems involving cylindrical coordinates and we shall have to deal with it in our study of solutions to certain partial differential equations. The quantity $\nu$ in equation (1.25) is called a parameter and can be any real number. The Bessel equation (1.25) has a regular singular point at $t = 0$ and so one can assume a Frobenius‡ type solution $z = \sum_{n=0}^{\infty} c_n t^{\nu+n}$. In this way one can calculate solutions $J_\nu(t)$ and $J_{-\nu}(t)$, for $\nu$ not an integer, where

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(\nu + m + 1)} \left(\frac{t}{2}\right)^{2m+\nu}$$

(1.26)

is defined as a Bessel function of the first kind of order $\nu$. In equation (1.26) the function $\Gamma$ is called a Gamma function. It is defined

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

(1.27)

with the properties

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \quad \text{and} \quad \Gamma(n + m + 1) = (n + m)!$$

(1.28)

for $m$ and $n$ integers. For the special values $n = 0, -1, -2, \ldots$ the Gamma function $\Gamma(n)$ is undefined. For these special values the function $1/\Gamma(n)$ is defined to be zero. A sketch of the Gamma function is illustrated in the figure 1-1.

![Figure 1-1 The Gamma function for $-4 \leq x \leq 6$.](image)

† See Appendix C
For $\nu = m$ an integer, one finds the functions $J_m(x)$ and $J_{-m}(x)$ satisfy the relation $J_{-m}(x) = (-1)^m J_m(x)$ and so these functions are no longer linearly independent functions. For any value of $\nu$ the Weber† and Schlafi† ratio

$$Y_\nu(t) = \frac{J_\nu(t) \cos \nu \pi - J_{-\nu}(t)}{\sin \nu \pi}$$  \hspace{1cm} (1.29)

is a linear combination of $J_\nu(t)$ and $J_{-\nu}(t)$ which can be used to define Bessel functions of the second kind of order $\nu$. The limiting process $Y_n(t) = \lim_{\nu \to n} Y_\nu(t)$ is used to define Bessel functions of the second kind with integer values $n$. Some texts refer to Bessel functions of the second kind of order $\nu$ as Neumann† functions of order $\nu$ and are denoted using the notation $N_\nu(x)$. Graphs of selected Bessel functions are given in the figure 1-2 and figure 1-3. Note the zeros of the Bessel functions are not equally spaced. If we denote by $\xi_{kn}$ the $n$th zero of the $k$th order Bessel function, then one can write $J_k(\xi_{kn}) = 0$, for $n = 1, 2, 3, \ldots$.

\[\begin{array}{c}
\text{y} \\
J_0(x) \quad J_1(x) \quad X
\end{array}\]

Figure 1-2 Bessel functions of the first kind.

\[\begin{array}{c}
\text{y} \\
Y_0(x) \quad Y_1(x) \quad X
\end{array}\]

Figure 1-3 Bessel functions of the second kind.

† See Appendix C