§2.2 DYNAMICS

Dynamics is concerned with studying the motion of particles and rigid bodies. By studying the motion of a single hypothetical particle, one can discern the motion of a system of particles. This in turn leads to the study of the motion of individual points in a continuous deformable medium.

Particle Movement

The trajectory of a particle in a generalized coordinate system is described by the parametric equations

$$x^i = x^i(t), \quad i = 1, \dots, N \tag{2.2.1}$$

where t is a time parameter. If the coordinates are changed to a barred system by introducing a coordinate transformation

$$\overline{x}^i = \overline{x}^i(x^1, x^2, \dots, x^N), \quad i = 1, \dots, N$$

then the trajectory of the particle in the barred system of coordinates is

$$\overline{x}^i = \overline{x}^i(x^1(t), x^2(t), \dots, x^N(t)), \quad i = 1, \dots, N. \tag{2.2.2}$$

The generalized velocity of the particle in the unbarred system is defined by

$$v^i = \frac{dx^i}{dt}, \quad i = 1, \dots, N. \tag{2.2.3}$$

By the chain rule differentiation of the transformation equations (2.2.2) one can verify that the velocity in the barred system is

$$\overline{v}^r = \frac{d\overline{x}^r}{dt} = \frac{\partial \overline{x}^r}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial \overline{x}^r}{\partial x^j} v^j, \quad r = 1, \dots, N. \tag{2.2.4}$$

Consequently, the generalized velocity v^i is a first order contravariant tensor. The speed of the particle is obtained from the magnitude of the velocity and is

$$v^2 = g_{ij} v^i v^j.$$

The generalized acceleration f^i of the particle is defined as the intrinsic derivative of the generalized velocity. The generalized acceleration has the form

$$f^i = \frac{\delta v^i}{\delta t} = v^i_{,n}\frac{dx^n}{dt} = \frac{dv^i}{dt} + \left\{ {i \atop m\, n} \right\} v^m v^n = \frac{d^2 x^i}{dt^2} + \left\{ {i \atop m\, n} \right\} \frac{dx^m}{dt}\frac{dx^n}{dt} \tag{2.2.5}$$

and the magnitude of the acceleration is

$$f^2 = g_{ij} f^i f^j.$$

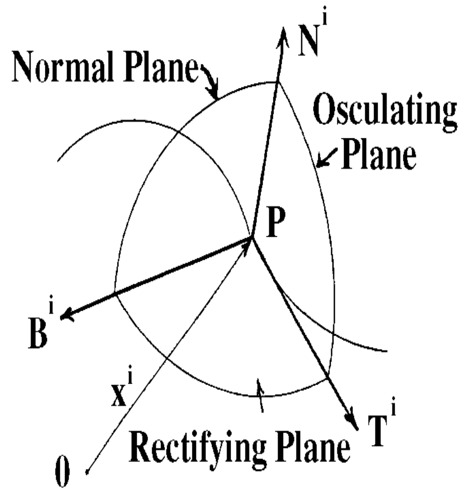

Figure 2.2-1 Tangent, normal and binormal to point P on curve.

Frenet-Serret Formulas

The parametric equations (2.2.1) describe a curve in our generalized space. With reference to the figure 2.2-1 we wish to define at each point P of the curve the following orthogonal unit vectors:

$$T^i = \text{unit tangent vector at each point } P.$$

$$N^i = \text{unit normal vector at each point } P.$$

$$B^i = \text{unit binormal vector at each point } P.$$

These vectors define the osculating, normal and rectifying planes illustrated in the figure 2.2-1.

In the generalized coordinates the arc length squared is

$$ds^2 = g_{ij}dx^i dx^j.$$

Define $T^i = \frac{dx^i}{ds}$ as the tangent vector to the parametric curve defined by equation (2.2.1). This vector is a unit tangent vector because if we write the element of arc length squared in the form

$$1 = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = g_{ij}T^iT^j, \tag{2.2.6}$$

we obtain the generalized dot product for T^i. This generalized dot product implies that the tangent vector is a unit vector. Differentiating the equation (2.2.6) intrinsically with respect to arc length s along the curve produces

$$g_{mn}\frac{\delta T^m}{\delta s}T^n + g_{mn}T^m\frac{\delta T^n}{\delta s} = 0,$$

which simplifies to

$$g_{mn}T^n\frac{\delta T^m}{\delta s} = 0. \tag{2.2.7}$$

The equation (2.2.7) is a statement that the vector $\frac{\delta T^m}{\delta s}$ is orthogonal to the vector T^m. The unit normal vector is defined as

$$N^i = \frac{1}{\kappa}\frac{\delta T^i}{\delta s} \qquad \text{or} \qquad N_i = \frac{1}{\kappa}\frac{\delta T_i}{\delta s}, \tag{2.2.8}$$

where κ is a scalar called the curvature and is chosen such that the magnitude of N^i is unity. The reciprocal of the curvature is $R = \frac{1}{\kappa}$, which is called the radius of curvature. The curvature of a straight line is zero while the curvature of a circle is a constant. The curvature measures the rate of change of the tangent vector as the arc length varies.

The equation (2.2.7) can be expressed in the form

$$g_{ij}T^iN^j = 0. \tag{2.2.9}$$

Taking the intrinsic derivative of equation (2.2.9) with respect to the arc length s produces

$$g_{ij}T^i\frac{\delta N^j}{\delta s} + g_{ij}\frac{\delta T^i}{\delta s}N^j = 0$$

or

$$g_{ij}T^i\frac{\delta N^j}{\delta s} = -g_{ij}\frac{\delta T^i}{\delta s}N^j = -\kappa g_{ij}N^iN^j = -\kappa. \tag{2.2.10}$$

The generalized dot product can be written

$$g_{ij}T^iT^j = 1,$$

and consequently we can express equation (2.2.10) in the form

$$g_{ij}T^i\frac{\delta N^j}{\delta s} = -\kappa g_{ij}T^iT^j \qquad \text{or} \qquad g_{ij}T^i\left(\frac{\delta N^j}{\delta s} + \kappa T^j\right) = 0. \tag{2.2.11}$$

Consequently, the vector

$$\frac{\delta N^j}{\delta s} + \kappa T^j \tag{2.2.12}$$

is orthogonal to T^i. In a similar manner, we can use the relation $g_{ij}N^iN^j = 1$ and differentiate intrinsically with respect to the arc length s to show that

$$g_{ij}N^i\frac{\delta N^j}{\delta s} = 0.$$

This in turn can be expressed in the form

$$g_{ij}N^i\left(\frac{\delta N^j}{\delta s} + \kappa T^j\right) = 0.$$

This form of the equation implies that the vector represented in equation (2.2.12) is also orthogonal to the unit normal N^i. We define the unit binormal vector as

$$B^i = \frac{1}{\tau}\left(\frac{\delta N^i}{\delta s} + \kappa T^i\right) \qquad \text{or} \qquad B_i = \frac{1}{\tau}\left(\frac{\delta N_i}{\delta s} + \kappa T_i\right) \tag{2.2.13}$$

where τ is a scalar called the torsion. The torsion is chosen such that the binormal vector is a unit vector. The torsion measures the rate of change of the osculating plane and consequently, the torsion τ is a measure of the twisting of the curve out of a plane. The value $\tau = 0$ corresponds to a plane curve. The vectors $T^i, N^i, B^i, i = 1,2,3$ satisfy the cross product relation

$$B^i = \epsilon^{ijk} T_j N_k.$$

If we differentiate this relation intrinsically with respect to arc length s we find

$$\begin{aligned} \frac{\delta B^i}{\delta s} &= \epsilon^{ijk} \left(T_j \frac{\delta N_k}{\delta s} + \frac{\delta T_j}{\delta s} N_k \right) \\ &= \epsilon^{ijk} \left[T_j (\tau B_k - \kappa T_k) + \kappa N_j N_k \right] \\ &= \tau \epsilon^{ijk} T_j B_k = -\tau \epsilon^{ikj} B_k T_j = -\tau N^i. \end{aligned} \tag{2.2.14}$$

The relations (2.2.8),(2.2.13) and (2.2.14) are now summarized and written

$$\begin{aligned} \frac{\delta T^i}{\delta s} &= \kappa N^i \\ \frac{\delta N^i}{\delta s} &= \tau B^i - \kappa T^i \\ \frac{\delta B^i}{\delta s} &= -\tau N^i. \end{aligned} \tag{2.2.15}$$

These equations are known as the Frenet-Serret formulas of differential geometry.

Velocity and Acceleration

Chain rule differentiation of the generalized velocity is expressible in the form

$$v^i = \frac{dx^i}{dt} = \frac{dx^i}{ds} \frac{ds}{dt} = T^i v, \tag{2.2.16}$$

where $v = \frac{ds}{dt}$ is the speed of the particle and is the magnitude of v^i. The vector T^i is the unit tangent vector to the trajectory curve at the time t. The equation (2.2.16) is a statement of the fact that the velocity of a particle is always in the direction of the tangent vector to the curve and has the speed v.

By chain rule differentiation, the generalized acceleration is expressible in the form

$$\begin{aligned} f^r = \frac{\delta v^r}{\delta t} &= \frac{dv}{dt} T^r + v \frac{\delta T^r}{\delta t} \\ &= \frac{dv}{dt} T^r + v \frac{\delta T^r}{\delta s} \frac{ds}{dt} \\ &= \frac{dv}{dt} T^r + \kappa v^2 N^r. \end{aligned} \tag{2.2.17}$$

The equation (2.2.17) states that the acceleration lies in the osculating plane. Further, the equation (2.2.17) indicates that the tangential component of the acceleration is $\frac{dv}{dt}$, while the normal component of the acceleration is κv^2.

Work and Potential Energy

Define M as the constant mass of the particle as it moves along the curve defined by equation (2.2.1). Also let Q^r denote the components of a force vector (in appropriate units of measurements) which acts upon the particle. Newton's second law of motion can then be expressed in the form

$$Q^r = Mf^r \qquad \text{or} \qquad Q_r = Mf_r. \tag{2.2.18}$$

The work done W in moving a particle from a point P_0 to a point P_1 along a curve $x^r = x^r(t), r = 1, 2, 3$, with parameter t, is represented by a summation of the tangential components of the forces acting along the path and is defined as the line integral

$$W = \int_{P_0}^{P_1} Q_r \frac{dx^r}{ds} ds = \int_{P_0}^{P_1} Q_r\, dx^r = \int_{t_0}^{t_1} Q_r \frac{dx^r}{dt}\, dt = \int_{t_0}^{t_1} Q_r v^r\, dt \tag{2.2.19}$$

where $Q_r = g_{rs}Q^s$ is the covariant form of the force vector, t is the time parameter and s is arc length along the curve.

Conservative Systems

If the force vector is conservative it means that the force is derivable from a scalar potential function

$$V = V(x^1, x^2, \dots, x^N) \qquad \text{such that} \qquad Q_r = -V_{,r} = -\frac{\partial V}{\partial x^r}, \quad r = 1, \dots, N. \tag{2.2.20}$$

In this case the equation (2.2.19) can be integrated and we find that to within an additive constant we will have $V = -W$. The potential function V is called the potential energy of the particle and the work done becomes the change in potential energy between the starting and end points and is independent of the path connecting the points.

Lagrange's Equations of Motion

The kinetic energy T of the particle is defined as one half the mass times the velocity squared and can be expressed in any of the forms

$$T = \frac{1}{2}M\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}Mv^2 = \frac{1}{2}Mg_{mn}v^m v^n = \frac{1}{2}Mg_{mn}\dot{x}^m \dot{x}^n, \tag{2.2.21}$$

where the dot notation denotes differentiation with respect to time. It is an easy exercise to calculate the derivatives

$$\begin{aligned}\frac{\partial T}{\partial \dot{x}^r} &= Mg_{rm}\dot{x}^m \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) &= M\left[g_{rm}\ddot{x}^m + \frac{\partial g_{rm}}{\partial x^n}\dot{x}^n\dot{x}^m\right] \\ \frac{\partial T}{\partial x^r} &= \frac{1}{2}M\frac{\partial g_{mn}}{\partial x^r}\dot{x}^m\dot{x}^n,\end{aligned} \tag{2.2.22}$$

and thereby verify the relation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} = Mf_r = Q_r, \quad r = 1, \dots, N. \tag{2.2.23}$$

This equation is called the Lagrange's form of the equations of motion.

EXAMPLE 2.2-1. (Equations of motion in spherical coordinates) Find the Lagrange's form of the equations of motion in spherical coordinates.

Solution: Let $x^1 = \rho$, $x^2 = \theta$, $x^3 = \phi$ then the element of arc length squared in spherical coordinates has the form

$$ds^2 = (d\rho)^2 + \rho^2 (d\theta)^2 + \rho^2 \sin^2\theta (d\phi)^2.$$

The element of arc length squared can be used to construct the kinetic energy. For example,

$$T = \frac{1}{2} M \left(\frac{ds}{dt}\right)^2 = \frac{1}{2} M \left[(\dot\rho)^2 + \rho^2(\dot\theta)^2 + \rho^2 \sin^2\theta (\dot\phi)^2\right].$$

The Lagrange form of the equations of motion of a particle are found from the relations (2.2.23) and are calculated to be:

$$\begin{aligned}
M f_1 = Q_1 &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\rho}\right) - \frac{\partial T}{\partial \rho} = M\left[\ddot\rho - \rho(\dot\theta)^2 - \rho \sin^2\theta(\dot\phi)^2\right] \\
M f_2 = Q_2 &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = M\left[\frac{d}{dt}\left(\rho^2\dot\theta\right) - \rho^2 \sin\theta\cos\theta(\dot\phi)^2\right] \\
M f_3 = Q_3 &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) - \frac{\partial T}{\partial \phi} = M\left[\frac{d}{dt}\left(\rho^2 \sin^2\theta\dot\phi\right)\right].
\end{aligned}$$

In terms of physical components we have

$$\begin{aligned}
Q_\rho &= M\left[\ddot\rho - \rho(\dot\theta)^2 - \rho\sin^2\theta(\dot\phi)^2\right] \\
Q_\theta &= \frac{M}{\rho}\left[\frac{d}{dt}\left(\rho^2\dot\theta\right) - \rho^2\sin\theta\cos\theta(\dot\phi)^2\right] \\
Q_\phi &= \frac{M}{\rho\sin\theta}\left[\frac{d}{dt}\left(\rho^2\sin^2\theta\dot\phi\right)\right].
\end{aligned}$$

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Euler-Lagrange Equations of Motion

Starting with the Lagrange's form of the equations of motion from equation (2.2.23), we assume that the external force Q_r is derivable from a potential function V as specified by the equation (2.2.20). That is, we assume the system is conservative and express the equations of motion in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot x^r}\right) - \frac{\partial T}{\partial x^r} = -\frac{\partial V}{\partial x^r} = Q_r, \quad r = 1, \ldots, N \tag{2.2.24}$$

The Lagrangian is defined by the equation

$$L = T - V = T(x^1, \ldots, x^N, \dot x^1, \ldots, \dot x^N) - V(x^1, \ldots, x^N) = L(x^i, \dot x^i). \tag{2.2.25}$$

Employing the defining equation (2.2.25), it is readily verified that the equations of motion are expressible in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x^r}\right) - \frac{\partial L}{\partial x^r} = 0, \qquad r = 1, \ldots, N, \tag{2.2.26}$$

which are called the Euler-Lagrange form for the equations of motion.

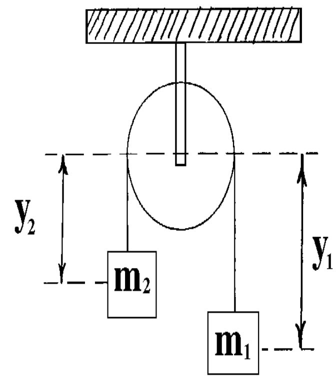

Figure 2.2-2 Simply pulley system

EXAMPLE 2.2-2. (Simple pulley system) Find the equation of motion for the simply pulley system illustrated in the figure 2.2-2.

Solution: The given system has only one degree of freedom, say y_1. It is assumed that

$$y_1 + y_2 = \ell = \text{a constant}.$$

The kinetic energy of the system is

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2.$$

Let y_1 increase by an amount dy_1 and show the work done by gravity can be expressed as

$$\begin{aligned}
dW &= m_1 g\, dy_1 + m_2 g\, dy_2 \\
dW &= m_1 g\, dy_1 - m_2 g\, dy_1 \\
dW &= (m_1 - m_2) g\, dy_1 = Q_1\, dy_1.
\end{aligned}$$

Here $Q_1 = (m_1 - m_2)g$ is the external force acting on the system where g is the acceleration of gravity. The Lagrange equation of motion is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) - \frac{\partial T}{\partial y_1} = Q_1$$

or

$$(m_1 + m_2)\ddot{y}_1 = (m_1 - m_2)g.$$

Initial conditions must be applied to y_1 and $\dot{y}_1$ before this equation can be solved.

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EXAMPLE 2.2-3. (Simple pendulum) Find the equation of motion for the pendulum system illustrated in the figure 2.2-3.

Solution: Choose the angle θ illustrated in the figure 2.2-3 as the generalized coordinate. If the pendulum is moved from a vertical position through an angle θ, we observe that the mass m moves up a distance $h = \ell - \ell\cos\theta$. The work done in moving this mass a vertical distance h is

$$W = -mgh = -mg\ell(1-\cos\theta),$$

since the force is $-mg$ in this coordinate system. In moving the pendulum through an angle θ, the arc length s swept out by the mass m is $s = \ell\theta$. This implies that the kinetic energy can be expressed

$$T = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}m\left(\ell\dot{\theta}\right)^2 = \frac{1}{2}m\ell^2(\dot{\theta})^2.$$

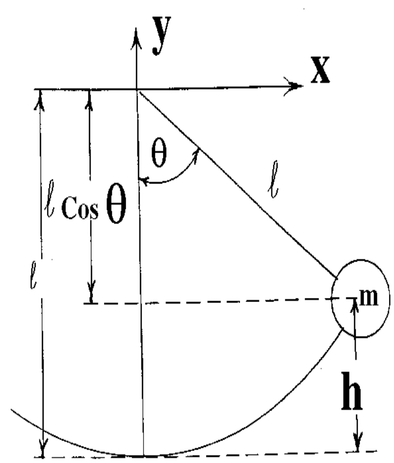

Figure 2.2-3 Simple pendulum system

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\ell^2(\dot{\theta})^2 - mg\ell(1-\cos\theta)$$

and from this we find the equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \qquad \text{or} \qquad \frac{d}{dt}\left(m\ell^2\dot{\theta}\right) - mg\ell(-\sin\theta) = 0.$$

This in turn simplifies to the equation

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.$$

This equation together with a set of initial conditions for θ and $\dot{\theta}$ represents the nonlinear differential equation which describes the motion of a pendulum without damping.

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EXAMPLE 2.2-4. (Compound pendulum) Find the equations of motion for the compound pendulum illustrated in the figure 2.2-4.

Solution: Choose for the generalized coordinates the angles $x^1 = \theta_1$ and $x^2 = \theta_2$ illustrated in the figure 2.2-4. To find the potential function V for this system we consider the work done as the masses m_1 and m_2 are moved. Consider independent motions of the angles θ_1 and θ_2. Imagine the compound pendulum initially in the vertical position as illustrated in the figure 2.2-4(a). Now let m_1 be displaced due to a change in θ_1 and obtain the figure 2.2-4(b). The work done to achieve this position is

$$W_1 = -(m_1 + m_2)gh_1 = -(m_1 + m_2)gL_1(1 - \cos\theta_1).$$

Starting from the position in figure 2.2-4(b) we now let θ_2 undergo a displacement and achieve the configuration in the figure 2.2-4(c).

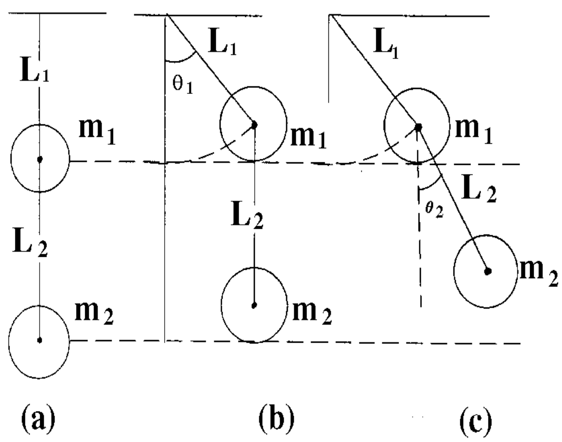

Figure 2.2-4 Compound pendulum

The work done due to the displacement θ_2 can be represented

$$W_2 = -m_2gh_2 = -m_2gL_2(1 - \cos\theta_2).$$

Since the potential energy V satisfies $V = -W$ to within an additive constant, we can write

$$V = -W = -W_1 - W_2 = -(m_1 + m_2)gL_1\cos\theta_1 - m_2gL_2\cos\theta_2 + constant,$$

where the constant term in the potential energy has been neglected since it does not contribute anything to the equations of motion. (i.e. the derivative of a constant is zero.)

The kinetic energy term for this system can be represented

$$\begin{aligned} T &= \frac{1}{2}m_1\left(\frac{ds_1}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{ds_2}{dt}\right)^2 \\ T &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2), \end{aligned} \tag{2.2.27}$$

where

$$
\begin{aligned}
(x_1, y_1) &= (L_1 \sin\theta_1\,,\ -L_1\cos\theta_1)\\
(x_2, y_2) &= (L_1 \sin\theta_1 + L_2\sin\theta_2, -L_1\cos\theta_1 - L_2\cos\theta_2)
\end{aligned}
\tag{2.2.28}
$$

are the coordinates of the masses m_1 and m_2 respectively. Substituting the equations (2.2.28) into equation (2.2.27) and simplifying produces the kinetic energy expression

$$
T = \frac{1}{2}(m_1 + m_2)L_1^2\dot\theta_1^2 + m_2 L_1 L_2 \dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2) + \frac{1}{2}m_2 L_2^2\dot\theta_2^2. \tag{2.2.29}
$$

Writing the Lagrangian as $L = T - V$, the equations describing the motion of the compound pendulum are obtained from the Lagrangian equations

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 \qquad\text{and}\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right) - \frac{\partial L}{\partial\theta_2} = 0.
$$

Calculating the necessary derivatives, substituting them into the Lagrangian equations of motion and then simplifying we derive the equations of motion

$$
\begin{aligned}
L_1\ddot\theta_1 + \frac{m_2}{m_1+m_2}L_2\ddot\theta_2\cos(\theta_1-\theta_2) + \frac{m_2}{m_1+m_2}L_2(\dot\theta_2)^2\sin(\theta_1-\theta_2) + g\sin\theta_1 &= 0\\
L_1\ddot\theta_1\cos(\theta_1-\theta_2) + L_2\ddot\theta_2 - L_1(\dot\theta_1)^2\sin(\theta_1-\theta_2) + g\sin\theta_2 &= 0.
\end{aligned}
$$

These equations are a set of coupled, second order nonlinear ordinary differential equations. These equations are subject to initial conditions being imposed upon the angular displacements (θ_1, θ_2) and the angular velocities $(\dot\theta_1, \dot\theta_2)$.

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Alternative Derivation of Lagrange's Equations of Motion

Let c denote a given curve represented in the parametric form

$$
x^i = x^i(t), \quad i = 1, \ldots, N, \quad t_0 \le t \le t_1
$$

and let P_0, P_1 denote two points on this curve corresponding to the parameter values t_0 and t_1 respectively. Let $\overline{c}$ denote another curve which also passes through the two points P_0 and P_1 as illustrated in the figure 2.2-5.

The curve $\overline{c}$ is represented in the parametric form

$$
\overline{x}^i = \overline{x}^i(t) = x^i(t) + \epsilon\eta^i(t), \quad i = 1, \ldots, N, \quad t_0 \le t \le t_1
$$

in terms of a parameter ϵ. In this representation the function $\eta^i(t)$ must satisfy the end conditions

$$
\eta^i(t_0) = 0 \quad\text{and}\quad \eta^i(t_1) = 0 \quad i = 1, \ldots, N
$$

since the curve $\overline{c}$ is assumed to pass through the end points P_0 and P_1.

Consider the line integral

$$
I(\epsilon) = \int_{t_0}^{t_1} L(t, x^i + \epsilon\eta^i, \dot x^i + \epsilon\dot\eta^i)\,dt, \tag{2.2.30}
$$

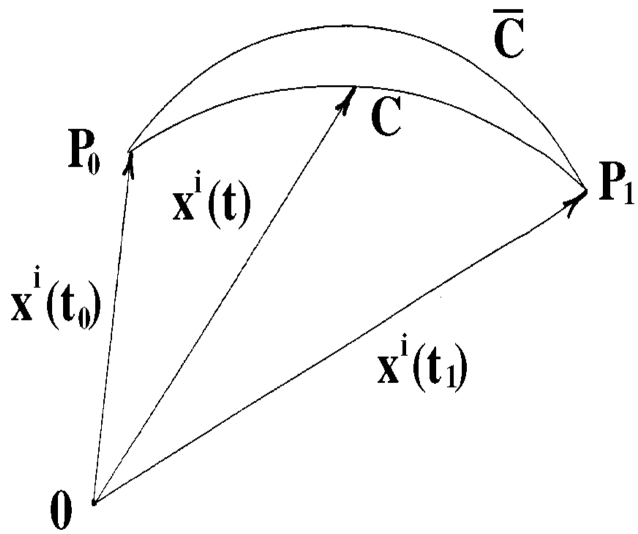

Figure 2.2-5. Motion along curves c and $\overline{c}$

where

$$L = T - V = L(t, \overline{x}^i, \dot{\overline{x}}^i)$$

is the Lagrangian evaluated along the curve $\overline{c}$. We ask the question, "What conditions must be satisfied by the curve c in order that the integral $I(\epsilon)$ have an extremum value when ϵ is zero?"If the integral $I(\epsilon)$ has a minimum value when ϵ is zero it follows that its derivative with respect to ϵ will be zero at this value and we will have

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0.$$

Employing the definition

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \lim_{\epsilon\to 0}\frac{I(\epsilon)-I(0)}{\epsilon} = I'(0) = 0$$

we expand the Lagrangian in equation (2.2.30) in a series about the point $\epsilon = 0$. Substituting the expansion

$$L(t, x^i + \epsilon\eta^i, \dot{x}^i + \epsilon\dot{\eta}^i) = L(t, x^i, \dot{x}^i) + \epsilon\left[\frac{\partial L}{\partial x^i}\eta^i + \frac{\partial L}{\partial \dot{x}^i}\dot{\eta}^i\right] + \epsilon^2[\quad] + \cdots$$

into equation (2.2.30) we calculate the derivative

$$I'(0) = \lim_{\epsilon\to 0}\frac{I(\epsilon)-I(0)}{\epsilon} = \lim_{\epsilon\to 0}\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x^i}\eta^i(t) + \frac{\partial L}{\partial \dot{x}^i}\dot{\eta}^i(t)\right]dt + \epsilon[\quad] + \cdots = 0,$$

where we have neglected higher order powers of ϵ since ϵ is approaching zero. Analysis of this equation informs us that the integral I has a minimum value at $\epsilon = 0$ provided that the integral

$$\delta I = \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x^i}\eta^i(t) + \frac{\partial L}{\partial \dot{x}^i}\dot{\eta}^i(t)\right]dt = 0 \tag{2.2.31}$$

is satisfied. Integrating the second term of this integral by parts we find

$$\delta I = \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}\eta^i \, dt + \left[\frac{\partial L}{\partial \dot{x}^i}\,\eta^i(t)\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\eta^i(t)\, dt = 0. \tag{2.2.32}$$

The end condition on $\eta^i(t)$ makes the middle term in equation (2.2.32) vanish and we are left with the integral

$$\delta I = \int_{t_0}^{t_1} \eta^i(t)\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right] dt = 0, \tag{2.2.33}$$

which must equal zero for all $\eta^i(t)$. Since $\eta^i(t)$ is arbitrary, the only way the integral in equation (2.2.33) can be zero for all $\eta^i(t)$ is for the term inside the brackets to vanish. This produces the result that the integral of the Lagrangian is an extremum when the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i = 1, \ldots, N \tag{2.2.34}$$

are satisfied. This is a necessary condition for the integral $I(\epsilon)$ to have a minimum value.

In general, any line integral of the form

$$I = \int_{t_0}^{t_1} \phi(t, x^i, \dot{x}^i)\, dt \tag{2.2.35}$$

has an extremum value if the curve c defined by $x^i = x^i(t)$, $i = 1, \ldots, N$ satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{x}^i}\right) - \frac{\partial \phi}{\partial x^i} = 0, \quad i = 1, \ldots, N. \tag{2.2.36}$$

The above derivation is a special case of (2.2.36) when $\phi = L$. Note that the equations of motion equations (2.2.34) are just another form of the equations (2.2.24). Note also that

$$\frac{\delta T}{\delta t} = \frac{\delta}{\delta t}\left(\frac{1}{2} m g_{ij} v^i v^j\right) = m g_{ij} v^i f^j = m f_i v^i = m f_i \dot{x}^i$$

and if we assume that the force Q_i is derivable from a potential function V, then $m f_i = Q_i = -\dfrac{\partial V}{\partial x^i}$, so that $\dfrac{\delta T}{\delta t} = m f_i \dot{x}^i = Q_i \dot{x}^i = -\dfrac{\partial V}{\partial x^i}\dot{x}^i = -\dfrac{\delta V}{\delta t}$ or $\dfrac{\delta}{\delta t}(T + V) = 0$ or $T + V = h =$ constant called the energy constant of the system.

Action Integral

The equations of motion (2.2.34) or (2.2.24) are interpreted as describing geodesics in a space whose line-element is

$$ds^2 = 2m(h - V) g_{jk} dx^j dx^k$$

where V is the potential function for the force system and $T + V = h$ is the energy constant of the motion. The integral of ds along a curve C between two points P_1 and P_2 is called an action integral and is

$$A = \sqrt{2m} \int_{P_1}^{P_2} \left\{(h - V) g_{jk} \frac{dx^j}{d\tau}\frac{dx^k}{d\tau}\right\}^{1/2} d\tau$$

where τ is a parameter used to describe the curve C. The principle of stationary action states that of all curves through the points P_1 and P_2 the one which makes the action an extremum is the curve specified by Newton's second law. The extremum is usually a minimum. To show this let

$$\phi = \sqrt{2m}\left\{(h-V)g_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}\right\}^{1/2}$$

in equation (2.2.36). Using the notation $\dot{x}^k = \frac{dx^k}{d\tau}$ we find that

$$\begin{aligned}\frac{\partial\phi}{\partial\dot{x}^i} =& \frac{2m}{\phi}(h-V)g_{ik}\dot{x}^k \\ \frac{\partial\phi}{\partial x^i} =& \frac{2m}{2\phi}(h-V)\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k - \frac{2m}{2\phi}\frac{\partial V}{\partial x^i}g_{jk}\dot{x}^j\dot{x}^k.\end{aligned}$$

The equation (2.2.36) which describe the extremum trajectories are found to be

$$\frac{d}{dt}\left[\frac{2m}{\phi}(h-V)g_{ik}\dot{x}^k\right] - \frac{2m}{2\phi}(h-V)\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k + \frac{2m}{\phi}\frac{\partial V}{\partial x^i}g_{jk}\dot{x}^j\dot{x}^k = 0.$$

By changing variables from τ to t where $\frac{dt}{d\tau} = \frac{\sqrt{m}\phi}{\sqrt{2}(h-V)}$ we find that the trajectory for an extremum must satisfy the equation

$$m\frac{d}{dt}\left(g_{ik}\frac{dx^k}{dt}\right) - \frac{m}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{dx^j}{dt}\frac{dx^k}{dt} + \frac{\partial V}{\partial x^i} = 0$$

which are the same equations as (2.2.24). (i.e. See also the equations (2.2.22).)

Dynamics of Rigid Body Motion

Let us derive the equations of motion of a rigid body which is rotating due to external forces acting upon it. We neglect any translational motion of the body since this type of motion can be discerned using our knowledge of particle dynamics. The derivation of the equations of motion is restricted to Cartesian tensors and rotational motion.

Consider a system of N particles rotating with angular velocity ω_i, $i = 1, 2, 3$, about a line L through the center of mass of the system. Let $\vec{V}^{(\alpha)}$ denote the velocity of the αth particle which has mass $m_{(\alpha)}$ and position $x_i^{(\alpha)}, i = 1, 2, 3$ with respect to an origin on the line L. Without loss of generality we can assume that the origin of the coordinate system is also at the center of mass of the system of particles, as this choice of an origin simplifies the derivation. The velocity components for each particle is obtained by taking cross products and we can write

$$\vec{V}^{(\alpha)} = \vec{\omega}\times\vec{r}^{\,(\alpha)} \qquad \text{or} \qquad V_i^{(\alpha)} = e_{ijk}\omega_j x_k^{(\alpha)}. \tag{2.2.37}$$

The kinetic energy of the system of particles is written as the sum of the kinetic energies of each individual particle and is

$$T = \frac{1}{2}\sum_{\alpha=1}^{N} m_{(\alpha)}V_i^{(\alpha)}V_i^{(\alpha)} = \frac{1}{2}\sum_{\alpha=1}^{N} m_{(\alpha)}e_{ijk}\omega_j x_k^{(\alpha)}e_{imn}\omega_m x_n^{(\alpha)}. \tag{2.2.38}$$

Employing the $e-\delta$ identity the equation (2.2.38) can be simplified to the form

$$T = \frac{1}{2}\sum_{\alpha=1}^{N} m_{(\alpha)}\left(\omega_m\omega_m x_k^{(\alpha)}x_k^{(\alpha)} - \omega_n\omega_k x_k^{(\alpha)}x_n^{(\alpha)}\right).$$

Define the second moments and products of inertia by the equation

$$I_{ij} = \sum_{\alpha=1}^{N} m_{(\alpha)}\left(x_k^{(\alpha)}x_k^{(\alpha)}\delta_{ij} - x_i^{(\alpha)}x_j^{(\alpha)}\right) \tag{2.2.39}$$

and write the kinetic energy in the form

$$T = \frac{1}{2}I_{ij}\omega_i\omega_j. \tag{2.2.40}$$

Similarly, the angular momentum of the system of particles can also be represented in terms of the second moments and products of inertia. The angular momentum of a system of particles is defined as a summation of the moments of the linear momentum of each individual particle and is

$$H_i = \sum_{\alpha=1}^{N} m_{(\alpha)}e_{ijk}x_j^{(\alpha)}v_k^{(\alpha)} = \sum_{\alpha=1}^{N} m_{(\alpha)}e_{ijk}x_j^{(\alpha)}e_{kmn}\omega_m x_n^{(\alpha)}. \tag{2.2.41}$$

The $e-\delta$ identity simplifies the equation (2.2.41) to the form

$$H_i = \omega_j\sum_{\alpha=1}^{N} m_{(\alpha)}\left(x_n^{(\alpha)}x_n^{(\alpha)}\delta_{ij} - x_j^{(\alpha)}x_i^{(\alpha)}\right) = \omega_j I_{ji}. \tag{2.2.42}$$

The equations of motion of a rigid body is obtained by applying Newton's second law of motion to the system of N particles. The equation of motion of the αth particle is written

$$m_{(\alpha)}\ddot{x}_i^{(\alpha)} = F_i^{(\alpha)}. \tag{2.2.43}$$

Summing equation (2.2.43) over all particles gives the result

$$\sum_{\alpha=1}^{N} m_{(\alpha)}\ddot{x}_i^{(\alpha)} = \sum_{\alpha=1}^{N} F_i^{(\alpha)}. \tag{2.2.44}$$

This represents the translational equations of motion of the rigid body. The equation (2.2.44) represents the rate of change of linear momentum being equal to the total external force acting upon the system. Taking the cross product of equation (2.2.43) with the position vector $x_j^{(\alpha)}$ produces

$$m_{(\alpha)}\ddot{x}_t^{(\alpha)}e_{rst}x_s^{(\alpha)} = e_{rst}x_s^{(\alpha)}F_t^{(\alpha)}$$

and summing over all particles we find the equation

$$\sum_{\alpha=1}^{N} m_{(\alpha)}e_{rst}x_s^{(\alpha)}\ddot{x}_t^{(\alpha)} = \sum_{\alpha=1}^{N} e_{rst}x_s^{(\alpha)}F_t^{(\alpha)}. \tag{2.2.45}$$

The equations (2.2.44) and (2.2.45) represent the conservation of linear and angular momentum and can be written in the forms

$$\frac{d}{dt}\left(\sum_{\alpha=1}^{N} m_{(\alpha)}\dot{x}_r^{(\alpha)}\right) = \sum_{\alpha=1}^{N} F_r^{(\alpha)} \tag{2.2.46}$$

and

$$\frac{d}{dt}\left(\sum_{\alpha=1}^{N} m_{(\alpha)}e_{rst}x_s^{(\alpha)}\dot{x}_t^{(\alpha)}\right) = \sum_{\alpha=1}^{N} e_{rst}x_s^{(\alpha)}F_t^{(\alpha)}. \tag{2.2.47}$$

By definition we have $G_r = \sum m_{(\alpha)}\dot{x}_r^{(\alpha)}$ representing the linear momentum, $F_r = \sum F_r^{(\alpha)}$ the total force acting on the system of particles, $H_r = \sum m_{(\alpha)}e_{rst}x_s^{(\alpha)}\dot{x}_t^{(\alpha)}$ is the angular momentum of the system relative to the origin, and $M_r = \sum e_{rst}x_s^{(\alpha)}F_t^{(\alpha)}$ is the total moment of the system relative to the origin. We can therefore express the equations (2.2.46) and (2.2.47) in the form

$$\frac{dG_r}{dt} = F_r \tag{2.2.48}$$

and

$$\frac{dH_r}{dt} = M_r. \tag{2.2.49}$$

The equation (2.2.49) expresses the fact that the rate of change of angular momentum is equal to the moment of the external forces about the origin. These equations show that the motion of a system of particles can be studied by considering the motion of the center of mass of the system (translational motion) and simultaneously considering the motion of points about the center of mass (rotational motion).

We now develop some relations in order to express the equations (2.2.49) in an alternate form. Toward this purpose we consider first the concepts of relative motion and angular velocity.

Relative Motion and Angular Velocity

Consider two different reference frames denoted by $\overline{S}$ and S. Both reference frames are Cartesian coordinates with axes $\overline{x}_i$ and x_i , $i = 1,2,3$, respectively. The reference frame S is fixed in space and is called an inertial reference frame or space-fixed reference system of axes. The reference frame $\overline{S}$ is fixed to and rotates with the rigid body and is called a body-fixed system of axes. Again, for convenience, it is assumed that the origins of both reference systems are fixed at the center of mass of the rigid body. Further, we let the system $\overline{S}$ have the basis vectors $\widehat{\overline{\mathbf{e}}}_i, i = 1,2,3$, while the reference system S has the basis vectors $\hat{\mathbf{e}}_i\,, i = 1,2,3$. The transformation equations between the two sets of reference axes are the affine transformations

$$\overline{x}_i = \ell_{ji}x_j \qquad \text{and} \qquad x_i = \ell_{ij}\overline{x}_j \tag{2.2.50}$$

where $\ell_{ij} = \ell_{ij}(t)$ are direction cosines which are functions of time t (i.e. the ℓ_{ij} are the cosines of the angles between the barred and unbarred axes where the barred axes are rotating relative to the space-fixed unbarred axes.) The direction cosines satisfy the relations

$$\ell_{ij}\ell_{ik} = \delta_{jk} \qquad \text{and} \qquad \ell_{ij}\ell_{kj} = \delta_{ik}. \tag{2.2.51}$$

EXAMPLE 2.2-5. (Euler angles ϕ, θ, ψ) Consider the following sequence of transformations which are used in celestial mechanics. First a rotation about the x_3 axis taking the x_i axes to the y_i axes

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where the rotation angle ϕ is called the longitude of the ascending node. Second, a rotation about the y_1 axis taking the y_i axes to the y_i' axes

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where the rotation angle θ is called the angle of inclination of the orbital plane. Finally, a rotation about the y_3' axis taking the y_i' axes to the $\bar{x}_i$ axes

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix}$$

where the rotation angle ψ is called the argument of perigee. The Euler angle θ is the angle $\bar{x}_3 0 x_3$, the angle ϕ is the angle $x_1 0 y_1$ and ψ is the angle $y_1 0 \bar{x}_1$. These angles are illustrated in the figure 2.2-6. Note also that the rotation vectors associated with these transformations are vectors of magnitude $\dot{\phi}, \dot{\theta}, \dot{\psi}$ in the directions indicated in the figure 2.2-6.

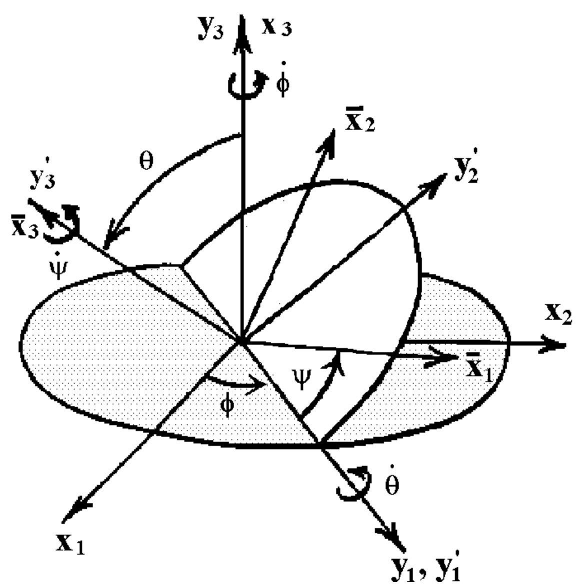

Figure 2.2-6. Euler angles.

By combining the above transformations there results the transformation equations (2.2.50)

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

It is left as an exercise to verify that the transformation matrix is orthogonal and the components ℓ_{ji} satisfy the relations (2.2.51).

■

Consider the velocity of a point which is rotating with the rigid body. Denote by $v_i = v_i(S)$, for $i = 1,2,3$, the velocity components relative to the S reference frame and by $\overline{v}_i = \overline{v}_i(\overline{S})$, $i = 1,2,3$ the velocity components of the same point relative to the body-fixed axes. In terms of the basis vectors we can write

$$\vec{V} = v_1(S)\,\hat{\mathbf{e}}_1 + v_2(S)\,\hat{\mathbf{e}}_2 + v_3(S)\,\hat{\mathbf{e}}_3 = \frac{dx_i}{dt}\,\hat{\mathbf{e}}_i \tag{2.2.52}$$

as the velocity in the S reference frame. Similarly, we write

$$\vec{\overline{V}} = \overline{v}_1(\overline{S})\widehat{\overline{\mathbf{e}}}_1 + \overline{v}_2(\overline{S})\widehat{\overline{\mathbf{e}}}_2 + \overline{v}_3(\overline{S})\widehat{\overline{\mathbf{e}}}_3 = \frac{d\overline{x}_i}{dt}\widehat{\overline{\mathbf{e}}}_i \tag{2.2.53}$$

as the velocity components relative to the body-fixed reference frame. There are occasions when it is desirable to represent $\vec{\overline{V}}$ in the S frame of reference and $\vec{V}$ in the $\overline{S}$ frame of reference. In these instances we can write

$$\vec{V} = v_1(\overline{S})\widehat{\overline{\mathbf{e}}}_1 + v_2(\overline{S})\widehat{\overline{\mathbf{e}}}_2 + v_3(\overline{S})\widehat{\overline{\mathbf{e}}}_3 \tag{2.2.54}$$

and

$$\vec{\overline{V}} = \overline{v}_1(S)\,\hat{\mathbf{e}}_1 + \overline{v}_2(S)\,\hat{\mathbf{e}}_2 + \overline{v}_3(S)\,\hat{\mathbf{e}}_3. \tag{2.2.55}$$

Here we have adopted the notation that $v_i(S)$ are the velocity components relative to the S reference frame and $v_i(\overline{S})$ are the same velocity components relative to the $\overline{S}$ reference frame. Similarly, $\overline{v}_i(\overline{S})$ denotes the velocity components relative to the $\overline{S}$ reference frame, while $\overline{v}_i(S)$ denotes the same velocity components relative to the S reference frame.

Here both $\vec{V}$ and $\vec{\overline{V}}$ are vectors and so their components are first order tensors and satisfy the transformation laws

$$\overline{v}_i(S) = \ell_{ji} v_j(S) = \ell_{ji}\dot{x}_j \qquad \text{and} \qquad v_i(\overline{S}) = \ell_{ij}\overline{v}_j(\overline{S}) = \ell_{ij}\dot{\overline{x}}_j. \tag{2.2.56}$$

The equations (2.2.56) define the relative velocity components as functions of time t. By differentiating the equations (2.2.50) we obtain

$$\frac{d\overline{x}_i}{dt} = \overline{v}_i(\overline{S}) = \ell_{ji}\dot{x}_j + \dot{\ell}_{ji}x_j \tag{2.2.57}$$

and

$$\frac{dx_i}{dt} = v_i(S) = \ell_{ij}\dot{\overline{x}}_j + \dot{\ell}_{ij}\overline{x}_j. \tag{2.2.58}$$

Multiply the equation (2.2.57) by ℓ_{mi} and multiply the equation (2.2.58) by ℓ_{im} and derive the relations

$$v_m(\overline{S}) = v_m(S) + \ell_{mi}\dot{\ell}_{ji}x_j \tag{2.2.59}$$

and

$$\overline{v}_m(S) = \overline{v}_m(\overline{S}) + \ell_{im}\dot{\ell}_{ij}\overline{x}_j. \tag{2.2.60}$$

The equations (2.2.59) and (2.2.60) describe the transformation laws of the velocity components upon changing from the S to the $\overline{S}$ reference frame. These equations can be expressed in terms of the angular velocity by making certain substitutions which are now defined.

The first order angular velocity vector ω_i is related to the second order skew-symmetric angular velocity tensor ω_{ij} by the defining equation

$$\omega_{mn} = e_{imn}\omega_i. \tag{2.2.61}$$

The equation (2.2.61) implies that ω_i and ω_{ij} are dual tensors and

$$\omega_i = \frac{1}{2} e_{ijk} \omega_{jk}.$$

Also the velocity of a point which is rotating about the origin relative to the S frame of reference is $v_i(S) = e_{ijk}\omega_j x_k$ which can also be written in the form $v_m(S) = -\omega_{mk} x_k$. Since the barred axes rotate with the rigid body, then a particle in the barred reference frame will have $v_m(\overline{S}) = 0$, since the coordinates of a point in the rigid body will be constants with respect to this reference frame. Consequently, we write equation (2.2.59) in the form $0 = v_m(S) + \ell_{mi}\dot{\ell}_{ji} x_j$ which implies that

$$v_m(S) = -\ell_{mi}\dot{\ell}_{ji} x_j = -\omega_{mk} x_k \quad \text{or} \quad \omega_{mj} = \omega_{mj}(\overline{S}, S) = \ell_{mi}\dot{\ell}_{ji}.$$

This equation is interpreted as describing the angular velocity tensor of $\overline{S}$ relative to S. Since ω_{ij} is a tensor, it can be represented in the barred system by

$$\begin{aligned} \overline{\omega}_{mn}(\overline{S}, S) &= \ell_{im}\ell_{jn}\omega_{ij}(\overline{S}, S) \\ &= \ell_{im}\ell_{jn}\ell_{is}\dot{\ell}_{js} \\ &= \delta_{ms}\ell_{jn}\dot{\ell}_{js} \\ &= \ell_{jn}\dot{\ell}_{jm} \end{aligned} \tag{2.2.62}$$

By differentiating the equations (2.2.51) it is an easy exercise to show that ω_{ij} is skew-symmetric. The second order angular velocity tensor can be used to write the equations (2.2.59) and (2.2.60) in the forms

$$\begin{aligned} v_m(\overline{S}) &= v_m(S) + \omega_{mj}(\overline{S}, S) x_j \\ \overline{v}_m(S) &= \overline{v}_m(\overline{S}) + \overline{\omega}_{jm}(\overline{S}, S)\overline{x}_j \end{aligned} \tag{2.2.63}$$

The above relations are now employed to derive the celebrated Euler's equations of motion of a rigid body.

Euler's Equations of Motion

We desire to find the equations of motion of a rigid body which is subjected to external forces. These equations are the formulas (2.2.49), and we now proceed to write these equations in a slightly different form. Similar to the introduction of the angular velocity tensor, given in equation (2.2.61), we now introduce the following tensors

1. The fourth order moment of inertia tensor I_{mnst} which is related to the second order moment of inertia tensor I_{ij} by the equations

$$I_{mnst} = \frac{1}{2} e_{jmn} e_{ist} I_{ij} \quad \text{or} \quad I_{ij} = \frac{1}{2} I_{pqrs} e_{ipq} e_{jrs} \tag{2.2.64}$$

2. The second order angular momentum tensor H_{jk} which is related to the angular momentum vector H_i by the equation

$$H_i = \frac{1}{2} e_{ijk} H_{jk} \quad \text{or} \quad H_{jk} = e_{ijk} H_i \tag{2.2.65}$$

3. The second order moment tensor M_{jk} which is related to the moment M_i by the relation

$$M_i = \frac{1}{2} e_{ijk} M_{jk} \quad \text{or} \quad M_{jk} = e_{ijk} M_i. \tag{2.2.66}$$

Now if we multiply equation (2.2.49) by e_{rjk}, then it can be written in the form

$$\frac{dH_{ij}}{dt} = M_{ij}. \tag{2.2.67}$$

Similarly, if we multiply the equation (2.2.42) by e_{imn}, then it can be expressed in the alternate form

$$H_{mn} = e_{imn}\omega_j I_{ji} = I_{mnst}\omega_{st}$$

and because of this relation the equation (2.2.67) can be expressed as

$$\frac{d}{dt}\left(I_{ijst}\omega_{st}\right) = M_{ij}. \tag{2.2.68}$$

We write this equation in the barred system of coordinates where $\overline{I}_{pqrs}$ will be a constant and consequently its derivative will be zero. We employ the transformation equations

$$\begin{aligned} I_{ijst} &= \ell_{ip}\ell_{jq}\ell_{sr}\ell_{tk}\overline{I}_{pqrk} \\ \overline{\omega}_{ij} &= \ell_{si}\ell_{tj}\omega_{st} \\ \overline{M}_{pq} &= \ell_{ip}\ell_{jq}M_{ij} \end{aligned}$$

and then multiply the equation (2.2.68) by $\ell_{ip}\ell_{jq}$ and simplify to obtain

$$\ell_{ip}\ell_{jq}\frac{d}{dt}\left(\ell_{i\alpha}\ell_{j\beta}\overline{I}_{\alpha\beta rk}\overline{\omega}_{rk}\right) = \overline{M}_{pq}.$$

Expand all terms in this equation and take note that the derivative of the $\overline{I}_{\alpha\beta rk}$ is zero. The expanded equation then simplifies to

$$\overline{I}_{pqrk}\frac{d\overline{\omega}_{rk}}{dt} + \left(\delta_{\alpha u}\delta_{pv}\delta_{\beta q} + \delta_{p\alpha}\delta_{\beta u}\delta_{qv}\right)\overline{I}_{\alpha\beta rk}\overline{\omega}_{rk}\overline{\omega}_{uv} = \overline{M}_{pq}. \tag{2.2.69}$$

Substitute into equation (2.2.69) the relations from equations (2.2.61),(2.2.64) and (2.2.66), and then multiply by e_{mpq} and simplify to obtain the Euler's equations of motion

$$\overline{I}_{im}\frac{d\overline{\omega}_i}{dt} - e_{tmj}\overline{I}_{ij}\overline{\omega}_i\overline{\omega}_t = \overline{M}_m. \tag{2.2.70}$$

Dropping the bar notation and performing the indicated summations over the range 1,2,3 we find the Euler equations have the form

$$\begin{aligned} &I_{11}\frac{d\omega_1}{dt} + I_{21}\frac{d\omega_2}{dt} + I_{31}\frac{d\omega_3}{dt} + \left(I_{13}\omega_1 + I_{23}\omega_2 + I_{33}\omega_3\right)\omega_2 - \left(I_{12}\omega_1 + I_{22}\omega_2 + I_{32}\omega_3\right)\omega_3 = M_1 \\ &I_{12}\frac{d\omega_1}{dt} + I_{22}\frac{d\omega_2}{dt} + I_{32}\frac{d\omega_3}{dt} + \left(I_{11}\omega_1 + I_{21}\omega_2 + I_{31}\omega_3\right)\omega_3 - \left(I_{13}\omega_1 + I_{23}\omega_2 + I_{33}\omega_3\right)\omega_1 = M_2 \\ &I_{13}\frac{d\omega_1}{dt} + I_{23}\frac{d\omega_2}{dt} + I_{33}\frac{d\omega_3}{dt} + \left(I_{12}\omega_1 + I_{22}\omega_2 + I_{32}\omega_3\right)\omega_1 - \left(I_{11}\omega_1 + I_{21}\omega_2 + I_{31}\omega_3\right)\omega_2 = M_3. \end{aligned} \tag{2.2.71}$$

In the special case where the barred axes are principal axes, then $I_{ij} = 0$ for $i \neq j$ and the Euler's equations reduces to the system of nonlinear differential equations

$$\begin{aligned} I_{11}\frac{d\omega_1}{dt} + (I_{33} - I_{22})\omega_2\omega_3 &= M_1 \\ I_{22}\frac{d\omega_2}{dt} + (I_{11} - I_{33})\omega_3\omega_1 &= M_2 \\ I_{33}\frac{d\omega_3}{dt} + (I_{22} - I_{11})\omega_1\omega_2 &= M_3. \end{aligned} \tag{2.2.72}$$

In the case of constant coefficients and constant moments the solutions of the above differential equations can be expressed in terms of Jacobi elliptic functions.

EXERCISE 2.2

▶ **1.** Find a set of parametric equations for the straight line which passes through the points $P_1(1,1,1)$ and $P_2(2,3,4)$. Find the unit tangent vector to any point on this line.

▶ **2.** Consider the space curve $x = \frac{1}{2}\sin^2 t$, $y = \frac{1}{2}t - \frac{1}{4}\sin 2t$, $z = \sin t$ where t is a parameter. Find the unit vectors $T^i, B^i, N^i, i = 1,2,3$ at the point where $t = \pi$.

▶ **3.** A claim has been made that the space curve $x = t$, $y = t^2$, $z = t^3$ intersects the plane 11x-6y+z=6 in three distinct points. Determine if this claim is true or false. Justify your answer and find the three points of intersection if they exist.

▶ **4.** Find a set of parametric equations $x_i = x_i(s_1, s_2), i = 1,2,3$ for the plane which passes through the points $P_1(3,0,0)$, $P_2(0,4,0)$ and $P_3(0,0,5)$. Find a unit normal to this plane.

▶ **5.** For the helix $x = \sin t \quad y = \cos t \quad z = \dfrac{2}{\pi}t$ find the equation of the tangent plane to the curve at the point where $t = \pi/4$. Find the equation of the tangent line to the curve at the point where $t = \pi/4$.

▶ **6.** Verify the derivative $\dfrac{\partial T}{\partial \dot{x}^r} = M g_{rm}\dot{x}^m$.

▶ **7.** Verify the derivative $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}^r}\right) = M\left[g_{rm}\ddot{x}^m + \dfrac{\partial g_{rm}}{\partial x^n}\dot{x}^n\dot{x}^m\right]$.

▶ **8.** Verify the derivative $\dfrac{\partial T}{\partial x^r} = \dfrac{1}{2}M\dfrac{\partial g_{mn}}{\partial x^r}\dot{x}^m\dot{x}^n$.

▶ **9.** Use the results from problems 6,7 and 8 to derive the Lagrange's form for the equations of motion defined by equation (2.2.23).

▶ **10.** Express the generalized velocity and acceleration in cylindrical coordinates $(x^1, x^2, x^3) = (r, \theta, z)$ and show

$$\begin{aligned} V^1 &= \frac{dx^1}{dt} = \frac{dr}{dt} \qquad & f^1 &= \frac{\delta V^1}{\delta t} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \\ V^2 &= \frac{dx^2}{dt} = \frac{d\theta}{dt} \qquad & f^2 &= \frac{\delta V^2}{\delta t} = \frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} \\ V^3 &= \frac{dx^3}{dt} = \frac{dz}{dt} \qquad & f^3 &= \frac{\delta V^3}{\delta t} = \frac{d^2z}{dt^2} \end{aligned}$$

Find the physical components of velocity and acceleration in cylindrical coordinates and show

$$\begin{aligned} V_r &= \frac{dr}{dt} \qquad & f_r &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \\ V_\theta &= r\frac{d\theta}{dt} \qquad & f_\theta &= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \\ V_z &= \frac{dz}{dt} \qquad & f_z &= \frac{d^2z}{dt^2} \end{aligned}$$

▶ **11.** Express the generalized velocity and acceleration in spherical coordinates $(x^2, x^2, x^3) = (\rho, \theta, \phi)$ and show

$$
\begin{aligned}
V^1 =& \frac{dx^1}{dt} = \frac{d\rho}{dt} & f^1 =& \frac{\delta V^1}{\delta t} = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2 - \rho\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 \\
V^2 =& \frac{dx^2}{dt} = \frac{d\theta}{dt} & f^2 =& \frac{\delta V^2}{\delta t} = \frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{2}{\rho}\frac{d\rho}{dt}\frac{d\theta}{dt} \\
V^3 =& \frac{dx^3}{dt} = \frac{d\phi}{dt} & f^3 =& \frac{\delta V^3}{\delta t} = \frac{d^2\phi}{dt^2} + \frac{2}{\rho}\frac{d\rho}{dt}\frac{d\phi}{dt} + 2\cot\theta\frac{d\theta}{dt}\frac{d\phi}{dt}
\end{aligned}
$$

Find the physical components of velocity and acceleration in spherical coordinates and show

$$
\begin{aligned}
V_\rho =& \frac{d\rho}{dt} & f_\rho =& \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2 - \rho\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 \\
V_\theta =& \rho\frac{d\theta}{dt} & f_\theta =& \rho\frac{d^2\theta}{dt^2} - \rho\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 + 2\frac{d\rho}{dt}\frac{d\theta}{dt} \\
V_\phi =& \rho\sin\theta\frac{d\phi}{dt} & f_\phi =& \rho\sin\theta\frac{d^2\phi}{dt^2} + 2\sin\theta\frac{d\rho}{dt}\frac{d\phi}{dt} + 2\rho\cos\theta\frac{d\theta}{dt}\frac{d\phi}{dt}
\end{aligned}
$$

▶ **12.** Expand equation (2.2.39) and write out all the components of the moment of inertia tensor I_{ij}.

▶ **13.** For ρ the density of a continuous material and $d\tau$ an element of volume inside a region R where the material is situated, we write $\rho d\tau$ as an element of mass inside R. Find an equation which describes the center of mass of the region R.

▶ **14.** Use the equation (2.2.68) to derive the equation (2.2.69).

▶ **15.** Drop the bar notation and expand the equation (2.2.70) and derive the equations (2.2.71).

▶ **16.** Verify the Euler transformation, given in example 2.2-5, is orthogonal.

▶ **17.** For the pulley and mass system illustrated in the figure 2.2-7 let

$$
\begin{aligned}
a &= \text{the radius of each pulley.} \\
\ell_1 &= \text{the length of the upper chord.} \\
\ell_2 &= \text{the length of the lower chord.}
\end{aligned}
$$

Neglect the weight of the pulley and find the equations of motion for the pulley mass system.

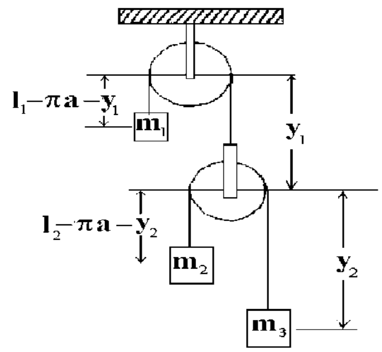

Figure 2.2-7. Pulley and mass system

► **18.** Let $\phi = \frac{ds}{dt}$, where s is the arc length between two points on a curve in generalized coordinates.

(a) Write the arc length in general coordinates as $ds = \sqrt{g_{mn}\dot{x}^m\dot{x}^n}dt$ and show the integral I, defined by equation (2.2.35), represents the distance between two points on a curve.

(b) Using the Euler-Lagrange equations (2.2.36) show that the shortest distance between two points in a generalized space is the curve defined by the equations: $\ddot{x}^i + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \dot{x}^j\dot{x}^k = \dot{x}^i \frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}}$

(c) Show in the special case $t = s$ the equations in part (b) reduce to $\dfrac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \dfrac{dx^j}{ds}\dfrac{dx^k}{ds} = 0$, for $i = 1, \ldots, N$. An examination of equation (1.5.51) shows that the above curves are geodesic curves.

(d) Show that the shortest distance between two points in a plane is a straight line.

(e) Consider two points on the surface of a cylinder of radius a. Let $u^1 = \theta$ and $u^2 = z$ denote surface coordinates in the two dimensional space defined by the surface of the cylinder. Show that the shortest distance between the points where $\theta = 0$, $z = 0$ and $\theta = \pi$, $z = H$ is $L = \sqrt{a^2\pi^2 + H^2}$.

► **19.** For $T = \frac{1}{2}mg_{ij}v^iv^j$ the kinetic energy of a particle and V the potential energy of the particle show that $T + V = constant$.

Hint: $mf_i = Q_i = -\frac{\partial V}{\partial x^i}, \quad i = 1, 2, 3$ and $\frac{dx^i}{dt} = \dot{x}^i = v^i, i = 1, 2, 3.$

► **20.** Define $H = T + V$ as the sum of the kinetic energy and potential energy of a particle. The quantity $H = H(x^r, p_r)$ is called the Hamiltonian of the particle and it is expressed in terms of:

- the particle position x^i and
- the particle momentum $p_i = mv_i = mg_{ij}\dot{x}^j$. Here $x^r$and p_r are treated as independent variables.

(a) Show that the particle momentum is a covariant tensor of rank 1.

(b) Express the kinetic energy T in terms of the particle momentum.

(c) Show that $p_i = \dfrac{\partial T}{\partial \dot{x}^i}$.

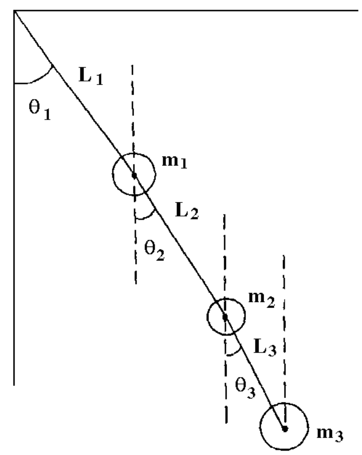

Figure 2.2-8. Compound pendulum

(d) Show that $\dfrac{dx^i}{dt} = \dfrac{\partial H}{\partial p_i}$ and $\dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x^i}$. These are a set of differential equations describing the position change and momentum change of the particle and are known as Hamilton's equations of motion for a particle.

► **21.** Let $\frac{\delta T^i}{\delta s} = \kappa N^i$ and $\frac{\delta N^i}{\delta s} = \tau B^i - \kappa T^i$ and calculate the intrinsic derivative of the cross product $B^i = \epsilon^{ijk} T_j N_k$ and find $\frac{\delta B^i}{\delta s}$ in terms of the unit normal vector.

► **22.** For T the kinetic energy of a particle and V the potential energy of a particle, define the Lagrangian $L = L(x^i, \dot{x}^i) = T - V = \dfrac{1}{2} M g_{ij} \dot{x}^i \dot{x}^j - V$ as a function of the independent variables $x^i, \dot{x}^i$. Define the Hamiltonian $H = H(x^i, p_i) = T + V = \dfrac{1}{2M} g^{ij} p_i p_j + V$, as a function of the independent variables x^i, p_i, where p_i is the momentum vector of the particle and M is the mass of the particle.

(a) Show that $p_i = \dfrac{\partial T}{\partial \dot{x}^i}$.

(b) Show that $\dfrac{\partial H}{\partial x^i} = -\dfrac{\partial L}{\partial x^i}$

► **23.** When the Euler angles, figure 2.2-6, are applied to the motion of rotating objects, θ is the angle of nutation, ϕ is the angle of precession and ψ is the angle of spin. Take projections and show that the time derivative of the Euler angles are related to the angular velocity vector components $\omega_x, \omega_y, \omega_z$ by the relations

$$\omega_x = \dot{\theta} \cos\psi + \dot{\phi} \sin\theta \sin\psi$$

$$\omega_y = -\dot{\theta} \sin\psi + \dot{\phi} \sin\theta \cos\psi$$

$$\omega_z = \dot{\psi} + \dot{\phi} \cos\theta$$

where $\omega_x, \omega_y, \omega_z$ are the angular velocity components along the $\overline{x}_1, \overline{x}_2, \overline{x}_3$ axes.

► **24.** Find the equations of motion for the compound pendulum illustrated in the figure 2.2-8.

► **25.** Let $\vec{F} = -\dfrac{GMm}{r^3}\vec{r}$ denote the inverse square law force of attraction between the earth and sun, with G a universal constant, M the mass of the sun, m the mass of the earth and $\frac{\vec{r}}{r}$ a unit vector from origin at the center of the sun pointing toward the earth. (a) Write down Newton's second law, in both vector and tensor form, which describes the motion of the earth about the sun. (b) Show that $\dfrac{d}{dt}(\vec{r} \times \vec{v}) = \vec{0}$ and consequently $\vec{r} \times \vec{v} = \vec{r} \times \frac{d\vec{r}}{dt} = \vec{h} =$ a constant.

► **26.** Construct a set of axes fixed and attached to an airplane. Let the x axis be a longitudinal axis running from the rear to the front of the plane along its center line. Let the y axis run between the wing tips and let the z axis form a right-handed system of coordinates. The y axis is called a lateral axis and the z axis is called a normal axis. Define *pitch* as any angular motion about the lateral axis. Define *roll* as any angular motion about the longitudinal axis. Define *yaw* as any angular motion about the normal axis. Consider two sets of axes. One set is the x, y, z axes attached to and moving with the aircraft. The other set of axes is denoted X, Y, Z and is fixed in space (an inertial set of axes). Describe the pitch, roll and yaw of an aircraft with respect to the inertial set of axes. Show the transformation is orthogonal. Hint: Consider pitch with respect to the fixed axes, then consider roll with respect to the pitch axes and finally consider yaw with respect to the roll axes. This produces three separate transformation matrices which can then be combined to describe the motions of pitch, roll and yaw of an aircraft.

► **27.** In Cartesian coordinates let $F_i = F_i(x^1, x^2, x^3)$ denote a force field and let $x^i = x^i(t)$ denote a curve C. (a) Show Newton's second law implies that along the curve C $\dfrac{d}{dt}\left(\dfrac{1}{2}m\left(\dfrac{dx^i}{dt}\right)^2\right) = F_i(x^1, x^2, x^3)\dfrac{dx^i}{dt}$ (no summation on i) and hence

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2\right)\right] = \frac{d}{dt}\left[\frac{1}{2}mv^2\right] = F_1\frac{dx^1}{dt} + F_2\frac{dx^2}{dt} + F_3\frac{dx^3}{dt}$$

(b) Consider two points on the curve C, say point A, $x^i(t_A)$ and point B, $x^i(t_B)$ and show that the work done in moving from A to B in the force field F_i is

$$\frac{1}{2}mv^2\Big]_{t_A}^{t_B} = \int_A^B F_i dx^1 + F_2 dx^2 + F_3 dx^3$$

where the right hand side is a line integral along the path C from A to B. (c) Show that if the force field is derivable from a potential function $U(x^1, x^2, x^3)$ by taking the gradient, then the work done is independent of the path C and depends only upon the end points A and B.

► **28.** Find the Lagrangian equations of motion of a spherical pendulum which consists of a bob of mass m suspended at the end of a wire of length ℓ, which is free to swing in any direction subject to the constraint that the wire length is constant. Neglect the weight of the wire and show that for the wire attached to the origin of a right handed x, y, z coordinate system, with the z axis downward, ϕ the angle between the wire and the z axis and θ the angle of rotation of the bob from the y axis, that there results the equations of motion $\quad \dfrac{d}{dt}\left(\sin^2\phi\,\dfrac{d\theta}{dt}\right) = 0 \quad$ and $\quad \dfrac{d^2\phi}{dt^2} - \left(\dfrac{d\theta}{dt}\right)^2 \sin\phi\cos\phi + \dfrac{g}{\ell}\sin\phi = 0$

► **29.** In Cartesian coordinates show the Frenet formulas can be written

$$\frac{d\vec{T}}{ds} = \vec{\delta} \times \vec{T}, \qquad \frac{d\vec{N}}{ds} = \vec{\delta} \times \vec{N}, \qquad \frac{d\vec{B}}{ds} = \vec{\delta} \times \vec{B}$$

where $\vec{\delta}$ is the Darboux vector and is defined $\vec{\delta} = \tau \vec{T} + \kappa \vec{B}$.

► **30.** Consider the following two cases for rigid body rotation.

Case 1: Rigid body rotation about a fixed line which is called the fixed axis of rotation. Select a point 0 on this fixed axis and denote by $\widehat{\mathbf{e}}$ a unit vector from 0 in the direction of the fixed line and denote by $\hat{\mathbf{e}}_R$ a unit vector which is perpendicular to the fixed axis of rotation. The position vector of a general point in the rigid body can then be represented by a position vector from the point 0 given by $\vec{r} = h\,\widehat{\mathbf{e}} + r_0\,\hat{\mathbf{e}}_R$ where h, r_0 and $\widehat{\mathbf{e}}$ are all constants and the vector $\hat{\mathbf{e}}_R$ is fixed in and rotating with the rigid body. Denote by $\omega = \dfrac{d\theta}{dt}$ the scalar angular change with respect to time of the vector $\hat{\mathbf{e}}_R$ as it rotates about the fixed line and define the vector angular velocity as $\vec{\omega} = \dfrac{d}{dt}(\theta\,\widehat{\mathbf{e}}) = \dfrac{d\theta}{dt}\widehat{\mathbf{e}}$ where $\theta\,\widehat{\mathbf{e}}$ is defined as the vector angle of rotation.

(a) Show that $\dfrac{d\,\hat{\mathbf{e}}_R}{d\theta} = \widehat{\mathbf{e}} \times \hat{\mathbf{e}}_R$.

(b) Show that $\vec{V} = \dfrac{d\vec{r}}{dt} = r_0\dfrac{d\,\hat{\mathbf{e}}_R}{dt} = r_0\dfrac{d\,\hat{\mathbf{e}}_R}{d\theta}\dfrac{d\theta}{dt} = \vec{\omega} \times (r_0\,\hat{\mathbf{e}}_R) = \vec{\omega} \times (h\,\widehat{\mathbf{e}} + r_0\,\hat{\mathbf{e}}_R) = \vec{\omega} \times \vec{r}$.

Case 2: Rigid body rotation about a fixed point 0. Construct at point 0 the unit vector $\hat{\mathbf{e}}_1$ which is fixed in and rotating with the rigid body. From pages 80,87 we know that $\dfrac{d\,\hat{\mathbf{e}}_1}{dt}$ must be perpendicular to $\hat{\mathbf{e}}_1$ and so we can define the vector $\hat{\mathbf{e}}_2$ as a unit vector which is in the direction of $\dfrac{d\,\hat{\mathbf{e}}_1}{dt}$ such that $\dfrac{d\,\hat{\mathbf{e}}_1}{dt} = \alpha\,\hat{\mathbf{e}}_2$ for some constant α. We can then define the unit vector $\hat{\mathbf{e}}_3$ from $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$.

(a) Show that $\dfrac{d\,\hat{\mathbf{e}}_3}{dt}$, which must be perpendicular to $\hat{\mathbf{e}}_3$, is also perpendicular to $\hat{\mathbf{e}}_1$.

(b) Show that $\dfrac{d\,\hat{\mathbf{e}}_3}{dt}$ can be written as $\dfrac{d\,\hat{\mathbf{e}}_3}{dt} = \beta\,\hat{\mathbf{e}}_2$ for some constant β.

(c) From $\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1$ show that $\dfrac{d\,\hat{\mathbf{e}}_2}{dt} = (\alpha\,\hat{\mathbf{e}}_3 - \beta\,\hat{\mathbf{e}}_1) \times \hat{\mathbf{e}}_2$

(d) Define $\vec{\omega} = \alpha\,\hat{\mathbf{e}}_3 - \beta\,\hat{\mathbf{e}}_1$ and show that $\dfrac{d\,\hat{\mathbf{e}}_1}{dt} = \vec{\omega} \times \hat{\mathbf{e}}_1, \qquad \dfrac{d\,\hat{\mathbf{e}}_2}{dt} = \vec{\omega} \times \hat{\mathbf{e}}_2, \qquad \dfrac{d\,\hat{\mathbf{e}}_3}{dt} = \vec{\omega} \times \hat{\mathbf{e}}_3$

(e) Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote an arbitrary point within the rigid body with respect to the point 0. Show that $\dfrac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$.

Note that in Case 2 the direction of $\vec{\omega}$ is not fixed as the unit vectors $\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_1$ are constantly changing. In this case the direction $\vec{\omega}$ is called an instantaneous axis of rotation and $\vec{\omega}$, which also can change in magnitude and direction, is called the instantaneous angular velocity.