§2.5 CONTINUUM MECHANICS (FLUIDS)

Let us consider a fluid medium and use Cartesian tensors to derive the mathematical equations that describe how a fluid behaves. A fluid continuum, like a solid continuum, is characterized by equations describing:

1. Conservation of linear momentum
$$\sigma_{ij,j} + \varrho b_i = \varrho \dot{v}_i \tag{2.5.1}$$
2. Conservation of angular momentum $\sigma_{ij} = \sigma_{ji}$.
3. Conservation of mass (continuity equation)
$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x_i} v_i + \varrho \frac{\partial v_i}{\partial x_i} = 0 \quad \text{or} \quad \frac{D\varrho}{Dt} + \varrho \nabla \cdot \vec{V} = 0. \tag{2.5.2}$$

In the above equations $v_i, i = 1,2,3$ is a velocity field, ϱ is the density of the fluid, σ_{ij} is the stress tensor and b_j is an external force per unit mass. In the cgs system of units of measurement, the above quantities have dimensions

$$[\dot{v}_j] = \text{cm/sec}^2, \qquad [b_j] = dynes/g, \quad [\sigma_{ij}] = dyne/cm^2, \quad [\varrho] = g/cm^3. \tag{2.5.3}$$

The displacement field $u_i, i = 1,2,3$ can be represented in terms of the velocity field $v_i, i = 1,2,3$, by the relation

$$u_i = \int_0^t v_i \, dt. \tag{2.5.4}$$

The strain tensor components of the medium can then be represented in terms of the velocity field as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \int_0^t \frac{1}{2}(v_{i,j} + v_{j,i}) \, dt = \int_0^t D_{ij} \, dt, \tag{2.5.5}$$

where

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \tag{2.5.6}$$

is called the *rate of deformation tensor* , *velocity strain tensor*, or *rate of strain tensor.*

Note the difference in the equations describing a solid continuum compared with those for a fluid continuum. In describing a solid continuum we were primarily interested in calculating the displacement field $u_i, i = 1,2,3$ when the continuum was subjected to external forces. In describing a fluid medium, we calculate the velocity field $v_i, i = 1,2,3$ when the continuum is subjected to external forces. We therefore replace the strain tensor relations by the velocity strain tensor relations in all future considerations concerning the study of fluid motion.

Constitutive Equations for Fluids

In addition to the above basic equations, we will need a set of constitutive equations which describe the material properties of the fluid. Toward this purpose consider an arbitrary point within the fluid medium and pass an imaginary plane through the point. The orientation of the plane is determined by a unit normal n_i , $i = 1,2,3$ to the planar surface. For a fluid at rest we wish to determine the stress vector $t_i^{(n)}$ acting on the plane element passing through the selected point P. We desire to express $t_i^{(n)}$ in terms of the stress tensor σ_{ij}. The superscript (n) on the stress vector is to remind you that the stress acting on the planar element depends upon the orientation of the plane through the point.

We make the assumption that $t_i^{(n)}$ is colinear with the normal vector to the surface passing through the selected point. It is also assumed that for fluid elements at rest, there are no shear forces acting on the planar element through an arbitrary point and therefore the stress tensor σ_{ij} should be independent of the orientation of the plane. That is, we desire for the stress vector σ_{ij} to be an isotropic tensor. This requires σ_{ij} to have a specific form. To find this specific form we let σ_{ij} denote the stress components in a general coordinate system x^i, $i = 1, 2, 3$ and let $\overline{\sigma}_{ij}$ denote the components of stress in a barred coordinate system $\overline{x}^i, i = 1, 2, 3$. Since σ_{ij} is a tensor, it must satisfy the transformation law

$$\overline{\sigma}_{mn} = \sigma_{ij} \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n}, \quad i, j, m, n = 1, 2, 3. \tag{2.5.7}$$

We desire for the stress tensor σ_{ij} to be an invariant under an arbitrary rotation of axes. Consider therefore the special coordinate transformations illustrated in the figures 2.5-1(a) and (b).

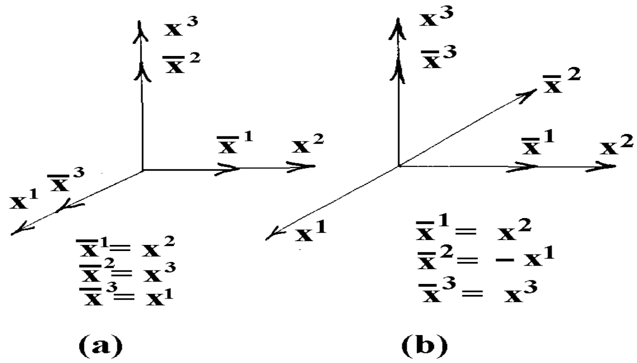

Figure 2.5-1. Coordinate transformations due to rotations

For the transformation equations given in figure 2.5-1(a), the stress tensor in the barred system of coordinates is

$$\begin{array}{lll} \overline{\sigma}_{11} = \sigma_{22} & \overline{\sigma}_{21} = \sigma_{32} & \overline{\sigma}_{31} = \sigma_{12} \\ \overline{\sigma}_{12} = \sigma_{23} & \overline{\sigma}_{22} = \sigma_{33} & \overline{\sigma}_{32} = \sigma_{13} \\ \overline{\sigma}_{13} = \sigma_{21} & \overline{\sigma}_{23} = \sigma_{31} & \overline{\sigma}_{33} = \sigma_{11}. \end{array} \tag{2.5.8}$$

If σ_{ij} is to be isotropic, we desire that $\overline{\sigma}_{11} = \sigma_{11}$, $\overline{\sigma}_{22} = \sigma_{22}$ and $\overline{\sigma}_{33} = \sigma_{33}$. If the equations (2.5.8) are to produce these results, we require that σ_{11}, σ_{22} and σ_{33} must be equal. We denote these common values by $(-p)$. In particular, the equations (2.5.8) show that if $\overline{\sigma}_{11} = \sigma_{11}$, $\overline{\sigma}_{22} = \sigma_{22}$ and $\overline{\sigma}_{33} = \sigma_{33}$, then we must require that $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$. If $\overline{\sigma}_{12} = \sigma_{12}$ and $\overline{\sigma}_{23} = \sigma_{23}$, then we also require that $\sigma_{12} = \sigma_{23} = \sigma_{31}$. We note that if $\overline{\sigma}_{13} = \sigma_{13}$ and $\overline{\sigma}_{32} = \sigma_{32}$, then we require that $\sigma_{21} = \sigma_{32} = \sigma_{13}$. If the equations (2.5.7) are expanded using the transformation given in figure 2.5-1(b), we obtain the additional requirements that

$$\begin{array}{lll} \overline{\sigma}_{11} = \sigma_{22} & \overline{\sigma}_{21} = -\sigma_{12} & \overline{\sigma}_{31} = \sigma_{32} \\ \overline{\sigma}_{12} = -\sigma_{21} & \overline{\sigma}_{22} = \sigma_{11} & \overline{\sigma}_{32} = -\sigma_{31} \\ \overline{\sigma}_{13} = \sigma_{23} & \overline{\sigma}_{23} = -\sigma_{13} & \overline{\sigma}_{33} = \sigma_{33}. \end{array} \tag{2.5.9}$$

Analysis of these equations implies that if σ_{ij} is to be isotropic, then $\overline{\sigma}_{21} = \sigma_{21} = -\sigma_{12} = -\sigma_{21}$

$$\text{or } \sigma_{21} = 0 \text{ which implies} \quad \sigma_{12} = \sigma_{23} = \sigma_{31} = \sigma_{21} = \sigma_{32} = \sigma_{13} = 0. \tag{2.5.10}$$

The above analysis demonstrates that if the stress tensor σ_{ij} is to be isotropic, it must have the form

$$\sigma_{ij} = -p\delta_{ij}. \tag{2.5.11}$$

Use the traction condition (2.3.11), and express the stress vector as

$$t_j^{(n)} = \sigma_{ij} n_i = -p n_j. \tag{2.5.12}$$

This equation is interpreted as representing the stress vector at a point on a surface with outward unit normal n_i, where p is the pressure (hydrostatic pressure) stress magnitude assumed to be positive. The negative sign in equation (2.5.12) denotes a compressive stress.

Imagine a submerged object in a fluid medium. We further imagine the object to be covered with unit normal vectors emanating from each point on its surface. The equation (2.5.12) shows that the hydrostatic pressure always acts on the object in a compressive manner. A force results from the stress vector acting on the object. The direction of the force is opposite to the direction of the unit outward normal vectors. It is a compressive force at each point on the surface of the object.

The above considerations were for a fluid at rest (hydrostatics). For a fluid in motion (hydrodynamics) a different set of assumptions must be made. Hydrodynamical experiments show that the shear stress components are not zero and so we assume a stress tensor having the form

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \quad i,j = 1,2,3, \tag{2.5.13}$$

where τ_{ij} is called the viscous stress tensor. Note that all real fluids are both viscous and compressible.

Definition: (Viscous/inviscid fluid) If the viscous stress tensor τ_{ij} is zero for all i, j, then the fluid is called an inviscid, nonviscous, ideal or perfect fluid. The fluid is called viscous when τ_{ij} is different from zero.

In these notes it is assumed that the equation (2.5.13) represents the basic form for constitutive equations describing fluid motion.

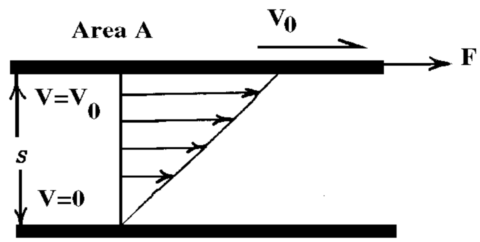

Figure 2.5-2. Viscosity experiment.

Viscosity

Most fluids are characterized by the fact that they cannot resist shearing stresses. That is, if you put a shearing stress on the fluid, the fluid gives way and flows. Consider the experiment illustrated in the figure 2.5-2 which illustrates a fluid moving between two parallel plane surfaces. Let S denote the distance between the two planes. Now keep the lower surface fixed or stationary and move the upper surface parallel to the lower surface with a constant velocity $\vec{V}_0$. If you measure the force F required to maintain the constant velocity of the upper surface, you discover that the force F varies directly as the area A of the surface and the ratio V_0/S. This is expressed in the form

$$\frac{F}{A} = \mu^* \frac{V_0}{S}. \tag{2.5.14}$$

The constant μ^* is a proportionality constant called the coefficient of viscosity. The viscosity usually depends upon temperature, but throughout our discussions we will assume the temperature is constant. A dimensional analysis of the equation (2.5.14) implies that the basic dimension of the viscosity is $[\mu^*] = ML^{-1}T^{-1}$. For example, $[\mu^*] = \text{gm}/(\text{cm sec})$ in the cgs system of units. The viscosity is usually measured in units of centipoise where one centipoise represents one-hundredth of a poise, where the unit of 1 poise= 1 gram per centimeter per second. The result of the above experiment shows that the stress is proportional to the change in velocity with change in distance or gradient of the velocity.

Linear Viscous Fluids

The above experiment with viscosity suggest that the viscous stress tensor τ_{ij} is dependent upon both the gradient of the fluid velocity and the density of the fluid.

In Cartesian coordinates, the simplest model suggested by the above experiment is that the viscous stress tensor τ_{ij} is proportional to the velocity gradient $v_{i,j}$ and so we write

$$\tau_{ik} = c_{ikmp} v_{m,p}, \tag{2.5.15}$$

where c_{ikmp} is a proportionality constant which is dependent upon the fluid density.

The viscous stress tensor must be independent of any reference frame, and hence we assume that the proportionality constants c_{ikmp} can be represented by an isotropic tensor. Recall that an isotropic tensor has the basic form

$$c_{ikmp} = \lambda^* \delta_{ik}\delta_{mp} + \mu^*(\delta_{im}\delta_{kp} + \delta_{ip}\delta_{km}) + \nu^*(\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}) \tag{2.5.16}$$

where λ^*, μ^* and ν^* are constants. Examining the results from equations (2.5.11) and (2.5.13) we find that if the viscous stress is symmetric, then $\tau_{ij} = \tau_{ji}$. This requires ν^* be chosen as zero. Consequently, the viscous stress tensor reduces to the form

$$\tau_{ik} = \lambda^* \delta_{ik} v_{p,p} + \mu^* (v_{k,i} + v_{i,k}). \tag{2.5.17}$$

The coefficient μ^* is called the first coefficient of viscosity and the coefficient λ^* is called the second coefficient of viscosity. Sometimes it is convenient to define

$$\zeta = \lambda^* + \frac{2}{3}\mu^* \tag{2.5.18}$$

as "another second coefficient of viscosity," or "bulk coefficient of viscosity." The condition of zero bulk viscosity is known as Stokes hypothesis. Many fluids problems assume the Stoke's hypothesis. This requires that the bulk coefficient be zero or very small. Under these circumstances the second coefficient of viscosity is related to the first coefficient of viscosity by the relation $\lambda^* = -\frac{2}{3}\mu^*$. In the study of shock waves and acoustic waves the Stoke's hypothesis is not applicable.

There are many tables and empirical formulas where the viscosity of different types of fluids or gases can be obtained. For example, in the study of the kinetic theory of gases the viscosity can be calculated from the Sutherland formula $\mu^* = \dfrac{C_1 g T^{3/2}}{T + C_2}$ where C_1, C_2 are constants for a specific gas. These constants can be found in certain tables. The quantity g is the gravitational constant and T is the temperature in degrees Rankine ($^oR = 460 + {}^oF$). Many other empirical formulas like the above exist. Also many graphs and tabular values of viscosity can be found. The table 5.1 lists the approximate values of the viscosity of some selected fluids and gases.

Table 5.1	Viscosity of selected fluids and gases in units of $\frac{\text{gram}}{\text{cm}-\text{sec}}$ = Poise at Atmospheric Pressure.			
Substance	0°C	20°C	60°C	100°C
Water	0.01798	0.01002	0.00469	0.00284
Alcohol	0.01773			
Ethyl Alcohol		0.012	0.00592	
Glycol		0.199	0.0495	0.0199
Mercury	0.017	0.0157	0.013	0.0100
Air	$1.708(10^{-4})$			$2.175(10^{-4})$
Helium	$1.86(10^{-4})$	$1.94(10^{-4})$		$2.28(10^{-4})$
Nitrogen	$1.658(10^{-4})$	$1.74(10^{-4})$	$1.92(10^{-4})$	$2.09(10^{-4})$

The viscous stress tensor given in equation (2.5.17) may also be expressed in terms of the rate of deformation tensor defined by equation (2.5.6). This representation is

$$\tau_{ij} = \lambda^* \delta_{ij} D_{kk} + 2\mu^* D_{ij}, \tag{2.5.19}$$

where $2D_{ij} = v_{i,j} + v_{j,i}$ and $D_{kk} = D_{11} + D_{22} + D_{33} = v_{1,1} + v_{2,2} + v_{3,3} = v_{i,i} = \Theta$ is the rate of change of the dilatation considered earlier. In Cartesian form, with velocity components u, v, w, the viscous stress tensor components are

$$\tau_{xx} =(\lambda^* + 2\mu^*)\frac{\partial u}{\partial x} + \lambda^*\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad \tau_{yx} = \tau_{xy} =\mu^*\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\tau_{yy} =(\lambda^* + 2\mu^*)\frac{\partial v}{\partial y} + \lambda^*\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \qquad \tau_{zx} = \tau_{xz} =\mu^*\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$
$$\tau_{zz} =(\lambda^* + 2\mu^*)\frac{\partial w}{\partial z} + \lambda^*\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad \tau_{zy} = \tau_{yz} =\mu^*\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

In cylindrical form, with velocity components v_r, v_θ, v_z, the viscous stess tensor components are

$$\tau_{rr} =2\mu^*\frac{\partial v_r}{\partial r} + \lambda^*\nabla\cdot\vec{V}$$
$$\tau_{\theta\theta} =2\mu^*\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \lambda^*\nabla\cdot\vec{V}$$
$$\tau_{zz} =2\mu^*\frac{\partial v_z}{\partial z} + \lambda^*\nabla\cdot\vec{V}$$
$$\text{where}\quad \nabla\cdot\vec{V} =\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$
$$\tau_{\theta r} = \tau_{r\theta} =\mu^*\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$
$$\tau_{rz} = \tau_{zr} =\mu^*\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$
$$\tau_{z\theta} = \tau_{\theta z} =\mu^*\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)$$

In spherical coordinates, with velocity components v_ρ, v_θ, v_ϕ, the viscous stress tensor components have the form

$$\tau_{\rho\rho} =2\mu^*\frac{\partial v_\rho}{\partial \rho} + \lambda^*\nabla\cdot\vec{V}$$
$$\tau_{\theta\theta} =2\mu^*\left(\frac{1}{\rho}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\rho}{\rho}\right) + \lambda^*\nabla\cdot\vec{V}$$
$$\tau_{\phi\phi} =2\mu^*\left(\frac{1}{\rho\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\rho}{\rho} + \frac{v_\theta\cot\theta}{\rho}\right) + \lambda^*\nabla\cdot\vec{V}$$
$$\text{where}\quad \nabla\cdot\vec{V} =\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 v_\rho\right) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{\rho\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$
$$\tau_{\rho\theta} = \tau_{\theta\rho} =\mu^*\left(\rho\frac{\partial}{\partial \rho}\left(\frac{v_\theta}{\rho}\right) + \frac{1}{\rho}\frac{\partial v_\rho}{\partial \theta}\right)$$
$$\tau_{\phi\rho} = \tau_{\rho\phi} =\mu^*\left(\frac{1}{\rho\sin\theta}\frac{\partial v_r}{\partial \phi} + \rho\frac{\partial}{\partial \rho}\left(\frac{v_\theta}{\rho}\right)\right)$$
$$\tau_{\theta\phi} = \tau_{\phi\theta} =\mu^*\left(\frac{\sin\theta}{\rho}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{\rho\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)$$

Note that the viscous stress tensor is a linear function of the rate of deformation tensor D_{ij}. Such a fluid is called a *Newtonian fluid.* In cases where the viscous stress tensor is a nonlinear function of D_{ij} the fluid is called *non-Newtonian.*

Definition: (Newtonian Fluid) If the viscous stress tensor τ_{ij} is expressible as a linear function of the rate of deformation tensor D_{ij}, the fluid is called a Newtonian fluid. Otherwise, the fluid is called a non-Newtonian fluid.

Important note: Do not assume an arbitrary form for the constitutive equations unless there is experimental evidence to support your assumption. A constitutive equation is a very important step in the modeling processes as it describes the material you are working with. One cannot arbitrarily assign a form to the viscous stress and expect the mathematical equations to describe the correct fluid behavior. The form of the viscous stress is an important part of the modeling process and by assigning different forms to the viscous stress tensor then various types of materials can be modeled. We restrict our study in these notes to Newtonian fluids.

In Cartesian coordinates the rate of deformation-stress constitutive equations for a Newtonian fluid can be written as

$$\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}D_{kk} + 2\mu^* D_{ij} \tag{2.5.20}$$

which can also be written in the alternative form

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} v_{k,k} + \mu^*(v_{i,j} + v_{j,i}) \tag{2.5.21}$$

involving the gradient of the velocity.

Upon transforming from a Cartesian coordinate system $y^i, i = 1, 2, 3$ to a more general system of coordinates $\overline{x}^i, i = 1, 2, 3$, we write

$$\overline{\sigma}_{mn} = \sigma_{ij} \frac{\partial y^i}{\partial \overline{x}^m} \frac{\partial y^j}{\partial \overline{x}^n}. \tag{2.5.22}$$

Now using the divergence from equation (2.1.3) and substituting equation (2.5.21) into equation (2.5.22) we obtain a more general expression for the constitutive equation. Performing the indicated substitutions there results

$$\begin{aligned}\overline{\sigma}_{mn} &= \left[-p\delta_{ij} + \lambda^* \delta_{ij} v^k_{,k} + \mu^*(v_{i,j} + v_{j,i})\right] \frac{\partial y^i}{\partial \overline{x}^m} \frac{\partial y^j}{\partial \overline{x}^n} \\ \overline{\sigma}_{mn} &= -p\overline{g}_{mn} + \lambda^* \overline{g}_{mn} \overline{v}^k_{,k} + \mu^*(\overline{v}_{m,n} + \overline{v}_{n,m}).\end{aligned}$$

Dropping the bar notation, the stress-velocity strain relationships in the general coordinates $x^i, i = 1, 2, 3$, is

$$\sigma_{mn} = -pg_{mn} + \lambda^* g_{mn} g^{ik} v_{i,k} + \mu^*(v_{m,n} + v_{n,m}). \tag{2.5.23}$$

Summary

The basic equations which describe the motion of a Newtonian fluid are :
Continuity equation (Conservation of mass)

$$\frac{\partial \varrho}{\partial t} + \left(\varrho v^i\right)_{,i} = 0, \quad \text{or} \quad \frac{D\varrho}{Dt} + \varrho \nabla \cdot \vec{V} = 0 \qquad \text{1 equation.} \tag{2.5.24}$$

Conservation of linear momentum $\quad \sigma^{ij}_{\ ,j} + \varrho b^i = \varrho \dot{v}^i, \qquad$ 3 equations

$$\text{or in vector form} \quad \varrho \frac{D\vec{V}}{Dt} = \varrho \vec{b} + \nabla \cdot \boldsymbol{\sigma} = \varrho \vec{b} - \nabla p + \nabla \cdot \boldsymbol{\tau} \tag{2.5.25}$$

where $\boldsymbol{\sigma} = \sum_{i=1}^3 \sum_{j=1}^3 (-p\delta_{ij} + \tau_{ij})\, \hat{e}_i\, \hat{e}_j$ and $\boldsymbol{\tau} = \sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij}\, \hat{e}_i\, \hat{e}_j$ are second order tensors. Conservation of angular momentum $\sigma^{ij} = \sigma^{ji}$, (Reduces the set of equations (2.5.23) to 6 equations.) Rate of deformation tensor (Velocity strain tensor)

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right), \qquad \text{6 equations.} \tag{2.5.26}$$

Constitutive equations

$$\sigma_{mn} = -pg_{mn} + \lambda^* g_{mn} g^{ik} v_{i,k} + \mu^*(v_{m,n} + v_{n,m}), \qquad \text{6 equations.} \tag{2.5.27}$$

In the cgs system of units the above quantities have the following units of measurements in Cartesian coordinates

$$
\begin{array}{rlr}
v_i & \text{is the velocity field , } i=1,2,3, & [v_i]=\text{cm/sec} \\
\sigma_{ij} & \text{is the stress tensor, } i,j=1,2,3, & [\sigma_{ij}]=\text{dyne/cm}^2 \\
\varrho & \text{is the fluid density} & [\varrho]=\text{gm/cm}^3 \\
b^i & \text{is the external body forces per unit mass} & [b^i]=\text{dyne/gm} \\
D_{ij} & \text{is the rate of deformation tensor} & [D_{ij}]=\text{sec}^{-1} \\
p & \text{is the pressure} & [p]=\text{dyne/cm}^2 \\
\lambda^*,\mu^* & \text{are coefficients of viscosity} & [\lambda^*]=[\mu^*]=\text{Poise}
\end{array}
$$

$$\text{where } 1 \text{ Poise} = 1\text{gm/cm sec}$$

If we assume the external body forces per unit mass are known, then the equations (2.5.24), (2.5.25), (2.5.26), and (2.5.27) represent 16 equations in the 16 unknowns

$$\varrho, v_1, v_2, v_3, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}, D_{11}, D_{12}, D_{13}, D_{22}, D_{23}, D_{33}.$$

Navier-Stokes-Duhem Equations of Fluid Motion

Substituting the stress tensor from equation (2.5.27) into the linear momentum equation (2.5.25), and assuming that the viscosity coefficients are constants, we obtain the Navier-Stokes-Duhem equations for fluid motion. In Cartesian coordinates these equations can be represented in any of the equivalent forms

$$
\begin{aligned}
\varrho \dot{v}_i &= \varrho b_i - p_{,j}\delta_{ij} + (\lambda^* + \mu^*) v_{k,ki} + \mu^* v_{i,jj} \\
\varrho \frac{\partial v_i}{\partial t} + \varrho v_j v_{i,j} &= \varrho b_i + (-p\delta_{ij} + \tau_{ij})_{,j} \\
\frac{\partial \varrho v_i}{\partial t} + (\varrho v_i v_j + p\delta_{ij} - \tau_{ij})_{,j} &= \varrho b_i \\
\varrho \frac{D\vec{v}}{Dt} &= \varrho \vec{b} - \nabla p + (\lambda^* + \mu^*)\nabla\,(\nabla\cdot\vec{v}) + \mu^*\nabla^2\vec{v}
\end{aligned}
\tag{2.5.28}
$$

where $\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}$ is the material derivative, substantial derivative or convective derivative. This derivative is represented as

$$\dot{v}_i = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x^j} v^j = \frac{\partial v_i}{\partial t} + v_{i,j} v^j. \tag{2.5.29}$$

In the vector form of equations (2.5.28), the terms on the right-hand side of the equation represent force terms. The term $\varrho\vec{b}$ represents external body forces per unit volume. If these forces are derivable from a potential function ϕ, then the external forces are conservative and can be represented in the form $-\varrho\nabla\phi$. The term $-\nabla p$ is the gradient of the pressure and represents a force per unit volume due to hydrostatic pressure. The above statement is verified in the exercises that follow this section. The remaining terms can be written

$$\vec{f}_{viscous} = (\lambda^* + \mu^*)\nabla\,(\nabla\cdot\vec{v}) + \mu^*\nabla^2\vec{v} \tag{2.5.30}$$

and are given the physical interpretation of an internal force per unit volume. These internal forces arise from the shearing stresses in the moving fluid. If $\vec{f}_{viscous}$ is zero the vector equation in (2.5.28) is called Euler's equation.

If the viscosity coefficients are nonconstant, then the Navier-Stokes equations can be written in the Cartesian form

$$\begin{aligned}\varrho[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}] =&\varrho b_i + \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + \lambda^*\delta_{ij}\frac{\partial v_k}{\partial x_k} + \mu^*\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]\\ =&\varrho b_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda^*\frac{\partial v_k}{\partial x_k}\right) + \frac{\partial}{\partial x^j}\left[\mu^*\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]\end{aligned}$$

which can also be written in terms of the bulk coefficient of viscosity $\zeta = \lambda^* + \frac{2}{3}\mu^*$ as

$$\begin{aligned}\varrho[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}] =&\varrho b_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left((\zeta - \frac{2}{3}\mu^*)\frac{\partial v_k}{\partial x_k}\right) + \frac{\partial}{\partial x^j}\left[\mu^*\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]\\ =&\varrho b_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\zeta\frac{\partial v_k}{\partial x_k}\right) + \frac{\partial}{\partial x^j}\left[\mu^*\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right)\right]\end{aligned}$$

These equations form the basics of viscous flow theory.

In the case of orthogonal coordinates, where $g_{(i)(i)} = h_i^2$ (no summation) and $g_{ij} = 0$ for $i \neq j$, general expressions for the Navier-Stokes equations in terms of the physical components $v(1), v(2), v(3)$ are:

Navier-Stokes-Duhem equations for compressible fluid in terms of physical components: $(i \neq j \neq k)$

$$\begin{aligned}&\varrho\left[\frac{\partial v(i)}{\partial t} + \frac{v(1)}{h_1}\frac{\partial v(i)}{\partial x_1} + \frac{v(2)}{h_2}\frac{\partial v(i)}{\partial x_2} + \frac{v(3)}{h_3}\frac{\partial v(i)}{\partial x_3}\right.\\
&\qquad\left. - \frac{v(j)}{h_ih_j}\left(v(j)\frac{\partial h_j}{\partial x_i} - v(i)\frac{\partial h_i}{\partial x_j}\right) + \frac{v(k)}{h_ih_k}\left(v(i)\frac{\partial h_i}{\partial x_k} - v(k)\frac{\partial h_k}{\partial x_i}\right)\right] =\\
&\varrho\frac{b(i)}{h_i} - \frac{1}{h_i}\frac{\partial p}{\partial x_i} + \frac{1}{h_i}\frac{\partial}{\partial x_i}\left(\lambda^*\nabla\cdot\vec{V}\right) + \frac{\mu^*}{h_ih_j}\left[\frac{h_j}{h_i}\frac{\partial}{\partial x_i}\left(\frac{v(j)}{h_j}\right) + \frac{h_i}{h_j}\frac{\partial}{\partial x_j}\left(\frac{v(i)}{h_i}\right)\right]\frac{\partial h_i}{\partial h_j}\\
&+\frac{\mu^*}{h_ih_k}\left[\frac{h_i}{h_k}\frac{\partial}{\partial x_k}\left(\frac{v(i)}{h_i}\right) + \frac{h_k}{h_i}\frac{\partial}{\partial x_i}\left(\frac{v(k)}{h_k}\right)\right]\frac{\partial h_i}{\partial x_k} - \frac{2\mu^*}{h_ih_j}\left[\frac{1}{h_j}\frac{\partial v(j)}{\partial x_j} + \frac{v(k)}{h_jh_k}\frac{\partial h_j}{\partial x_k} + \frac{v(i)}{h_ih_j}\frac{\partial h_j}{\partial x_i}\right]\\
&-\frac{2\mu^*}{h_ih_k}\left[\frac{1}{h_k}\frac{\partial v(k)}{\partial x_k} + \frac{v(i)}{h_ih_k}\frac{\partial h_k}{\partial x_i} + \frac{v(k)}{h_kh_j}\frac{\partial h_k}{\partial x_i}\right]\frac{\partial h_k}{\partial x_i} + \frac{1}{h_ih_jh_k}\left[\frac{\partial}{\partial x_i}\left\{2\mu^*h_jh_k\left(\frac{1}{h_i}\frac{\partial v(i)}{\partial x_i} + \frac{v(j)}{h_ih_j}\frac{\partial h_i}{\partial h_j} + \frac{v(k)}{h_ih_k}\frac{\partial h_i}{\partial x_k}\right)\right\}\right.\\
&\left.+\frac{\partial}{\partial x_j}\left\{\mu^*h_ih_k\left(\frac{h_j}{h_i}\frac{\partial}{\partial x_i}\left(\frac{v(j)}{h_j}\right) + \frac{h_i}{h_j}\frac{\partial}{\partial x_j}\left(\frac{v(i)}{h_i}\right)\right)\right\} + \frac{\partial}{\partial x_k}\left\{\mu^*h_ih_j\left(\frac{h_i}{h_k}\frac{\partial}{\partial x_k}\left(\frac{v(i)}{h_i}\right) + \frac{h_k}{h_i}\frac{\partial}{\partial x_i}\left(\frac{v(k)}{h_k}\right)\right)\right\}\right]\end{aligned}\tag{2.5.31}$$

where $\nabla\cdot\vec{v}$ is found in equation (2.1.4).

In the above equation, cyclic values are assigned to i, j and k. That is, for the x_1 components assign the values $i = 1, j = 2, k = 3$; for the x_2 components assign the values $i = 2, j = 3, k = 1$; and for the x_3 components assign the values $i = 3, j = 1, k = 2$.

The tables 5.2, 5.3 and 5.4 show the expanded form of the Navier-Stokes equations in Cartesian, cylindrical and spherical coordinates respectively.

$$\varrho\frac{DV_x}{Dt} = \varrho b_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu^*\frac{\partial V_x}{\partial x} + \lambda^*\nabla\cdot\vec{V}\right] + \frac{\partial}{\partial y}\left[\mu^*\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu^*\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)\right]$$

$$\varrho\frac{DV_y}{Dt} = \varrho b_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu^*\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu^*\frac{\partial V_y}{\partial y} + \lambda^*\nabla\cdot\vec{V}\right] + \frac{\partial}{\partial z}\left[\mu^*\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)\right]$$

$$\varrho\frac{DV_z}{Dt} = \varrho b_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu^*\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu^*\left(\frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[2\mu^*\frac{\partial V_z}{\partial z} + \lambda^*\nabla\cdot\vec{V}\right]$$

where $\frac{D}{Dt}(\,) = \frac{\partial(\,)}{\partial t} + V_x\frac{\partial(\,)}{\partial x} + V_y\frac{\partial(\,)}{\partial y} + V_z\frac{\partial(\,)}{\partial z}$

and $\nabla\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$(2.5.31a)$

Table 5.2 Navier-Stokes equations for compressible fluids in Cartesian coordinates.

$$\varrho\left[\frac{DV_r}{Dt} - \frac{V_\theta^2}{r}\right] = \varrho b_r - \frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left[2\mu^*\frac{\partial V_r}{\partial r} + \lambda^*\nabla\cdot\vec{V}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu^*\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu^*\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)\right] + \frac{2\mu^*}{r}\left(\frac{\partial V_r}{\partial r} - \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r}\right)$$

$$\varrho\left[\frac{DV_\theta}{Dt} + \frac{V_rV_\theta}{r}\right] = \varrho b_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\partial}{\partial r}\left[\mu^*\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}\right)\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[2\mu^*\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right) + \lambda^*\nabla\cdot\vec{V}\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu^*\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right)\right] + \frac{2\mu^*}{r}\left[\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}\right]$$

$$\varrho\frac{DV_z}{Dt} = \varrho b_z - \frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[\mu^* r\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu^*\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[2\mu^*\frac{\partial V_z}{\partial z} + \lambda^*\nabla\cdot\vec{V}\right]$$

where $\frac{D}{Dt}(\,) = \frac{\partial(\,)}{\partial t} + V_r\frac{\partial(\,)}{\partial r} + \frac{V_\theta}{r}\frac{\partial(\,)}{\partial \theta} + V_z\frac{\partial(\,)}{\partial z}$

and $\nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$

$(2.5.31b)$

Table 5.3 Navier-Stokes equations for compressible fluids in cylindrical coordinates.

Observe that for incompressible flow $\frac{D\varrho}{Dt} = 0$ which implies $\nabla \cdot \vec{V} = 0$. Therefore, the assumptions of constant viscosity and incompressibility of the flow will simplify the above equations. If on the other hand the viscosity is temperature dependent and the flow is compressible, then one should add to the above equations the continuity equation, an energy equation and an equation of state. The energy equation comes from the first law of thermodynamics applied to a control volume within the fluid and will be considered in the sections ahead. The equation of state is a relation between thermodynamic variables which is added so that the number of equations equals the number of unknowns. Such a system of equations is known as a closed system. An example of an equation of state is the ideal gas law where pressure p is related to gas density ϱ and temperature T by the relation $p = \varrho RT$ where R is the universal gas constant.

$$
\begin{aligned}
\varrho\left[\frac{DV_\rho}{Dt} - \frac{V_\theta^2+V_\phi^2}{\rho}\right] = & \varrho b_\rho - \frac{\partial p}{\partial \rho} + \frac{\partial}{\partial \rho}\left[2\mu^* \frac{\partial V_\rho}{\partial \rho} + \lambda^* \nabla\cdot\vec{V}\right] + \frac{1}{\rho}\frac{\partial}{\partial \theta}\left[\mu^*\rho\frac{\partial}{\partial \rho}\left(\frac{V_\theta}{\rho}\right) + \frac{\mu^*}{\rho}\frac{\partial V_\rho}{\partial \theta}\right] \\
& + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\mu^*}{\rho\sin\theta}\frac{\partial V_\rho}{\partial \phi} + \mu^*\rho\frac{\partial}{\partial\rho}\left(\frac{V_\phi}{\rho}\right)\right] \\
& + \frac{\mu^*}{\rho}\left[4\frac{\partial V_\rho}{\partial \rho} - \frac{2}{\rho}\frac{\partial V_\theta}{\partial\theta} - \frac{4V_\rho}{\rho} - \frac{2}{\rho\sin\theta}\frac{\partial V_\phi}{\partial\phi} - \frac{2V_\theta\cot\theta}{\rho} + \rho\cot\theta\frac{\partial}{\partial\rho}\left(\frac{V_\theta}{\rho}\right) + \frac{\cot\theta}{\rho}\frac{\partial V_\rho}{\partial\theta}\right]
\end{aligned}
$$

$$
\begin{aligned}
\varrho\left[\frac{DV_\theta}{Dt} + \frac{V_\rho V_\theta}{\rho} - \frac{V_\phi^2\cot\theta}{\rho}\right] = & \varrho b_\theta - \frac{1}{\rho}\frac{\partial p}{\partial\theta} + \frac{\partial}{\partial\rho}\left[\mu^*\rho\frac{\partial}{\partial\rho}\left(\frac{V_\theta}{\rho}\right) + \frac{\mu^*}{\rho}\frac{\partial V_\rho}{\partial\theta}\right] \\
& + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\frac{2\mu^*}{\rho}\left(\frac{\partial V_\theta}{\partial\theta} + V_\rho\right) + \lambda^*\nabla\cdot\vec{V}\right] \\
& + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\mu^*\sin\theta}{\rho}\frac{\partial}{\partial\theta}\left(\frac{V_\phi}{\sin\theta}\right) + \frac{\mu^*}{\rho\sin\theta}\frac{\partial V_\theta}{\partial\phi}\right] \\
& + \frac{\mu^*}{\rho}\left[2\cot\theta\left(\frac{1}{\rho}\frac{\partial V_\theta}{\partial\theta} - \frac{1}{\rho\sin\theta}\frac{\partial V_\phi}{\partial\phi} - \frac{V_\theta\cot\theta}{\rho}\right) + 3\left(\rho\frac{\partial}{\partial\rho}\left(\frac{V_\theta}{\rho}\right) + \frac{1}{\rho}\frac{\partial V_\rho}{\partial\theta}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\varrho\left[\frac{DV_\phi}{Dt} + \frac{V_\theta V_\phi}{\rho} + \frac{V_\theta V_\phi\cot\theta}{\rho}\right] = & \varrho b_\phi - \frac{1}{\rho\sin\theta}\frac{\partial p}{\partial\phi} + \frac{\partial}{\partial\rho}\left[\frac{\mu^*}{\rho\sin\theta}\frac{\partial V_\rho}{\partial\phi} + \mu^*\rho\frac{\partial}{\partial\rho}\left(\frac{V_\phi}{\rho}\right)\right] \\
& + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\frac{\mu^*\sin\theta}{\rho}\frac{\partial}{\partial\theta}\left(\frac{V_\phi}{\sin\theta}\right) + \frac{\mu^*}{\rho\sin\theta}\frac{\partial V_\theta}{\partial\phi}\right] \\
& + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{2\mu^*}{\rho}\left(\frac{1}{\sin\theta}\frac{\partial V_\phi}{\partial\phi} + V_\rho + V_\theta\cot\theta\right) + \lambda^*\nabla\cdot\vec{V}\right] \\
& + \frac{\mu^*}{\rho}\left[\frac{3}{\rho\sin\theta}\frac{\partial V_\rho}{\partial\phi} + 3\rho\frac{\partial}{\partial\rho}\left(\frac{V_\phi}{\rho}\right) + 2\cot\theta\left(\frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\left(\frac{V_\phi}{\sin\theta}\right) + \frac{1}{\rho\sin\theta}\frac{\partial V_\theta}{\partial\phi}\right)\right]
\end{aligned}
$$

$$
\text{where} \quad \frac{D}{Dt}(\,) = \frac{\partial(\,)}{\partial t} + V_\rho\frac{\partial(\,)}{\partial\rho} + \frac{V_\theta}{\rho}\frac{\partial(\,)}{\partial\theta} + \frac{V_\phi}{\rho\sin\theta}\frac{\partial(\,)}{\partial\phi}
$$

$$
\text{and} \quad \nabla\cdot\vec{V} = \frac{1}{\rho^2}\frac{\partial(\rho^2 V_\rho)}{\partial\rho} + \frac{1}{\rho\sin\theta}\frac{\partial V_\theta\sin\theta}{\partial\theta} + \frac{1}{\rho\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{2.5.31c}
$$

Table 5.4 Navier-Stokes equations for compressible fluids in spherical coordinates.

We now consider various special cases of the Navier-Stokes-Duhem equations.

Special Case 1: Assume that $\vec{b}$ is a conservative force such that $\vec{b} = -\nabla\,\phi$. Also assume that the viscous force terms are zero. Consider steady flow ($\frac{\partial \vec{v}}{\partial t} = 0$) and show that equation (2.5.28) reduces to the equation

$$(\vec{v} \cdot \nabla)\,\vec{v} = \frac{-1}{\varrho}\nabla\,p - \nabla\,\phi \quad \varrho \text{ is constant.} \tag{2.5.32}$$

Employing the vector identity

$$(\vec{v} \cdot \nabla)\,\vec{v} = (\nabla \times \vec{v}) \times \vec{v} + \frac{1}{2}\nabla(\vec{v} \cdot \vec{v}), \tag{2.5.33}$$

we take the dot product of equation (2.5.32) with the vector $\vec{v}$. Noting that $\vec{v} \cdot [(\nabla \times \vec{v}) \times \vec{v}] = \vec{0}$ we obtain

$$\vec{v} \cdot \nabla \left[\frac{p}{\varrho} + \phi + \frac{1}{2}v^2\right] = 0. \tag{2.5.34}$$

This equation shows that for steady flow we will have

$$\frac{p}{\varrho} + \phi + \frac{1}{2}v^2 = \text{constant} \tag{2.5.35}$$

along a streamline. This result is known as Bernoulli's theorem. In the special case where $\phi = gh$ is a force due to gravity, the equation (2.5.35) reduces to $\frac{p}{\varrho} + \frac{v^2}{2} + gh = constant$. This equation is known as Bernoulli's equation. It is a conservation of energy statement which has many applications in fluids.

Special Case 2: Assume that $\vec{b} = -\nabla\,\phi$ is conservative and define the quantity Ω by

$$\vec{\Omega} = \nabla\,\times \vec{v} = \operatorname{curl} \vec{v} \qquad\qquad \omega = \frac{1}{2}\Omega \tag{2.5.36}$$

as the vorticity vector associated with the fluid flow and observe that its magnitude is equivalent to twice the angular velocity of a fluid particle. Then using the identity from equation (2.5.33) we can write the Navier-Stokes-Duhem equations in terms of the vorticity vector. We obtain the hydrodynamic equations

$$\frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2}\nabla\,v^2 = -\frac{1}{\varrho}\nabla\,p - \nabla\,\phi + \frac{1}{\varrho}\vec{f}_{viscous}, \tag{2.5.37}$$

where $\vec{f}_{viscous}$ is defined by equation (2.5.30). In the special case of nonviscous flow this further reduces to the Euler equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2}\nabla\,v^2 = -\frac{1}{\varrho}\nabla\,p - \nabla\,\phi.$$

If the density ϱ is a function of the pressure only it is customary to introduce the function

$$P = \int_c^p \frac{dp}{\varrho} \quad \text{so that} \quad \nabla P = \frac{dP}{dp}\nabla p = \frac{1}{\varrho}\nabla p$$

then the Euler equation becomes

$$\frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} = -\nabla(P + \phi + \frac{1}{2}v^2).$$

Some examples of vorticies are smoke rings, hurricanes, tornadoes, and some sun spots. You can create a vortex by letting water stand in a sink and then remove the plug. Watch the water and you will see that a rotation or vortex begins to occur. Vortices are associated with circulating motion.

Pick an arbitrary simple closed curve C and place it in the fluid flow and define the line integral $K = \oint_C \vec{v} \cdot \hat{e}_t \, ds$, where ds is an element of arc length along the curve C, $\vec{v}$ is the vector field defining the velocity, and $\hat{e}_t$ is a unit tangent vector to the curve C. The integral K is called the circulation of the fluid around the closed curve C. The circulation is the summation of the tangential components of the velocity field along the curve C. The local vorticity at a point is defined as the limit

$$\lim_{\text{Area}\to 0} \frac{\text{Circulation around } C}{\text{Area inside } C} = \text{circulation per unit area.}$$

By Stokes theorem, if $\operatorname{curl} \vec{v} = \vec{0}$, then the fluid is called irrotational and the circulation is zero. Otherwise the fluid is rotational and possesses vorticity.

If we are only interested in the velocity field we can eliminate the pressure by taking the curl of both sides of the equation (2.5.37). If we further assume that the fluid is incompressible we obtain the special equations

$$\begin{aligned} \nabla \cdot \vec{v} &= 0 && \text{Incompressible fluid, } \varrho \text{ is constant.} \\ \vec{\Omega} &= \operatorname{curl} \vec{v} && \text{Definition of vorticity vector.} \\ \frac{\partial \vec{\Omega}}{\partial t} + \nabla \times (\vec{\Omega} \times \vec{v}) &= \frac{\mu^*}{\varrho} \nabla^2 \vec{\Omega} && \text{Results because curl of gradient is zero.} \end{aligned} \tag{2.5.38}$$

Note that when Ω is identically zero, we have irrotational motion and the above equations reduce to the Cauchy-Riemann equations. Note also that if the term $\nabla \times (\vec{\Omega} \times \vec{v})$ is neglected, then the last equation in equation (2.5.38) reduces to a diffusion equation. This suggests that the vorticity diffuses through the fluid once it is created.

Vorticity can be caused by a rigid rotation or by shear flow. For example, in cylindrical coordinates let $\vec{V} = r\omega \, \hat{e}_\theta$, with r, ω constants, denote a rotational motion, then $\operatorname{curl} \vec{V} = \nabla \times \vec{V} = 2\omega \, \hat{e}_z$, which shows the vorticity is twice the rotation vector. Shear can also produce vorticity. For example, consider the velocity field $\vec{V} = y \, \hat{e}_1$ with $y \geq 0$. Observe that this type of flow produces shear because $|\vec{V}|$ increases as y increases. For this flow field we have $\operatorname{curl} \vec{V} = \nabla \times \vec{V} = -\, \hat{e}_3$. The right-hand rule tells us that if an imaginary paddle wheel is placed in the flow it would rotate clockwise because of the shear effects.

Scaled Variables

In the Navier-Stokes-Duhem equations for fluid flow we make the assumption that the external body forces are derivable from a potential function ϕ and write $\vec{b} = -\nabla \, \phi \, [dyne/gm]$ We also want to write the Navier-Stokes equations in terms of scaled variables

$$\begin{array}{llll} \overline{\vec{v}} = \dfrac{\vec{v}}{v_0} & \overline{\varrho} = \dfrac{\varrho}{\varrho_0} & \overline{\phi} = \dfrac{\phi}{gL}, & \overline{y} = \dfrac{y}{L} \\ \overline{p} = \dfrac{p}{p_0} & \overline{t} = \dfrac{t}{\tau} & \overline{x} = \dfrac{x}{L} & \overline{z} = \dfrac{z}{L} \end{array}$$

which can be referred to as the barred system of dimensionless variables. Dimensionless variables are introduced by scaling each variable associated with a set of equations by an appropriate constant term called a characteristic constant associated with that variable. Usually the characteristic constants are chosen from various parameters used in the formulation of the set of equations. The characteristic constants assigned to each variable are not unique and so problems can be scaled in a variety of ways. The characteristic constants assigned to each variable are scales, of the appropriate dimension, which act as reference quantities which reflect the order of magnitude changes expected of that variable over a certain range or area of interest associated with the problem. An inappropriate magnitude selected for a characteristic constant can result in a scaling where significant information concerning the problem can be lost. This is analogous to selecting an inappropriate mesh size in a numerical method. The numerical method might give you an answer but details of the answer might be lost.

In the above scaling of the variables occurring in the Navier-Stokes equations we let v_0 denote some characteristic speed, p_0 a characteristic pressure, ϱ_0 a characteristic density, L a characteristic length, g the acceleration of gravity and τ a characteristic time (for example $\tau = L/v_0$), then the barred variables $\overline{v}$, $\overline{p}$, $\overline{\varrho}$,$\overline{\phi}$, $\overline{t}$, $\overline{x}$, $\overline{y}$ and $\overline{z}$ are dimensionless. Define the barred gradient operator by

$$\overline{\nabla} = \frac{\partial}{\partial \overline{x}}\,\hat{e}_1 + \frac{\partial}{\partial \overline{y}}\,\hat{e}_2 + \frac{\partial}{\partial \overline{z}}\,\hat{e}_3$$

where all derivatives are with respect to the barred variables. The above change of variables reduces the Navier-Stokes-Duhem equations

$$\varrho\frac{\partial \vec{v}}{\partial t} + \varrho(\vec{v}\cdot\nabla)\,\vec{v} = -\varrho\nabla\phi - \nabla\,p + (\lambda^* + \mu^*)\nabla\;(\nabla\cdot\vec{v}) + \mu^*\nabla^2\,\vec{v}, \tag{2.5.39}$$

to the form

$$\begin{aligned}\left(\frac{\varrho_0 v_0}{\tau}\right)\overline{\varrho}\frac{\partial \vec{\overline{v}}}{\partial \overline{t}} + \left(\frac{\varrho_0 v_0^2}{L}\right)\overline{\varrho}\left(\vec{\overline{v}}\cdot\overline{\nabla}\right)\vec{\overline{v}} = &-\varrho_0 g\overline{\varrho}\overline{\nabla}\,\overline{\phi} - \left(\frac{p_0}{L}\right)\overline{\nabla}\overline{p}\\ &+\frac{(\lambda^*+\mu^*)}{L^2}v_0\overline{\nabla}\left(\overline{\nabla}\cdot\vec{\overline{v}}\right) + \left(\frac{\mu^* v_0}{L^2}\right)\overline{\nabla}^2\vec{\overline{v}}.\end{aligned} \tag{2.5.40}$$

Now if each term in the equation (2.5.40) is divided by the coefficient $\varrho_0 v_0^2/L$, we obtain the equation

$$S\overline{\varrho}\frac{\partial \vec{\overline{v}}}{\partial \overline{t}} + \overline{\varrho}\left(\vec{\overline{v}}\cdot\overline{\nabla}\right)\vec{\overline{v}} = \frac{-1}{F}\overline{\varrho}\overline{\nabla}\,\overline{\phi} - E\overline{\nabla}\overline{p} + \left(\frac{\lambda^*}{\mu^*}+1\right)\frac{1}{R}\overline{\nabla}\left(\overline{\nabla}\cdot\vec{\overline{v}}\right) + \frac{1}{R}\overline{\nabla}^2\vec{\overline{v}} \tag{2.5.41}$$

which has the dimensionless coefficients

$E = \dfrac{p_0}{\varrho_0 v_0^2}$ = Euler number

$R = \dfrac{\varrho_0 V_0 L}{\mu^*}$ = Reynolds number

$F = \dfrac{v_0^2}{gL}$ = Froude number, g is acceleration of gravity

$S = \dfrac{L}{\tau v_0}$ = Strouhal number.

Dropping the bars over the symbols, we write the dimensionless equation using the above coefficients. The scaled equation is found to have the form

$$S\varrho\frac{\partial \vec{v}}{\partial t} + \varrho(\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{F}\varrho\nabla\phi - E\nabla p + \left(\frac{\lambda^*}{\mu^*}+1\right)\frac{1}{R}\nabla\left(\nabla\cdot\vec{v}\right) + \frac{1}{R}\nabla^2\vec{v} \tag{2.5.42}$$

Boundary Conditions

Fluids problems can be classified as internal flows or external flows. An example of an internal flow problem is that of fluid moving through a converging-diverging nozzle. An example of an external flow problem is fluid flow around the boundary of an aircraft. For both types of problems there is some sort of boundary which influences how the fluid behaves. In these types of problems the fluid is assumed to adhere to a boundary. Let $\vec{r}_b$ denote the position vector to a point on a boundary associated with a moving fluid, and let $\vec{r}$ denote the position vector to a general point in the fluid. Define $\vec{v}(\vec{r})$ as the velocity of the fluid at the point $\vec{r}$ and define $\vec{v}(\vec{r}_b)$ as the known velocity of the boundary. The boundary might be moving within the fluid or it could be fixed in which case the velocity at all points on the boundary is zero. We define the boundary condition associated with a moving fluid as an adherence boundary condition.

Definition: (Adherence Boundary Condition)
An adherence boundary condition associated with a fluid in motion is defined as the limit $\lim\limits_{\vec{r}\to\vec{r}_b} \vec{v}(\vec{r}) = \vec{v}(\vec{r}_b)$ where $\vec{r}_b$ is the position vector to a point on the boundary.

Sometimes, when no finite boundaries are present, it is necessary to impose conditions on the components of the velocity far from the origin. Such conditions are referred to as boundary conditions at infinity.

Summary and Additional Considerations

Throughout the development of the basic equations of continuum mechanics we have neglected thermodynamical and electromagnetic effects. The inclusion of thermodynamics and electromagnetic fields adds additional terms to the basic equations of a continua. These basic equations describing a continuum are:

Conservation of mass

The conservation of mass is a statement that the total mass of a body is unchanged during its motion. This is represented by the continuity equation

$$\frac{\partial \varrho}{\partial t} + (\varrho v^k)_{,k} = 0 \quad \text{or} \quad \frac{D\varrho}{Dt} + \varrho \nabla \cdot \vec{V} = 0$$

where ϱ is the mass density and v^k is the velocity.

Conservation of linear momentum

The conservation of linear momentum requires that the time rate of change of linear momentum equal the resultant of all forces acting on the body. In symbols, we write

$$\frac{D}{Dt}\int_{\mathcal{V}} \varrho v^i \, d\tau = \int_{\mathcal{S}} F^i_{(s)} n_i \, dS + \int_{\mathcal{V}} \varrho F^i_{(b)} \, d\tau + \sum_{\alpha=1}^{n} F^i_{(\alpha)} \tag{2.5.43}$$

where $\frac{Dv^i}{Dt} = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x^k} v^k$ is the material derivative, $F^i_{(s)}$ are the surface forces per unit area, $F^i_{(b)}$ are the body forces per unit mass and $F^i_{(\alpha)}$ represents isolated external forces. Here $\mathcal{S}$ represents the surface and $\mathcal{V}$ represents the volume of the control volume. The right-hand side of this conservation law represents the resultant force coming from the consideration of all surface forces and body forces acting on a control volume.

Surface forces acting upon the control volume involve such things as pressures and viscous forces, while body forces are due to such things as gravitational, magnetic and electric fields.

Conservation of angular momentum

The conservation of angular momentum states that the time rate of change of angular momentum (moment of linear momentum) must equal the total moment of all forces and couples acting upon the body. In symbols,

$$\frac{D}{Dt}\int_{\mathcal{V}} \varrho e_{ijk} x^j v^k \, d\tau = \int_{\mathcal{S}} e_{ijk} x^j F^k_{(s)} \, dS + \int_{\mathcal{V}} \varrho e_{ijk} x^j F^k_{(b)} \, d\tau + \sum_{\alpha=1}^{n} (e_{ijk} x^j_{(\alpha)} F^k_{(\alpha)} + M^i_{(\alpha)}) \tag{2.5.44}$$

where $M^i_{(\alpha)}$ represents concentrated couples and $F^k_{(\alpha)}$ represents isolated forces.

Conservation of energy

The conservation of energy law requires that the time rate of change of kinetic energy plus internal energies is equal to the sum of the rate of work from all forces and couples plus a summation of all external energies that enter or leave a control volume per unit of time. The energy equation results from the first law of thermodynamics and can be written

$$\frac{D}{Dt}(E+K) = \dot{W} + \dot{Q}_h \tag{2.5.45}$$

where E is the internal energy, K is the kinetic energy, $\dot{W}$ is the rate of work associated with surface and body forces, and $\dot{Q}_h$ is the input heat rate from surface and internal effects.

Let e denote the internal specific energy density within a control volume, then $E = \int_{\mathcal{V}} \varrho e \, d\tau$ represents the total internal energy of the control volume. The kinetic energy of the control volume is expressed as $K = \frac{1}{2}\int_{\mathcal{V}} \varrho g_{ij} v^i v^j \, d\tau$ where v^i is the velocity, ϱ is the density and $d\tau$ is a volume element. The energy (rate of work) associated with the body and surface forces is represented

$$\dot{W} = \int_{\mathcal{S}} g_{ij} F^i_{(s)} v^j \, dS + \int_{\mathcal{V}} \varrho g_{ij} F^i_{(b)} v^j \, d\tau + \sum_{\alpha=1}^{n} (g_{ij} F^i_{(\alpha)} v^j + g_{ij} M^i_{(\alpha)} \omega^j)$$

where ω^j is the angular velocity of the point $x^i_{(\alpha)}$, $F^i_{(\alpha)}$ are isolated forces, and $M^i_{(\alpha)}$ are isolated couples. Two external energy sources due to thermal sources are heat flow q^i and rate of internal heat production $\frac{\partial Q}{\partial t}$ per unit volume. The conservation of energy can thus be represented

$$\begin{aligned} \frac{D}{Dt}\int_{\mathcal{V}} \varrho (e + \frac{1}{2} g_{ij} v^i v^j) \, d\tau = &\int_{\mathcal{S}} (g_{ij} F^i_{(s)} v^j - q_i n^i) \, dS + \int_{\mathcal{V}} (\varrho g_{ij} F^i_{(b)} v^j + \frac{\partial Q}{\partial t}) \, d\tau \\ &+ \sum_{\alpha=1}^{n} (g_{ij} F^i_{(\alpha)} v^j + g_{ij} M^i_{(\alpha)} \omega^j + U_{(\alpha)}) \end{aligned} \tag{2.5.46}$$

where $U_{(\alpha)}$ represents all other energies resulting from thermal, mechanical, electric, magnetic or chemical sources which influx the control volume and D/Dt is the material derivative.

In equation (2.5.46) the left hand side is the material derivative of an integral of the total energy $e_t = \varrho(e + \frac{1}{2} g_{ij} v^i v^j)$ over the control volume. Material derivatives are not like ordinary derivatives and so we cannot interchange the order of differentiation and integration in this term. Here we must use the result that

$$\frac{D}{Dt}\int_{\mathcal{V}} e_t\,d\tau = \int_{\mathcal{V}}\left(\frac{\partial e_t}{\partial t} + \nabla\cdot(e_t\vec{V})\right)\,d\tau.$$

To prove this result we consider a more general problem. Let $\mathcal{A}$ denote the amount of some quantity per unit mass. The quantity $\mathcal{A}$ can be a scalar, vector or tensor. The total amount of this quantity inside the control volume is $A = \int_{\mathcal{V}} \varrho\mathcal{A}\,d\tau$ and therefore the rate of change of this quantity is

$$\frac{\partial A}{\partial t} = \int_{\mathcal{V}}\frac{\partial(\varrho\mathcal{A})}{\partial t}\,d\tau = \frac{D}{Dt}\int_{\mathcal{V}}\varrho\mathcal{A}\,d\tau - \int_{\mathcal{S}}\varrho\mathcal{A}\vec{V}\cdot\hat{n}\,dS,$$

which represents the rate of change of material within the control volume plus the influx into the control volume. The minus sign is because $\hat{n}$ is always a unit outward normal. By converting the surface integral to a volume integral, by the Gauss divergence theorem, and rearranging terms we find that

$$\frac{D}{Dt}\int_{\mathcal{V}}\varrho\mathcal{A}\,d\tau = \int_{\mathcal{V}}\left[\frac{\partial(\varrho\mathcal{A})}{\partial t} + \nabla\cdot(\varrho\mathcal{A}\vec{V})\right]\,d\tau.$$

In equation (2.5.46) we neglect all isolated external forces and substitute $F^i_{(s)} = \sigma^{ij}n_j$, $F^i_{(b)} = b^i$ where $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$. We then replace all surface integrals by volume integrals and find that the conservation of energy can be represented in the form

$$\frac{\partial e_t}{\partial t} + \nabla\cdot(e_t\vec{V}) = \nabla(\boldsymbol{\sigma}\cdot\vec{V}) - \nabla\cdot\vec{q} + \varrho\vec{b}\cdot\vec{V} + \frac{\partial Q}{\partial t} \tag{2.5.47}$$

where $e_t = \varrho e + \varrho(v_1^2 + v_2^2 + v_3^2)/2$ is the total energy and $\boldsymbol{\sigma} = \sum_{i=1}^3\sum_{j=1}^3 \sigma_{ij}\,\hat{e}_i\,\hat{e}_j$ is the second order stress tensor. Here

$$\boldsymbol{\sigma}\cdot\vec{V} = -p\vec{V} + \sum_{j=1}^3 \tau_{1j}v_j\;\hat{e}_1 + \sum_{j=1}^3 \tau_{2j}v_j\;\hat{e}_2 + \sum_{j=1}^3 \tau_{3j}v_j\;\hat{e}_3 = -p\vec{V} + \boldsymbol{\tau}\cdot\vec{V}$$

and $\tau_{ij} = \mu^*(v_{i,j} + v_{j,i}) + \lambda^*\delta_{ij}v_{k,k}$ is the viscous stress tensor. Using the identities

$$\varrho\frac{D(e_t/\varrho)}{Dt} = \frac{\partial e_t}{\partial t} + \nabla\cdot(e_t\vec{V}) \quad \text{and} \quad \varrho\frac{D(e_t/\varrho)}{Dt} = \varrho\frac{De}{Dt} + \varrho\frac{D(V^2/2)}{Dt}$$

together with the momentum equation (2.5.25) dotted with $\vec{V}$ as

$$\varrho\frac{D\vec{V}}{Dt}\cdot\vec{V} = \varrho\vec{b}\cdot\vec{V} - \nabla p\cdot\vec{V} + (\nabla\cdot\boldsymbol{\tau})\cdot\vec{V}$$

the energy equation (2.5.47) can then be represented in the form

$$\varrho\frac{D\,e}{Dt} + p(\nabla\cdot\vec{V}) = -\nabla\cdot\vec{q} + \frac{\partial Q}{\partial t} + \Phi \tag{2.5.48}$$

where Φ is the dissipation function and can be represented

$$\Phi = (\tau_{ij}v_i)_{,j} - v_i\tau_{ij,j} = \nabla\cdot(\boldsymbol{\tau}\cdot\vec{V}) - (\nabla\cdot\boldsymbol{\tau})\cdot\vec{V}.$$

As an exercise it can be shown that the dissipation function can also be represented as $\Phi = 2\mu^* D_{ij}D_{ij} + \lambda^*\Theta^2$ where Θ is the dilatation. The heat flow vector is determined from the Fourier law of heat conduction in terms of the temperature T as $\vec{q} = -\kappa\nabla T$, where κ is the thermal conductivity. Consequently, the energy equation can be written as

$$\varrho\frac{De}{Dt} + p(\nabla\cdot\vec{V}) = \frac{\partial Q}{\partial t} + \Phi + \nabla(k\nabla T). \tag{2.5.49}$$

In Cartesian coordinates (x, y, z) we use

$$\begin{aligned}
\frac{D}{Dt} =& \frac{\partial}{\partial t} + V_x\frac{\partial}{\partial x} + V_y\frac{\partial}{\partial y} + V_z\frac{\partial}{\partial z}\\
\nabla\cdot\vec{V} =& \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\\
\nabla\cdot(\kappa\nabla T) =& \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right)
\end{aligned}$$

In cylindrical coordinates (r, θ, z)

$$\begin{aligned}
\frac{D}{Dt} =& \frac{\partial}{\partial t} + V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + V_z\frac{\partial}{\partial z}\\
\nabla\cdot\vec{V} =& \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r^2}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}\\
\nabla\cdot(\kappa\nabla T) =& \frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\kappa\frac{\partial T}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right)
\end{aligned}$$

and in spherical coordinates (ρ, θ, ϕ)

$$\begin{aligned}
\frac{D}{Dt} =& \frac{\partial}{\partial t} + V_\rho\frac{\partial}{\partial\rho} + \frac{V_\theta}{\rho}\frac{\partial}{\partial\theta}\frac{V_\phi}{\rho\sin\theta}\frac{\partial}{\partial\phi}\\
\nabla\cdot\vec{V} =& \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{\rho\sin\theta}\frac{\partial V_\phi}{\partial\phi}\\
\nabla\cdot(\kappa\nabla T) =& \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\kappa\frac{\partial T}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\kappa\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\kappa\frac{\partial T}{\partial\phi}\right)
\end{aligned}$$

The combination of terms $h = e + p/\varrho$ is known as enthalpy and at times is used to express the energy equation in the form

$$\varrho\frac{D\,h}{Dt} = \frac{D\,p}{Dt} + \frac{\partial Q}{\partial t} - \nabla\cdot\vec{q} + \Phi.$$

The derivation of this equation is left as an exercise.

Conservative Systems

Let Q denote some physical quantity per unit volume. Here Q can be either a scalar, vector or tensor field. Place within this field an imaginary simple closed surface S which encloses a volume V. The total amount of Q within the surface is given by $\iiint_V Q\,d\tau$ and the rate of change of this amount with respect to time is $\frac{\partial}{\partial t}\iiint Q\,d\tau$. The total amount of Q within S changes due to sources (or sinks) within the volume and by transport processes. Transport processes introduce a quantity $\vec{J}$, called current, which represents a flow per unit area across the surface S. The inward flux of material into the volume is denoted $\iint_S -\vec{J}\cdot\hat{n}\,d\sigma$ ($\hat{n}$ is a unit outward normal.) The sources (or sinks) S_Q denotes a generation (or loss) of material per unit volume so that $\iiint_V S_Q\,d\tau$ denotes addition (or loss) of material to the volume. For a fixed volume we then have the material balance

$$\iiint_V \frac{\partial Q}{\partial t}\,d\tau = -\iint_S \vec{J}\cdot\hat{n}\,d\sigma + \iiint_V S_Q\,d\tau.$$

Using the divergence theorem of Gauss one can derive the general conservation law

$$\frac{\partial Q}{\partial t} + \nabla \cdot \vec{J} = S_Q \tag{2.5.50}$$

The continuity equation and energy equations are examples of a scalar conservation law in the special case where $S_Q = 0$. In Cartesian coordinates, we can represent the continuity equation by letting

$$Q = \varrho \quad \text{and} \quad \vec{J} = \varrho \vec{V} = \varrho(V_x\,\hat{e}_1 + V_y\,\hat{e}_2 + V_z\,\hat{e}_3) \tag{2.5.51}$$

The energy equation conservation law is represented by selecting $Q = e_t$ and neglecting the rate of internal heat energy we let

$$\begin{aligned}\vec{J} = &\left[(e_t + p)v_1 - \sum_{i=1}^{3} v_i \tau_{xi} + q_x\right]\hat{e}_1 + \\ &\left[(e_t + p)v_2 - \sum_{i=1}^{3} v_i \tau_{yi} + q_y\right]\hat{e}_2 + \\ &\left[(e_t + p)v_3 - \sum_{i=1}^{3} v_i \tau_{zi} + q_z\right]\hat{e}_3.\end{aligned} \tag{2.5.52}$$

In a general orthogonal system of coordinates (x_1, x_2, x_3) the equation (2.5.50) is written

$$\frac{\partial}{\partial t}\left((h_1h_2h_3Q)\right) + \frac{\partial}{\partial x_1}\left((h_2h_3J_1)\right) + \frac{\partial}{\partial x_2}\left((h_1h_3J_2)\right) + \frac{\partial}{\partial x_3}\left((h_1h_2J_3)\right) = 0,$$

where h_1, h_2, h_3 are scale factors obtained from the transformation equations to the general orthogonal coordinates.

The momentum equations are examples of a vector conservation law having the form

$$\frac{\partial \vec{a}}{\partial t} + \nabla \cdot (\boldsymbol{T}) = \varrho \vec{b} \tag{2.5.53}$$

where $\vec{a}$ is a vector and $\boldsymbol{T}$ is a second order symmetric tensor $\boldsymbol{T} = \sum_{k=1}^{3}\sum_{j=1}^{3} T_{jk}\,\hat{e}_j\,\hat{e}_k$. In Cartesian coordinates we let $\vec{a} = \varrho(V_x\,\hat{e}_1 + V_y\,\hat{e}_2 + V_z\,\hat{e}_3)$ and $T_{ij} = \varrho v_i v_j + p\delta_{ij} - \tau_{ij}$. In general coordinates (x_1, x_2, x_3) the momentum equations result by selecting $\vec{a} = \varrho\vec{V}$ and $T_{ij} = \varrho v_i v_j + p\delta_{ij} - \tau_{ij}$. In a general orthogonal system the conservation law (2.5.53) has the general form

$$\frac{\partial}{\partial t}\left((h_1h_2h_3\vec{a})\right) + \frac{\partial}{\partial x_1}\Big((h_2h_3\boldsymbol{T}\cdot\hat{e}_1)\Big) + \frac{\partial}{\partial x_2}\Big((h_1h_3\boldsymbol{T}\cdot\hat{e}_2)\Big) + \frac{\partial}{\partial x_3}\Big((h_1h_2\boldsymbol{T}\cdot\hat{e}_3)\Big) = \varrho\vec{b}. \tag{2.5.54}$$

Neglecting body forces and internal heat production, the continuity, momentum and energy equations can be expressed in the strong conservative form

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0 \tag{2.5.55}$$

where

$$U = \begin{bmatrix} \rho \\ \rho V_x \\ \rho V_y \\ \rho V_z \\ e_t \end{bmatrix} \tag{2.5.56}$$

$$E = \begin{bmatrix} \rho V_x \\ \rho V_x^2 + p - \tau_{xx} \\ \rho V_x V_y - \tau_{xy} \\ \rho V_x V_z - \tau_{xz} \\ (e_t + p)V_x - V_x\tau_{xx} - V_y\tau_{xy} - V_z\tau_{xz} + q_x \end{bmatrix} \tag{2.5.57}$$

$$F = \begin{bmatrix} \rho V_y \\ \rho V_x V_y - \tau_{xy} \\ \rho V_y^2 + p - \tau_{yy} \\ \rho V_y V_z - \tau_{yz} \\ (e_t + p)V_y - V_x\tau_{yx} - V_y\tau_{yy} - V_z\tau_{yz} + q_y \end{bmatrix} \tag{2.5.58}$$

$$G = \begin{bmatrix} \rho V_z \\ \rho V_x V_z - \tau_{xz} \\ \rho V_y V_z - \tau_{yz} \\ \rho V_z^2 + p - \tau_{zz} \\ (e_t + p)V_z - V_x\tau_{zx} - V_y\tau_{zy} - V_z\tau_{zz} + q_z \end{bmatrix} \tag{2.5.59}$$

where the shear stresses are $\tau_{ij} = \mu^*(V_{i,j} + V_{j,i}) + \delta_{ij}\lambda^* V_{k,k}$ for $i, j, k = 1, 2, 3$.

Computational Coordinates

To transform the conservative system (2.5.55) from a physical (x, y, z) domain to a computational (ξ, η, ζ) domain requires that a general change of variables take place. Consider the following general transformation of the independent variables

$$\xi = \xi(x, y, z) \qquad \eta = \eta(x, y, z) \qquad \zeta = \zeta(x, y, z) \tag{2.5.60}$$

with Jacobian different from zero. The chain rule for changing variables in equation (2.5.55) requires the operators

$$\begin{aligned} \frac{\partial(\ )}{\partial x} =& \frac{\partial(\ )}{\partial \xi}\xi_x + \frac{\partial(\ )}{\partial \eta}\eta_x + \frac{\partial(\ )}{\partial \zeta}\zeta_x \\ \frac{\partial(\ )}{\partial y} =& \frac{\partial(\ )}{\partial \xi}\xi_y + \frac{\partial(\ )}{\partial \eta}\eta_y + \frac{\partial(\ )}{\partial \zeta}\zeta_y \\ \frac{\partial(\ )}{\partial z} =& \frac{\partial(\ )}{\partial \xi}\xi_z + \frac{\partial(\ )}{\partial \eta}\eta_z + \frac{\partial(\ )}{\partial \zeta}\zeta_z \end{aligned} \tag{2.5.61}$$

The partial derivatives in these equations occur in the differential expressions

$$\begin{aligned} d\xi =& \xi_x\,dx + \xi_y\,dy + \xi_z\,dz \\ d\eta =& \eta_x\,dx + \eta_y\,dy + \eta_z\,dz \\ d\zeta =& \zeta_x\,dx + \zeta_y\,dy + \zeta_z\,dz \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{2.5.62}$$

In a similar mannaer from the inverse transformation equations

$$x = x(\xi, \eta, \zeta) \qquad y = y(\xi, \eta, \zeta) \qquad z = z(\xi, \eta, \zeta) \tag{2.5.63}$$

we can write the differentials

$$\begin{aligned} dx =& x_\xi\,d\xi + x_\eta\,d\eta + x_\zeta\,d\zeta \\ dy =& y_\xi\,d\xi + y_\eta\,d\eta + y_\zeta\,d\zeta \\ dz =& z_\xi\,d\xi + z_\zeta\,d\zeta + z_\zeta\,d\zeta \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} \tag{2.5.64}$$

The transformations (2.5.62) and (2.5.64) are inverses of each other and so we can write

$$\begin{aligned}\begin{bmatrix}\xi_x & \xi_y & \xi_z\\ \eta_x & \eta_y & \eta_z\\ \zeta_x & \zeta_y & \zeta_z\end{bmatrix} =& \begin{bmatrix}x_\xi & x_\eta & x_\zeta\\ y_\xi & y_\eta & y_\zeta\\ z_\xi & z_\eta & z_\zeta\end{bmatrix}^{-1}\\ =&J\begin{bmatrix}y_\eta z_\zeta - y_\zeta z_\eta & -(x_\eta z_\zeta - x_\zeta z_\eta) & x_\eta y_\zeta - x_\zeta y_\eta\\ -(y_\xi z_\zeta - y_\zeta z_\xi) & x_\xi z_\zeta - x_\zeta z_\xi & -(x_\xi y_\zeta - x_\zeta y_\xi)\\ y_\xi z_\eta - y_\eta z_\xi & -(x_\xi z_\eta - x_\eta z_\xi) & x_\xi y_\eta - x_\eta y_\xi\end{bmatrix}\end{aligned} \tag{2.5.65}$$

By comparing like elements in equation (2.5.65) we obtain the relations

$$\begin{aligned} \xi_x =&J(y_\eta z_\zeta - y_\zeta z_\eta) \qquad & \eta_x =&-J(y_\xi z_\zeta - y_\zeta z_\xi) \qquad & \zeta_x =&J(y_\xi z_\eta - y_\eta z_\xi)\\ \xi_y =&-J(x_\eta z_\zeta - x_\zeta z_\eta) & \eta_y =&J(x_\xi z_\zeta - z_\zeta z_\xi) & \zeta_y =&-J(x_\xi z_\eta - x_\eta z_\xi)\\ \xi_z =&J(x_\eta y_\zeta - x_\zeta y_\eta) & \eta_z =&-J(x_\xi y_\zeta - x_\zeta y_\xi) & \zeta_z =&J(x_\xi y_\eta - x_\eta y_\xi) \end{aligned} \tag{2.5.66}$$

The equations (2.5.55) can now be written in terms of the new variables (ξ, η, ζ) as

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial \xi}\xi_x + \frac{\partial E}{\partial \eta}\eta_x + \frac{\partial E}{\partial \zeta}\zeta_x + \frac{\partial F}{\partial \xi}\xi_y + \frac{\partial F}{\partial \eta}\eta_y + \frac{\partial F}{\partial \zeta}\zeta_y + \frac{\partial G}{\partial \xi}\xi_z + \frac{\partial G}{\partial \eta}\eta_z + \frac{\partial G}{\partial \zeta}\zeta_z = 0 \tag{2.5.67}$$

Now divide each term by the Jacobian J and write the equation (2.5.67) in the form

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{U}{J}\right) + &\frac{\partial}{\partial \xi}\left(\frac{E\xi_x + F\xi_y + G\xi_z}{J}\right)\\ &+\frac{\partial}{\partial \eta}\left(\frac{E\eta_x + F\eta_y + G\eta_z}{J}\right)\\ &+\frac{\partial}{\partial \zeta}\left(\frac{E\zeta_x + F\zeta_y + G\zeta_z}{J}\right)\\ &-E\left\{\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\zeta_x}{J}\right)\right\}\\ &-F\left\{\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\zeta_y}{J}\right)\right\}\\ &-G\left\{\frac{\partial}{\partial \xi}\left(\frac{\xi_z}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_z}{J}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\zeta_z}{J}\right)\right\} = 0 \end{aligned} \tag{2.5.68}$$

Using the relations given in equation (2.5.66) one can show that the curly bracketed terms above are all zero and so the transformed equations (2.5.55) can also be written in the conservative form

$$\frac{\partial \widehat{U}}{\partial t} + \frac{\partial \widehat{E}}{\partial \xi} + \frac{\partial \widehat{F}}{\partial \eta} + \frac{\partial \widehat{G}}{\partial \zeta} = 0 \tag{2.5.69}$$

where

$$\begin{aligned} \widehat{U} =&\frac{U}{J}\\ \widehat{E} =&\frac{E\xi_x + F\xi_y + G\xi z}{J}\\ \widehat{F} =&\frac{E\eta_x + F\eta_y + G\eta_z}{J}\\ \widehat{G} =&\frac{E\zeta_x + F\zeta_y + G\zeta_z}{J} \end{aligned} \tag{2.5.70}$$

Fourier law of heat conduction

The Fourier law of heat conduction can be written $q_i = -\kappa T_{,i}$ for isotropic material and $q_i = -\kappa_{ij} T_{,j}$ for anisotropic material. The Prandtl number is a nondimensional constant defined as $Pr = \frac{c_p \mu^*}{\kappa}$ so that the heat flow terms can be represented in Cartesian coordinates as

$$q_x = -\frac{c_p \mu^*}{Pr}\frac{\partial T}{\partial x} \qquad q_y = -\frac{c_p \mu^*}{Pr}\frac{\partial T}{\partial y} \qquad q_z = -\frac{c_p \mu^*}{Pr}\frac{\partial T}{\partial z}$$

Now one can employ the equation of state relations $P = \varrho e(\gamma - 1)$, $c_p = \frac{\gamma R}{\gamma - 1}$, $c_p T = \frac{\gamma RT}{\gamma - 1}$ and write the above equations in the alternate forms

$$q_x = -\frac{\mu^*}{Pr(\gamma-1)}\frac{\partial}{\partial x}\left(\frac{\gamma P}{\varrho}\right) \qquad q_y = -\frac{\mu^*}{Pr(\gamma-1)}\frac{\partial}{\partial y}\left(\frac{\gamma P}{\varrho}\right) \qquad q_z = -\frac{\mu^*}{Pr(\gamma-1)}\frac{\partial}{\partial z}\left(\frac{\gamma P}{\varrho}\right)$$

The speed of sound is given by $a = \sqrt{\dfrac{\gamma P}{\varrho}} = \sqrt{\gamma RT}$ and so one can substitute a^2 in place of the ratio $\dfrac{\gamma P}{\varrho}$ in the above equations.

Equilibrium and Nonequilibrium Thermodynamics

High temperature gas flows require special considerations. In particular, the specific heat for monotonic and diatomic gases are different and are in general a function of temperature. The energy of a gas can be written as $e = e_t + e_r + e_v + e_e + e_n$ where e_t represents translational energy, e_r is rotational energy, e_v is vibrational energy, e_e is electronic energy, and e_n is nuclear energy. The gases follow a Boltzmann distribution for each degree of freedom and consequently at very high temperatures the rotational, translational and vibrational degrees of freedom can each have their own temperature. Under these conditions the gas is said to be in a state of nonequilibrium. In such a situation one needs additional energy equations. The energy equation developed in these notes is for equilibrium thermodynamics where the rotational, translational and vibrational temperatures are the same.

Equation of state

It is assumed that an equation of state such as the universal gas law or perfect gas law $pV = nRT$ holds which relates pressure $p\,[N/m^2]$, volume $V\,[m^3]$, amount of gas $n\,[mol]$,and temperature $T\,[K]$ where $R\,[J/mol-K]$ is the universal molar gas constant. If the ideal gas law is represented in the form $p = \varrho RT$ where $\varrho\,[Kg/m^3]$ is the gas density, then the universal gas constant must be expressed in units of $[J/Kg-K]$ (See Appendix A). Many gases deviate from this ideal behavior. In order to account for the intermolecular forces associated with high density gases, an empirical equation of state of the form

$$p = \rho RT + \sum_{n=1}^{M_1} \beta_n \rho^{n+r_1} + e^{-\gamma_1\rho - \gamma_2\rho^2}\sum_{n=1}^{M_2} c_n \rho^{n+r_2}$$

involving constants $M_1, M_2, \beta_n, c_n, r_1, r_2, \gamma_1, \gamma_2$ is often used. For a perfect gas the relations

$$e = c_v T \quad \gamma = \frac{c_p}{c_v} \quad c_v = \frac{R}{\gamma - 1} \quad c_p = \frac{\gamma R}{\gamma - 1} \quad h = c_p T$$

hold, where R is the universal gas constant, c_v is the specific heat at constant volume, c_p is the specific heat at constant pressure, γ is the ratio of specific heats and h is the enthalpy. For c_v and c_p constants the relations $p = (\gamma - 1)\varrho e$ and $RT = (\gamma - 1)e$ can be verified.

EXAMPLE 2.5-1. (One-dimensional fluid flow)

Construct an x-axis running along the center line of a long cylinder with cross sectional area A. Consider the motion of a gas driven by a piston and moving with velocity $v_1 = u$ in the x-direction. From an Eulerian point of view we imagine a control volume fixed within the cylinder and assume zero body forces. We require the following equations be satisfied.

Conservation of mass $\frac{\partial \varrho}{\partial t} + \text{div}(\varrho \vec{V}) = 0$ which in one-dimension reduces to $\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho u) = 0.$

Conservation of momentum, equation (2.5.28) reduces to $\frac{\partial}{\partial t}(\varrho u) + \frac{\partial}{\partial x}(\varrho u^2) + \frac{\partial p}{\partial x} = 0.$

Conservation of energy, equation (2.5.48) in the absence of heat flow and internal heat production, becomes in one dimension $\varrho\left(\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x}\right) + p\frac{\partial u}{\partial x} = 0$. Using the conservation of mass relation this equation can be written in the form $\frac{\partial}{\partial t}(\varrho e) + \frac{\partial}{\partial x}(\varrho e u) + p\frac{\partial u}{\partial x} = 0.$

In contrast, from a Lagrangian point of view we let the control volume move with the flow and consider advection terms. This gives the following three equations which can then be compared with the above Eulerian equations of motion.

Conservation of mass $\frac{d}{dt}(\varrho J) = 0$ which in one-dimension is equivalent to $\frac{D\varrho}{Dt} + \varrho\frac{\partial u}{\partial x} = 0.$

Conservation of momentum, equation (2.5.25) in one-dimension $\varrho\frac{Du}{Dt} + \frac{\partial p}{\partial x} = 0.$

Conservation of energy, equation (2.5.48) in one-dimension $\varrho\frac{De}{Dt} + p\frac{\partial u}{\partial x} = 0$. In the above equations $\frac{D(\,)}{Dt} = \frac{\partial}{\partial t}(\,) + u\frac{\partial}{\partial x}(\,)$. The Lagrangian viewpoint gives three equations in the three unknowns ρ, u, e.

In both the Eulerian and Lagrangian equations the pressure p represents the total pressure $p = p_g + p_v$ where p_g is the gas pressure and p_v is the viscous pressure which causes loss of kinetic energy. The gas pressure is a function of ϱ, e and is determined from the ideal gas law $p_g = \varrho RT = \varrho(c_p - c_v)T = \varrho(\frac{c_p}{c_v} - 1)c_v T$ or $p_g = \varrho(\gamma - 1)e$. Some kind of assumption is usually made to represent the viscous pressure p_v as a function of e, u. The above equations are then subjected to boundary and initial conditions and are usually solved numerically.

■

Entropy inequality

Energy transfer is not always reversible. Many energy transfer processes are irreversible. The second law of thermodynamics allows energy transfer to be reversible only in special circumstances. In general, the second law of thermodynamics can be written as an entropy inequality, known as the Clausius-Duhem inequality. This inequality states that the time rate of change of the total entropy is greater than or equal to the total entropy change occurring across the surface and within the body of a control volume. The Clausius-Duhem inequality places restrictions on the constitutive equations. This inequality can be expressed in the form

$$\underbrace{\frac{D}{Dt}\int_{\mathcal{V}} \varrho s\, d\tau}_{\text{Rate of entropy increase}} \geq \underbrace{\int_{\mathcal{S}} s^i n_i\, dS + \int_{\mathcal{V}} \rho b\, d\tau + \sum_{\alpha=1}^{n} B_{(\alpha)}}_{\text{Entropy input rate into control volume}}$$

where s is the specific entropy density, s^i is an entropy flux, b is an entropy source and $B_{(\alpha)}$ are isolated entropy sources. Irreversible processes are characterized by the use of the inequality sign while for reversible

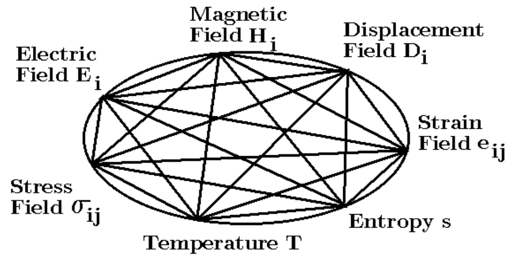

Figure 2.5-3. Interaction of various fields.

processes the equality sign holds. The Clausius-Duhem inequality is assumed to hold for all independent thermodynamical processes.

If in addition there are electric and magnetic fields to consider, then these fields place additional forces upon the material continuum and we must add all forces and moments due to these effects. In particular we must add the following equations

Gauss's law for magnetism $\qquad \nabla \cdot \vec{B} = 0 \qquad \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial x^i}(\sqrt{g}B^i) = 0.$

Gauss's law for electricity $\qquad \nabla \cdot \vec{D} = \varrho_e \qquad \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial x^i}(\sqrt{g}D^i) = \varrho_e.$

Faraday's law $\qquad \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \qquad \epsilon^{ijk}E_{k,j} = -\dfrac{\partial B^i}{\partial t}.$

Ampere's law $\qquad \nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \qquad \epsilon^{ijk}H_{k,j} = J^i + \dfrac{\partial D^i}{\partial t}.$

where ϱ_e is the charge density, J^i is the current density, $D_i = \epsilon_i^j E_j + P_i$ is the electric displacement vector, H_i is the magnetic field, $B_i = \mu_i^j H_j + M_i$ is the magnetic induction, E_i is the electric field, M_i is the magnetization vector and P_i is the polarization vector. Taking the divergence of Ampere's law produces the law of conservation of charge which requires that

$$\frac{\partial \varrho_e}{\partial t} + \nabla \cdot \vec{J} = 0 \qquad \frac{\partial \varrho_e}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}J^i) = 0.$$

The figure 2.5-3 is constructed to suggest some of the interactions that can occur between various variables which define the continuum. Pyroelectric effects occur when a change in temperature causes changes in the electrical properties of a material. Temperature changes can also change the mechanical properties of materials. Similarly, piezoelectric effects occur when a change in either stress or strain causes changes in the electrical properties of materials. Photoelectric effects are said to occur if changes in electric or mechanical properties effect the refractive index of a material. Such changes can be studied by modifying the constitutive equations to include the effects being considered.

From figure 2.5-3 we see that there can exist a relationship between the displacement field D_i and electric field E_i. When this relationship is linear we can write $D_i = \epsilon_{ji}E_j$ and $E_j = \beta_{jn}D_n$, where ϵ_{ji} are dielectric constants and β_{jn} are dielectric impermabilities. Similarly, when linear piezoelectric effects exist we can write linear relations between stress and electric fields such as $\sigma_{ij} = -g_{kij}E_k$ and $E_i = -e_{ijk}\sigma_{jk}$, where g_{kij} and e_{ijk} are called piezoelectric constants. If there is a linear relation between strain and an electric fields, this is another type of piezoelectric effect whereby $e_{ij} = d_{ijk}E_k$ and $E_k = -h_{ijk}e_{jk}$, where d_{ijk} and h_{ijk} are another set of piezoelectric constants. Similarly, entropy changes can cause pyroelectric effects. Piezooptical effects (photoelasticity) occurs when mechanical stresses change the optical properties of the material. Electrical and heat effects can also change the optical properties of materials. Piezoresistivity occurs when mechanical stresses change the electric resistivity of materials. Electric field changes can cause variations in temperature, another pyroelectric effect. When temperature effects the entropy of a material this is known as a heat capacity effect. When stresses effect the entropy in a material this is called a piezocaloric effect. Some examples of the representation of these additional effects are as follows. The piezoelectric effects are represented by equations of the form

$$\sigma_{ij} = -h_{mij}D_m \quad D_i = d_{ijk}\sigma_{jk} \quad e_{ij} = g_{kij}D_k \quad D_i = e_{ijk}e_{jk}$$

where h_{mij}, d_{ijk}, g_{kij} and e_{ijk} are piezoelectric constants.

Knowledge of the material or electric interaction can be used to help modify the constitutive equations. For example, the constitutive equations can be modified to included temperature effects by expressing the constitutive equations in the form

$$\sigma_{ij} = c_{ijkl}e_{kl} - \beta_{ij}\Delta T \qquad \text{and} \qquad e_{ij} = s_{ijkl}\sigma_{kl} + \alpha_{ij}\Delta T$$

where for isotropic materials the coefficients α_{ij} and β_{ij} are constants. As another example, if the strain is modified by both temperature and an electric field, then the constitutive equations would take on the form

$$e_{ij} = s_{ijkl}\sigma_{kl} + \alpha_{ij}\Delta T + d_{mij}E_m.$$

Note that these additional effects are additive under conditions of small changes. That is, we may use the principal of superposition to calculate these additive effects.

If the electric field and electric displacement are replaced by a magnetic field and magnetic flux, then piezomagnetic relations can be found to exist between the variables involved. One should consult a handbook to determine the order of magnitude of the various piezoelectric and piezomagnetic effects. For a large majority of materials these effects are small and can be neglected when the field strengths are weak.

The Boltzmann Transport Equation

The modeling of the transport of particle beams through matter, such as the motion of energetic protons or neutrons through bulk material, can be approached using ideas from the classical kinetic theory of gases. Kinetic theory is widely used to explain phenomena in such areas as: statistical mechanics, fluids, plasma physics, biological response to high-energy radiation, high-energy ion transport and various types of radiation shielding. The problem is basically one of describing the behavior of a system of interacting particles and their distribution in space, time and energy. The average particle behavior can be described by the Boltzmann equation which is essentially a continuity equation in a six-dimensional phase space (x, y, z, V_x, V_y, V_z). We will be interested in examining how the particles in a volume element of phase space change with time. We introduce the following notation:

(i) $\vec{r}$ the position vector of a typical particle of phase space and $d\tau = dxdydz$ the corresponding spatial volume element at this position.

(ii) $\vec{V}$ the velocity vector associated with a typical particle of phase space and $d\tau_v = dV_x dV_y dV_z$ the corresponding velocity volume element.

(iii) $\vec{\Omega}$ a unit vector in the direction of the velocity $\vec{V} = v\vec{\Omega}$.

(iv) $E = \frac{1}{2}mv^2$ kinetic energy of particle.

(v) $d\vec{\Omega}$ is a solid angle about the direction $\vec{\Omega}$ and $d\tau\, dE\, d\vec{\Omega}$ is a volume element of phase space involving the solid angle about the direction Ω.

(vi) $n = n(\vec{r}, E, \vec{\Omega}, t)$ the number of particles in phase space per unit volume at position $\vec{r}$ per unit velocity at position $\vec{V}$ per unit energy in the solid angle $d\vec{\Omega}$ at time t and $N = N(\vec{r}, E, \vec{\Omega}, t) = vn(\vec{r}, E, \vec{\Omega}, t)$ the number of particles per unit volume per unit energy in the solid angle $d\vec{\Omega}$ at time t. The quantity $N(\vec{r}, E, \vec{\Omega}, t)d\tau\, dE\, d\vec{\Omega}$ represents the number of particles in a volume element around the position $\vec{r}$ with energy between E and $E + dE$ having direction $\vec{\Omega}$ in the solid angle $d\vec{\Omega}$ at time t.

(vii) $\phi(\vec{r}, E, \vec{\Omega}, t) = vN(\vec{r}, E, \vec{\Omega}, t)$ is the particle flux (number of particles/cm^2 − Mev − sec).

(viii) $\Sigma(E' \to E, \vec{\Omega}' \to \vec{\Omega})$ a scattering cross-section which represents the fraction of particles with energy E' and direction $\vec{\Omega}'$ that scatter into the energy range between E and $E + dE$ having direction $\vec{\Omega}$ in the solid angle $d\vec{\Omega}$ per particle flux.

(ix) $\Sigma_s(E, \vec{r})$ fractional number of particles scattered out of volume element of phase space per unit volume per flux.

(x) $\Sigma_a(E, \vec{r})$ fractional number of particles absorbed in a unit volume of phase space per unit volume per flux.

Consider a particle at time t having a position $\vec{r}$ in phase space as illustrated in the figure 2.5-4. This particle has a velocity $\vec{V}$ in a direction $\vec{\Omega}$ and has an energy E. In terms of $d\tau = dx\, dy\, dz$, $\vec{\Omega}$ and E an element of volume of phase space can be denoted $d\tau dE d\vec{\Omega}$, where $d\vec{\Omega} = d\vec{\Omega}(\theta, \psi) = \sin\theta d\theta d\psi$ is a solid angle about the direction $\vec{\Omega}$.

The Boltzmann transport equation represents the rate of change of particle density in a volume element $d\tau\, dE\, d\vec{\Omega}$ of phase space and is written

$$\frac{d}{dt} N(\vec{r}, E, \vec{\Omega}, t)\, d\tau\, dE\, d\vec{\Omega} = D_C N(\vec{r}, E, \vec{\Omega}, t) \tag{2.5.71}$$

where D_C is a collision operator representing gains and losses of particles to the volume element of phase space due to scattering and absorption processes. The gains to the volume element are due to any sources $S(\vec{r}, E, \vec{\Omega}, t)$ per unit volume of phase space, with units of number of particles/sec per volume of phase space, together with any scattering of particles into the volume element of phase space. That is particles entering the volume element of phase space with energy E, which experience a collision, leave with some energy $E - \Delta E$ and thus will be lost from our volume element. Particles entering with energies $E' > E$ may,

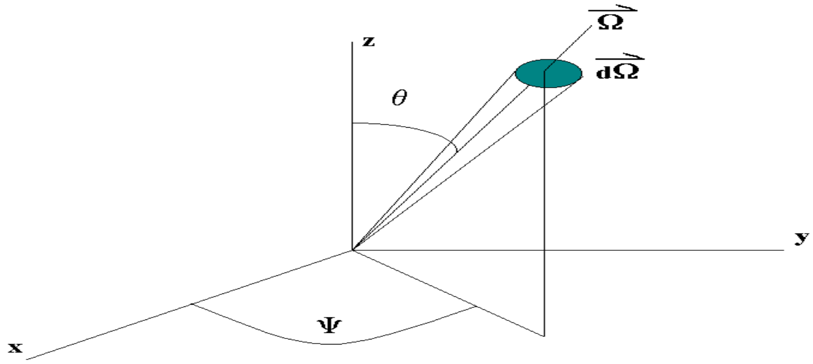

Figure 2.5-4. Volume element and solid angle about position $\vec{r}$.

depending upon the cross-sections, exit with energy $E' - \Delta E = E$ and thus will contribute a gain to the volume element. In terms of the flux ϕ the gains due to scattering into the volume element are denoted by

$$\int d\vec{\Omega}' \int dE' \Sigma(E' \to E, \vec{\Omega}' \to \vec{\Omega}) \phi(\vec{r}, E', \vec{\Omega}, t)\, d\tau\, dE\, d\vec{\Omega}$$

and represents the particles at position $\vec{r}$ experiencing a scattering collision with a particle of energy E' and direction $\vec{\Omega}'$ which causes the particle to end up with energy between E and $E + dE$ and direction $\vec{\Omega}$ in $d\vec{\Omega}$. The summations are over all possible initial energies.

In terms of ϕ the losses are due to those particles leaving the volume element because of scattering and are

$$\Sigma_s(E, \vec{r}) \phi(\vec{r}, E, \vec{\Omega}, t) d\tau\, dE\, d\vec{\Omega}.$$

The particles which are lost due to absorption processes are

$$\Sigma_a(E, \vec{r}) \phi(\vec{r}, E, \vec{\Omega}, t)\, d\tau\, dE\, d\vec{\Omega}.$$

The total change to the number of particles in an element of phase space per unit of time is obtained by summing all gains and losses. This total change is

$$\begin{aligned}\frac{dN}{dt}\, d\tau\, dE\, d\Omega = & \int d\vec{\Omega}' \int dE'\, \Sigma(E' \to E, \vec{\Omega}' \to \vec{\Omega}) \phi(\vec{r}, E', \vec{\Omega}, t)\, d\tau\, dE\, d\vec{\Omega} \\ & - \Sigma_s(E, \vec{r}) \phi(\vec{r}, E, \vec{\Omega}, t) d\tau\, dE\, d\Omega \\ & - \Sigma_a(E, \vec{r}) \phi(\vec{r}, E, \vec{\Omega}, t)\, d\tau\, dE\, d\vec{\Omega} \\ & + S(\vec{r}, E, \vec{\Omega}, t) d\tau\, dE\, d\vec{\Omega}.\end{aligned} \tag{2.5.72}$$

The rate of change $\frac{dN}{dt}$ on the left-hand side of equation (2.5.72) expands to

$$\begin{aligned}\frac{dN}{dt} = & \frac{\partial N}{\partial t} + \frac{\partial N}{\partial x}\frac{dx}{dt} + \frac{\partial N}{\partial y}\frac{dy}{dt} + \frac{\partial N}{\partial z}\frac{dz}{dt} \\ & + \frac{\partial N}{\partial V_x}\frac{dV_x}{dt} + \frac{\partial N}{\partial V_y}\frac{dV_y}{dt} + \frac{\partial N}{\partial V_z}\frac{dV_z}{dt}\end{aligned}$$

which can be written as

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \vec{V} \cdot \nabla_{\vec{r}} N + \frac{\vec{F}}{m} \cdot \nabla_{\vec{V}} N \tag{2.5.73}$$

where $\frac{d\vec{V}}{dt} = \frac{\vec{F}}{m}$ represents any forces acting upon the particles. The Boltzmann equation can then be expressed as

$$\frac{\partial N}{\partial t} + \vec{V} \cdot \nabla_{\vec{r}} N + \frac{\vec{F}}{m} \cdot \nabla_{\vec{V}} N = \text{Gains} - \text{Losses}. \tag{2.5.74}$$

If the right-hand side of the equation (2.5.74) is zero, the equation is known as the Liouville equation. In the special case where the velocities are constant and do not change with time the above equation (2.5.74) can be written in terms of the flux ϕ and has the form

$$\left[\frac{1}{v}\frac{\partial}{\partial t} + \vec{\Omega} \cdot \nabla_{\vec{r}} + \Sigma_s(E, \vec{r}) + \Sigma_a(E, \vec{r})\right] \phi(\vec{r}, E, \vec{\Omega}, t) = D_C \phi \tag{2.5.75}$$

where

$$D_C \phi = \int d\vec{\Omega}' \int dE' \Sigma(E' \to E, \vec{\Omega}' \to \vec{\Omega}) \phi(\vec{r}, E', \vec{\Omega}', t) + S(\vec{r}, E, \vec{\Omega}, t).$$

The above equation represents the Boltzmann transport equation in the case where all the particles are the same. In the case of atomic collisions of particles one must take into consideration the generation of secondary particles resulting from the collisions.

Let there be a number of particles of type j in a volume element of phase space. For example $j = p$ (protons) and $j = n$ (neutrons). We consider steady state conditions and define the quantities

(i) $\phi_j(\vec{r}, E, \vec{\Omega})$ as the flux of the particles of type j.

(ii) $\sigma_{jk}(\vec{\Omega}, \vec{\Omega}', E, E')$ the collision cross-section representing processes where particles of type k moving in direction $\vec{\Omega}'$ with energy E' produce a type j particle moving in the direction $\vec{\Omega}$ with energy E.

(iii) $\sigma_j(E) = \Sigma_s(E, \vec{r}) + \Sigma_a(E, \vec{r})$ the cross-section for type j particles.

The steady state form of the equation (2.5.64) can then be written as

$$\begin{aligned} \vec{\Omega} \cdot \nabla \phi_j(\vec{r}, E, \vec{\Omega}) + &\sigma_j(E) \phi_j(\vec{r}, E, \vec{\Omega}) \\ &= \sum_k \int \sigma_{jk}(\vec{\Omega}, \vec{\Omega}', E, E') \phi_k(\vec{r}, E', \vec{\Omega}') d\vec{\Omega}' dE' \end{aligned} \tag{2.5.76}$$

where the summation is over all particles $k \neq j$.

The Boltzmann transport equation can be represented in many different forms. These various forms are dependent upon the assumptions made during the derivation, the type of particles, and collision cross-sections. In general the collision cross-sections are dependent upon three components.

(1) *Elastic collisions.* Here the nucleus is not excited by the collision but energy is transferred by projectile recoil.

(2) *Inelastic collisions.* Here some particles are raised to a higher energy state but the excitation energy is not sufficient to produce any particle emissions due to the collision.

(3) *Non-elastic collisions.* Here the nucleus is left in an excited state due to the collision processes and some of its nucleons (protons or neutrons) are ejected. The remaining nucleons interact to form a stable structure and usually produce a distribution of low energy particles which is isotropic in character.

Various assumptions can be made concerning the particle flux. The resulting form of Boltzmann's equation must be modified to reflect these additional assumptions. As an example, we consider modifications to Boltzmann's equation in order to describe the motion of a massive ion moving into a region filled with a homogeneous material. Here it is assumed that the mean-free path for nuclear collisions is large in comparison with the mean-free path for ion interaction with electrons. In addition, the following assumptions are made

(i) All collision interactions are non-elastic.

(ii) The secondary particles produced have the same direction as the original particle. This is called the straight-ahead approximation.

(iii) Secondary particles never have kinetic energies greater than the original projectile that produced them.

(iv) A charged particle will eventually transfer all of its kinetic energy and stop in the media. This stopping distance is called the range of the projectile. The stopping power $S_j(E) = \frac{dE}{dx}$ represents the energy loss per unit length traveled in the media and determines the range by the relation $\frac{dR_j}{dE} = \frac{1}{S_j(E)}$ or $R_j(E) = \int_0^E \frac{dE'}{S_j(E')}$. Using the above assumptions Wilson, et.al.[1] show that the steady state linearized Boltzmann equation for homogeneous materials takes on the form

$$\begin{aligned} \vec{\Omega} \cdot \nabla \phi_j(\vec{r}, E, \vec{\Omega}) - \frac{1}{A_j} \frac{\partial}{\partial E}(S_j(E)\phi_j(\vec{r}, E, \vec{\Omega})) + \sigma_j(E)\phi_j(\vec{r}, E, \vec{\Omega}) \\ = \sum_{k \neq j} \int dE'\, d\vec{\Omega}'\, \sigma_{jk}(\vec{\Omega}, \vec{\Omega}', E, E')\phi_k(\vec{r}, E', \vec{\Omega}') \end{aligned} \tag{2.5.77}$$

where A_j is the atomic mass of the ion of type j and $\phi_j(\vec{r}, E, \vec{\Omega})$ is the flux of ions of type j moving in the direction $\vec{\Omega}$ with energy E.

Observe that in most cases the left-hand side of the Boltzmann equation represents the time rate of change of a distribution type function in a phase space while the right-hand side of the Boltzmann equation represents the time rate of change of this distribution function within a volume element of phase space due to scattering and absorption collision processes.

Boltzmann Equation for gases

Consider the Boltzmann equation in terms of a particle distribution function $f(\vec{r}, \vec{V}, t)$ which can be written as

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla_{\vec{r}} + \frac{\vec{F}}{m} \cdot \nabla_{\vec{V}} \right) f(\vec{r}, \vec{V}, t) = D_C f(\vec{r}, \vec{V}, t) \tag{2.5.78}$$

for a single species of gas particles where there is only scattering and no absorption of the particles. An element of volume in phase space (x, y, z, V_x, V_y, V_z) can be thought of as a volume element $d\tau = dxdydz$ for the spatial elements together with a volume element $d\tau_v = dV_x dV_y dV_z$ for the velocity elements. These elements are centered at position $\vec{r}$ and velocity $\vec{V}$ at time t. In phase space a constant velocity V_1 can be thought of as a sphere since $V_1^2 = V_x^2 + V_y^2 + V_z^2$. The phase space volume element $d\tau d\tau_v$ changes with time since the position $\vec{r}$ and velocity $\vec{V}$ change with time. The position vector $\vec{r}$ changes because of velocity

[1] John W. Wilson, Lawrence W. Townsend, Walter Schimmerling, Govind S. Khandelwal, Ferdous Kahn, John E. Nealy, Francis A. Cucinotta, Lisa C. Simonsen, Judy L. Shinn, and John W. Norbury, *Transport Methods and Interactions for Space Radiations*, NASA Reference Publication 1257, December 1991.

and the velocity vector changes because of the acceleration $\frac{\vec{F}}{m}$. Here $f(\vec{r},\vec{V},t)d\tau d\tau_v$ represents the expected number of particles in the phase space element $d\tau d\tau_v$ at time t.

Assume there are no collisions, then each of the gas particles in a volume element of phase space centered at position $\vec{r}$ and velocity $\vec{V}_1$ move during a time interval dt to a phase space element centered at position $\vec{r}+\vec{V}_1 dt$ and $\vec{V}_1+\frac{\vec{F}}{m}dt$. If there were no loss or gains of particles, then the number of particles must be conserved and so these gas particles must move smoothly from one element of phase space to another without any gains or losses of particles. Because of scattering collisions in $d\tau$ many of the gas particles move into or out of the velocity range $\vec{V}_1$ to $\vec{V}_1+d\vec{V}_1$. These collision scattering processes are denoted by the collision operator $D_C f(\vec{r},\vec{V},t)$ in the Boltzmann equation.

Consider two identical gas particles which experience a binary collision. Imagine that particle 1 with velocity $\vec{V}_1$ collides with particle 2 having velocity $\vec{V}_2$. Denote by $\sigma(\vec{V}_1 \to \vec{V}_1', \vec{V}_2 \to \vec{V}_2')\, d\tau_{V_1} d\tau_{V_2}$ the conditional probability that particle 1 is scattered from velocity $\vec{V}_1$ to between $\vec{V}_1'$ and $\vec{V}_1'+d\vec{V}_1'$ and the struck particle 2 is scattered from velocity $\vec{V}_2$ to between $\vec{V}_2'$ and $\vec{V}_2'+d\vec{V}_2'$. We will be interested in collisions of the type $(\vec{V}_1',\vec{V}_2')\to(\vec{V}_1,\vec{V}_2)$ for a fixed value of $\vec{V}_1$ as this would represent the number of particles scattered into $d\tau_{V_1}$. Also of interest are collisions of the type $(\vec{V}_1,\vec{V}_2)\to(\vec{V}_1',\vec{V}_2')$ for a fixed value $\vec{V}_1$ as this represents particles scattered out of $d\tau_{V_1}$. Imagine a gas particle in $d\tau$ with velocity $\vec{V}_1'$ subjected to a beam of particles with velocities $\vec{V}_2'$. The incident flux on the element $d\tau d\tau_{V_1'}$ is $|\vec{V}_1'-\vec{V}_2'|f(\vec{r},\vec{V}_2',t)d\tau_{V_2'}$ and hence

$$\sigma(\vec{V}_1 \to \vec{V}_1', \vec{V}_2 \to \vec{V}_2')\, d\tau_{V_1} d\tau_{V_2} dt\, |\vec{V}_1'-\vec{V}_2'| f(\vec{r},\vec{V}_2',t)\, d\tau_{V_2'} \tag{2.5.79}$$

represents the number of collisions, in the time interval dt, which scatter from $\vec{V}_1'$ to between $\vec{V}_1$ and $\vec{V}_1+d\vec{V}_1$ as well as scattering $\vec{V}_2'$ to between $\vec{V}_2$ and $\vec{V}_2+d\vec{V}_2$. Multiply equation (2.5.79) by the density of particles in the element $d\tau d\tau_{V_1'}$ and integrate over all possible initial velocities $\vec{V}_1'$,$\vec{V}_2'$ and final velocities $\vec{V}_2$ not equal to $\vec{V}_1$. This gives the number of particles in $d\tau$ which are scattered into $d\tau_{V_1} dt$ as

$$Ns_{in} = d\tau d\tau_{V_1} dt \int d\tau_{V_2} d\tau_{V_2'} \int d\tau_{V_1'}\, \sigma(\vec{V}_1' \to \vec{V}_1, \vec{V}_2' \to \vec{V}_2)|\vec{V}_1'-\vec{V}_2'| f(\vec{r},\vec{V}_1',t) f(\vec{r},\vec{V}_2',t). \tag{2.5.80}$$

In a similar manner the number of particles in $d\tau$ which are scattered out of $d\tau_{V_1} dt$ is

$$Ns_{out} = d\tau d\tau_{V_1} dt f(\vec{r},\vec{V}_1,t) \int d\tau_{V_2} \int d\tau_{V_2'} \int d\tau_{V_1'}\, \sigma(\vec{V}_1' \to \vec{V}_1, \vec{V}_2' \to \vec{V}_2)|\vec{V}_2-\vec{V}_1| f(\vec{r},\vec{V}_2,t). \tag{2.5.81}$$

Let

$$W(\vec{V}_1' \to \vec{V}_1, \vec{V}_2' \to \vec{V}_2) = |\vec{V}_1-\vec{V}_2|\, \sigma(\vec{V}_1' \to \vec{V}_1, \vec{V}_2' \to \vec{V}_2) \tag{2.5.82}$$

define a symmetric scattering kernel and use the relation $D_C f(\vec{r},\vec{V},t) = Ns_{in} - Ns_{out}$ to represent the Boltzmann equation for gas particles in the form

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla_{\vec{r}} + \frac{\vec{F}}{m}\cdot\nabla_{\vec{V}}\right) f(\vec{r},\vec{V}_1,t) = \\ &\int d\tau_{V_1'} \int d\tau_{V_2'} \int d\tau_{V_2} W(\vec{V}_1 \to \vec{V}_1', \vec{V}_2 \to \vec{V}_2') \left[f(\vec{r},\vec{V}_1',t)f(\vec{r},\vec{V}_2',t) - f(\vec{r},\vec{V}_1,t)f(\vec{r},\vec{V}_2,t)\right].\end{aligned} \tag{2.5.83}$$

Take the moment of the Boltzmann equation (2.5.83) with respect to an arbitrary function $\phi(\vec{V}_1)$. That is, multiply equation (2.5.83) by $\phi(\vec{V}_1)$ and then integrate over all elements of velocity space $d\tau_{V_1}$. Define the following averages and terminology:

- The particle density per unit volume

$$n = n(\vec{r}, t) = \int d\tau_V \, f(\vec{r}, \vec{V}, t) = \iiint\limits_{-\infty}^{+\infty} f(\vec{r}, \vec{V}, t) dV_x dV_y dV_z \tag{2.5.84}$$

where $\rho = nm$ is the mass density.

- The mean velocity

$$\overline{\vec{V}_1} = \vec{V} = \frac{1}{n} \iiint\limits_{-\infty}^{+\infty} \vec{V}_1 f(\vec{r}, \vec{V}_1, t) dV_{1x} dV_{1y} dV_{1z}$$

For any quantity $Q = Q(\vec{V}_1)$ define the barred quantity

$$\overline{Q} = \overline{Q(\vec{r}, t)} = \frac{1}{n(\vec{r}, t)} \int Q(\vec{V}) f(\vec{r}, \vec{V}, t) \, d\tau_V = \frac{1}{n} \iiint\limits_{-\infty}^{+\infty} Q(\vec{V}) f(\vec{r}, \vec{V}, t) dV_x dV_y dV_z. \tag{2.5.85}$$

Further, assume that $\frac{\vec{F}}{m}$ is independent of $\vec{V}$, then the moment of equation (2.5.83) produces the result

$$\frac{\partial}{\partial t} \left(n \overline{\phi} \right) + \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \left(n \overline{V_{1i} \phi} \right) - n \sum_{i=1}^{3} \frac{F_i}{m} \overline{\frac{\partial \phi}{\partial V_{1i}}} = 0 \tag{2.5.86}$$

known as the Maxwell transfer equation. The first term in equation (2.5.86) follows from the integrals

$$\int \frac{\partial f(\vec{r}, \vec{V}_1, t)}{\partial t} \phi(\vec{V}_1) d\tau_{V_1} = \frac{\partial}{\partial t} \int f(\vec{r}, \vec{V}_1, t) \phi(\vec{V}_1) \, d\tau_{V_1} = \frac{\partial}{\partial t} (n \overline{\phi}) \tag{2.5.87}$$

where differentiation and integration have been interchanged. The second term in equation (2.5.86) follows from the integral

$$\begin{aligned} \int \vec{V}_1 \nabla_{\vec{r}} f \, \phi(\vec{V}_1) d\tau_{V_1} &= \int \sum_{i=1}^{3} V_{1i} \frac{\partial f}{\partial x^i} \phi \, d\tau_{V_1} \\ &= \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \left(\int V_{1i} \phi f \, d\tau_{V_1} \right) \\ &= \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \left(n \overline{V_{1i} \phi} \right). \end{aligned} \tag{2.5.88}$$

The third term in equation (2.5.86) is obtained from the following integral where integration by parts is employed

$$\begin{aligned} \int \frac{\vec{F}}{m} \nabla_{\vec{V}_1} f \phi \, d\tau_{V_1} &= \int \sum_{i=1}^{3} \left(\frac{F_i}{m} \frac{\partial f}{\partial V_{1i}} \right) \phi \, d\tau_{V_1} \\ &= \iiint\limits_{-\infty}^{+\infty} \sum_{i=1}^{3} \phi \left(\frac{F_i}{m} \frac{\partial f}{\partial V_{1i}} \right) dV_{1x} dV_{1y} dV_{1y} \\ &= - \int \frac{\partial}{\partial V_{1i}} \left(\frac{F_i}{m} \phi \right) f \, d\tau_{V_1} \\ &= -n \overline{\frac{\partial}{\partial V_{1i}} \left(\frac{F_i}{m} \phi \right)} = - \frac{F_i}{m} \overline{\frac{\partial \phi}{\partial V_{1i}}} \end{aligned} \tag{2.5.89}$$

since F_i does not depend upon $\vec{V}_1$ and $f(\vec{r}, \vec{V}, t)$ equals zero for V_i equal to $\pm\infty$. The right-hand side of equation (2.5.86) represents the integral of $(D_C f)\phi$ over velocity space. This integral is zero because of the symmetries associated with the right-hand side of equation (2.5.83). Physically, the integral of $(D_c f)\phi$ over velocity space must be zero since collisions with only scattering terms cannot increase or decrease the number of particles per cubic centimeter in any element of phase space.

In equation (2.5.86) we write the velocities V_{1i} in terms of the mean velocities (u, v, w) and random velocities (U_r, V_r, W_r) with

$$V_{11} = U_r + u, \qquad V_{12} = V_r + v, \qquad V_{13} = W_r + w \tag{2.5.90}$$

or $\vec{V}_1 = \vec{V}_r + \vec{V}$ with $\overline{\vec{V}_1} = \overline{\vec{V}_r + \vec{V}} = \vec{V}$ since $\overline{\vec{V}_r} = 0$ (i.e. the average random velocity is zero.) For future reference we write equation (2.5.86) in terms of these random velocities and the material derivative. Substitution of the velocities from equation (2.5.90) in equation (2.5.86) gives

$$\frac{\partial(n\overline{\phi})}{\partial t} + \frac{\partial}{\partial x}\left(n\overline{(U_r + u)\phi}\right) + \frac{\partial}{\partial y}\left(n\overline{(V_r + v)\phi}\right) + \frac{\partial}{\partial z}\left(n\overline{(W_r + w)\phi}\right) - n\sum_{i=1}^{3}\frac{F_i}{m}\overline{\frac{\partial \phi}{\partial V_{1i}}} = 0 \tag{2.5.91}$$

or

$$\begin{aligned}\frac{\partial(n\overline{\phi})}{\partial t} &+ \frac{\partial}{\partial x}\left(n\overline{u\phi}\right) + \frac{\partial}{\partial y}\left(n\overline{v\phi}\right) + \frac{\partial}{\partial z}\left(n\overline{w\phi}\right)\\ &+ \frac{\partial}{\partial x}\left(n\overline{U_r\phi}\right) + \frac{\partial}{\partial y}\left(n\overline{V_r\phi}\right) + \frac{\partial}{\partial z}\left(n\overline{W_r\phi}\right) - n\sum_{i=1}^{3}\frac{F_i}{m}\overline{\frac{\partial \phi}{\partial V_{1i}}} = 0.\end{aligned} \tag{2.5.92}$$

Observe that

$$n\overline{u\phi} = \iiint_{-\infty}^{+\infty} u\phi f(\vec{r}, \vec{V}, t)\, dV_x dV_y dV_z = nu\overline{\phi} \tag{2.5.93}$$

and similarly $n\overline{v\phi} = nv\overline{\phi}$, $n\overline{w\phi} = nw\overline{\phi}$. This enables the equation (2.5.92) to be written in the form

$$\begin{aligned}n\frac{\partial\overline{\phi}}{\partial t} &+ nu\frac{\partial\overline{\phi}}{\partial x} + nv\frac{\partial\overline{\phi}}{\partial y} + nw\frac{\partial\overline{\phi}}{\partial z}\\ &+ \overline{\phi}\left[\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) + \frac{\partial}{\partial y}(nv) + \frac{\partial}{\partial z}(nw)\right]\\ &+ \frac{\partial}{\partial x}\left(n\overline{U_r\phi}\right) + \frac{\partial}{\partial y}\left(n\overline{V_r\phi}\right) + \frac{\partial}{\partial z}\left(n\overline{W_r\phi}\right) - n\sum_{i=1}^{3}\frac{F_i}{m}\overline{\frac{\partial \phi}{\partial V_{1i}}} = 0.\end{aligned} \tag{2.5.94}$$

The middle bracketed sum in equation (2.5.94) is recognized as the continuity equation when multiplied by m and hence is zero. The moment equation (2.5.86) now has the form

$$n\frac{D\overline{\phi}}{Dt} + \frac{\partial}{\partial x}\left(n\overline{U_r\phi}\right) + \frac{\partial}{\partial y}\left(n\overline{V_r\phi}\right) + \frac{\partial}{\partial z}\left(n\overline{W_r\phi}\right) - n\sum_{i=1}^{3}\frac{F_i}{m}\overline{\frac{\partial \phi}{\partial V_{1i}}} = 0. \tag{2.5.95}$$

Note that from the equations (2.5.86) or (2.5.95) one can derive the basic equations of fluid flow from continuum mechanics developed earlier. We consider the following special cases of the Maxwell transfer equation.

(i) In the special case $\phi = m$ the equation (2.5.86) reduces to the continuity equation for fluids. That is, equation (2.5.86) becomes

$$\frac{\partial}{\partial t}(nm) + \nabla \cdot (nm\overline{\vec{V}_1}) = 0 \tag{2.5.96}$$

which is the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \tag{2.5.97}$$

where ρ is the mass density and $\vec{V}$ is the mean velocity defined earlier.

(ii) In the special case $\phi = m\vec{V}_1$ is momentum, the equation (2.5.86) reduces to the momentum equation for fluids. To show this, we write equation (2.5.86) in terms of the dyadic $\vec{V}_1\vec{V}_1$ in the form

$$\frac{\partial}{\partial t}\left(nm\overline{\vec{V}_1}\right) + \nabla \cdot (nm\overline{\vec{V}_1\vec{V}_1}) - n\vec{F} = 0 \tag{2.5.98}$$

or

$$\frac{\partial}{\partial t}\left(\rho(\vec{V}_r + \vec{V})\right) + \nabla \cdot (\rho(\vec{V}_r + \vec{V})(\vec{V}_r + \vec{V})) - n\vec{F} = 0. \tag{2.5.99}$$

Let $\boldsymbol{\sigma} = -\rho\vec{V}_r\vec{V}_r$ denote a stress tensor which is due to the random motions of the gas particles and write equation (2.5.99) in the form

$$\rho\frac{\partial \vec{V}}{\partial t} + \vec{V}\frac{\partial \rho}{\partial t} + \rho\vec{V}(\nabla \cdot \vec{V}) + \vec{V}(\nabla \cdot (\rho\vec{V})) - \nabla \cdot \boldsymbol{\sigma} - n\vec{F} = 0. \tag{2.5.100}$$

The term $\vec{V}\left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{V})\right) = 0$ because of the continuity equation and so equation (2.5.100) reduces to the momentum equation

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\nabla \cdot \vec{V}\right) = n\vec{F} + \nabla \cdot \boldsymbol{\sigma}. \tag{2.5.101}$$

For $\vec{F} = q\vec{E} + q\vec{V} \times \vec{B} + m\vec{b}$, where q is charge, $\vec{E}$ and $\vec{B}$ are electric and magnetic fields, and $\vec{b}$ is a body force per unit mass, together with

$$\boldsymbol{\sigma} = \sum_{i=1}^{3}\sum_{j=1}^{3}(-p\delta_{ij} + \tau_{ij})\widehat{e}_i\widehat{e}_j \tag{2.5.102}$$

the equation (2.5.101) becomes the momentum equation

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{b} - \nabla p + \nabla \cdot \tau + nq(\vec{E} + \vec{V} \times \vec{B}). \tag{2.5.103}$$

In the special case were $\vec{E}$ and $\vec{B}$ vanish, the equation (2.5.103) reduces to the previous momentum equation (2.5.25) .

(iii) In the special case $\phi = \frac{m}{2}\vec{V}_1 \cdot \vec{V}_1 = \frac{m}{2}(V_{11}^2 + V_{12}^2 + V_{13}^2)$ is the particle kinetic energy, the equation (2.5.86) simplifies to the energy equation of fluid mechanics. To show this we substitute ϕ into equation (2.5.95) and simplify. Note that

$$\begin{aligned}\overline{\phi} =& \frac{m}{2}\left[\overline{(U_r + u)^2} + \overline{(V_r + v)^2} + \overline{(W_r + w)^2}\right]\\ \overline{\phi} =& \frac{m}{2}\left[\overline{U_r^2} + \overline{V_r^2} + \overline{W_r^2} + u^2 + v^2 + w^2\right]\end{aligned} \tag{2.5.104}$$

since $u\overline{U_r} = v\overline{V_r} = w\overline{W_r} = 0$. Let $V^2 = u^2 + v^2 + w^2$ and $\overline{C_r^2} = \overline{U_r^2} + \overline{V_r^2} + \overline{W_r^2}$ and write equation (2.5.104) in the form

$$\overline{\phi} = \frac{m}{2}\left(\overline{C_r^2} + V^2\right). \tag{2.5.105}$$

Also note that

$$\begin{aligned} n\overline{U_r\phi} =& \frac{nm}{2}\left[\overline{U_r(U_r+u)^2} + \overline{U_r(V_r+v)^2} + \overline{U_r(W_r+w)^2}\right] \\ =& \frac{nm}{2}\left[\frac{\overline{U_rC_r^2}}{2} + u\overline{U_r^2} + v\overline{U_rV_r} + w\overline{U_rW_r}\right] \end{aligned} \tag{2.5.106}$$

and that

$$n\overline{V_r\phi} = \frac{nm}{2}\left[\overline{V_rC_r^2} + u\overline{V_rU_r} + v\overline{V_r^2} + w\overline{V_rW_r}\right] \tag{2.5.107}$$

$$n\overline{W_r\phi} = \frac{nm}{2}\left[\overline{W_rC_r^2} + u\overline{W_rU_r} + v\overline{W_rV_r} + w\overline{W_r^2}\right] \tag{2.5.108}$$

are similar results.

We use $\frac{\partial}{\partial V_{1i}}(\phi) = mV_{1i}$ together with the previous results substituted into the equation (2.5.95), and find that the Maxwell transport equation can be expressed in the form

$$\begin{aligned} \rho\frac{D}{Dt}\left(\frac{\overline{C_r^2}}{2} + \frac{V^2}{2}\right) =& -\frac{\partial}{\partial x}\left(\rho[u\overline{U_r^2} + v\overline{U_rV_r} + w\overline{U_rW_r}]\right) \\ & -\frac{\partial}{\partial y}\left(\rho[u\overline{V_rU_r} + v\overline{V_r^2} + w\overline{V_rW_r}]\right) \\ & -\frac{\partial}{\partial z}\left(\rho[u\overline{W_rU_r} + v\overline{W_rV_r} + w\overline{W_r^2}]\right) \\ & -\frac{\partial}{\partial x}\left(\rho\frac{\overline{U_rC_r^2}}{2}\right) - \frac{\partial}{\partial y}\left(\rho\frac{\overline{V_rC_r^2}}{2}\right) - \frac{\partial}{\partial z}\left(\rho\frac{\overline{W_rC_r^2}}{2}\right) + n\vec{F}\cdot\vec{V}. \end{aligned} \tag{2.5.109}$$

Compare the equation (2.5.109) with the energy equation (2.5.48)

$$\rho\frac{De}{Dt} + \rho\frac{D}{Dt}\left(\frac{V^2}{2}\right) = \nabla(\boldsymbol{\sigma}\cdot\vec{V}) - \nabla\cdot\vec{q} + \rho\vec{b}\cdot\vec{V} \tag{2.5.110}$$

where the internal heat energy has been set equal to zero. Let $e = \frac{\overline{C_r^2}}{2}$ denote the internal energy due to random motion of the gas particles, $\vec{F} = m\vec{b}$, and let

$$\begin{aligned} \nabla\cdot\vec{q} =& -\frac{\partial}{\partial x}\left(\rho\frac{\overline{U_rC_r^2}}{2}\right) - \frac{\partial}{\partial y}\left(\rho\frac{\overline{V_rC_r^2}}{2}\right) - \frac{\partial}{\partial z}\left(\rho\frac{\overline{W_rC_r^2}}{2}\right) \\ =& -\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \end{aligned} \tag{2.5.111}$$

represent the heat conduction terms due to the transport of particle energy $\frac{mC_r^2}{2}$ by way of the random particle motion. The remaining terms are related to the rate of change of work and surface stresses giving

$$\begin{aligned} -\frac{\partial}{\partial x}\left(\rho[u\overline{U_r^2} + v\overline{U_rV_r} + w\overline{U_rW_r}]\right) =& \frac{\partial}{\partial x}\left(u\sigma_{xx} + v\sigma_{xy} + w\sigma_{xz}\right) \\ -\frac{\partial}{\partial y}\left(\rho[u\overline{V_rU_r} + v\overline{V_r^2} + w\overline{V_rW_r}]\right) =& \frac{\partial}{\partial y}\left(u\sigma_{yx} + v\sigma_{yy} + w\sigma_{yz}\right) \\ -\frac{\partial}{\partial z}\left(\rho[u\overline{W_rU_r} + v\overline{W_rV_r} + w\overline{W_r^2}]\right) =& \frac{\partial}{\partial z}\left(u\sigma_{zx} + v\sigma_{zy} + w\sigma_{zz}\right). \end{aligned} \tag{2.5.112}$$

This gives the stress relations due to random particle motion

$$\begin{aligned}
\sigma_{xx} &= -\,\rho\overline{U_r^2} & \sigma_{yx} &= -\,\rho\overline{V_r U_r} & \sigma_{zx} &= -\,\rho\overline{W_r U_r}\\
\sigma_{xy} &= -\,\rho\overline{U_r V_r} & \sigma_{yy} &= -\,\rho\overline{V_r^2} & \sigma_{zy} &= -\,\rho\overline{W_r V_r}\\
\sigma_{xz} &= -\,\rho\overline{U_r W_r} & \sigma_{yz} &= -\,\rho\overline{V_r W_r} & \sigma_{zz} &= -\,\rho\overline{W_r^2}.
\end{aligned} \tag{2.5.113}$$

The Boltzmann equation is a basic macroscopic model used for the study of individual particle motion where one takes into account the distribution of particles in both space, time and energy. The Boltzmann equation for gases assumes only binary collisions as three-body or multi-body collisions are assumed to rarely occur. Another assumption used in the development of the Boltzmann equation is that the actual time of collision is thought to be small in comparison with the time between collisions. The basic problem associated with the Boltzmann equation is to find a velocity distribution, subject to either boundary and/or initial conditions, which describes a given gas flow.

The continuum equations involve trying to obtain the macroscopic variables of density, mean velocity, stress, temperature and pressure which occur in the basic equations of continuum mechanics considered earlier. Note that the moments of the Boltzmann equation, derived for gases, also produced these same continuum equations and so they are valid for gases as well as liquids.

In certain situations one can assume that the gases approximate a Maxwellian distribution

$$f(\vec{r},\vec{V},t) \approx n(\vec{r},t)\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{m}{2kT}\vec{V}\cdot\vec{V}\right) \tag{2.5.114}$$

thereby enabling the calculation of the pressure tensor and temperature from statistical considerations.

In general, one can say that the Boltzmann integral-differential equation and the Maxwell transfer equation are two important formulations in the kinetic theory of gases. The Maxwell transfer equation depends upon some gas-particle property ϕ which is assumed to be a function of the gas-particle velocity. The Boltzmann equation depends upon a gas-particle velocity distribution function f which depends upon position $\vec{r}$, velocity $\vec{V}$ and time t. These formulations represent two distinct and important viewpoints considered in the kinetic theory of gases.

EXERCISE 2.5

▶ **1.** Let $p = p(x, y, z)$, [dyne/cm^2] denote the pressure at a point (x, y, z) in a fluid medium at rest (hydrostatics), and let ΔV denote an element of fluid volume situated at this point as illustrated in the figure 2.5-5.

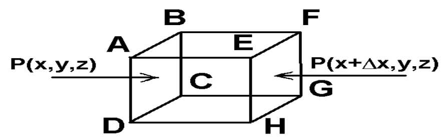

Figure 2.5-5. Pressure acting on a volume element.

(a) Show that the force acting on the face $ABCD$ is $p(x, y, z)\Delta y \Delta z\, \hat{e}_1$.
(b) Show that the force acting on the face $EFGH$ is

$$-p(x + \Delta x, y, z)\Delta y \Delta z\, \hat{e}_1 = -\left[p(x, y, z) + \frac{\partial p}{\partial x}\Delta x + \frac{\partial^2 p}{\partial x^2}\frac{(\Delta x)^2}{2!} + \cdots\right] \Delta y \Delta z\, \hat{e}_1.$$

(c) In part (b) neglect terms with powers of Δx greater than or equal to 2 and show that the resultant force in the x-direction is $-\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z\, \hat{e}_1$.
(d) What has been done in the x-direction can also be done in the y and z-directions. Show that the resultant forces in these directions are $-\frac{\partial p}{\partial y}\Delta x \Delta y \Delta z\, \hat{e}_2$ and $-\frac{\partial p}{\partial z}\Delta x \Delta y \Delta z\, \hat{e}_3$. (e) Show that $-\nabla p = -\left(\frac{\partial p}{\partial x}\hat{e}_1 + \frac{\partial p}{\partial y}\hat{e}_2 + \frac{\partial p}{\partial z}\hat{e}_3\right)$ is the force per unit volume acting at the point (x, y, z) of the fluid medium.

▶ **2.** Follow the example of exercise 1 above but use cylindrical coordinates and find the force per unit volume at a point (r, θ, z). Hint: An element of volume in cylindrical coordinates is given by $\Delta V = r\Delta r \Delta\theta \Delta z$.

▶ **3.** Follow the example of exercise 1 above but use spherical coordinates and find the force per unit volume at a point (ρ, θ, ϕ). Hint: An element of volume in spherical coordinates is $\Delta V = \rho^2 \sin\theta \Delta\rho \Delta\theta \Delta\phi$.

▶ **4.** Show that if the density $\varrho = \varrho(x, y, z, t)$ is a constant, then $v^r_{,r} = 0$.

▶ **5.** Assume that λ^* and μ^* are zero. Such a fluid is called a nonviscous or perfect fluid. (a) Show the Cartesian equations describing conservation of linear momentum are

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= b_x - \frac{1}{\varrho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= b_y - \frac{1}{\varrho}\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= b_z - \frac{1}{\varrho}\frac{\partial p}{\partial z}
\end{aligned}$$

where (u, v, w) are the physical components of the fluid velocity. (b) Show that the continuity equation can be written

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho u) + \frac{\partial}{\partial y}(\varrho v) + \frac{\partial}{\partial z}(\varrho w) = 0$$

► **6.** Assume $\lambda^* = \mu^* = 0$ so that the fluid is ideal or nonviscous. Use the results given in problem 5 and make the following additional assumptions:

- The density is constant and so the fluid is incompressible.
- The body forces are zero.
- Steady state flow exists.
- Only two dimensional flow in the x-$y$plane is considered such that $u = u(x,y)$, $v = v(x,y)$ and $w = 0$. (a) Employ the above assumptions and simplify the equations in problem 5 and verify the results

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\varrho}\frac{\partial p}{\partial x} = 0$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\varrho}\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(b) Make the additional assumption that the flow is irrotational and show that this assumption produces the results

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad \frac{1}{2}\left(u^2 + v^2\right) + \frac{1}{\varrho}p = \mathit{constant}.$$

(c) Point out the Cauchy-Riemann equations and Bernoulli's equation in the above set of equations.

► **7.** Assume the body forces are derivable from a potential function ϕ such that $b_i = -\phi_{,i}$. Show that for an ideal fluid with constant density the equations of fluid motion can be written in either of the forms

$$\frac{\partial v^r}{\partial t} + v^r_{\ ,s}v^s = -\frac{1}{\varrho}g^{rm}p_{,m} - g^{rm}\phi_{,m} \quad \text{or} \quad \frac{\partial v_r}{\partial t} + v_{r,s}v^s = -\frac{1}{\varrho}p_{,r} - \phi_{,r}$$

► **8.** The vector identities $\nabla^2\vec{v} = \nabla(\nabla\cdot\vec{v}) - \nabla\times(\nabla\times\vec{v})$ and $(\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\nabla(\vec{v}\cdot\vec{v}) - \vec{v}\times(\nabla\times\vec{v})$ are used to express the Navier-Stokes-Duhem equations in alternate forms involving the vorticity $\Omega = \nabla\times\vec{v}$. (a) Use Cartesian tensor notation and derive the above identities. (b) Show the second identity can be written in generalized coordinates as $v^j v^m_{\ ,j} = g^{mj}v^k v_{k,j} - \epsilon^{mnp}\epsilon^{ijk}g_{pi}v_n v_{k,j}$. Hint: Show that $\dfrac{\partial v^2}{\partial x^j} = 2v^k v_{k,j}$.

► **9.** Use problem 8 and show that the results in problem 7 can be written

$$\frac{\partial v^r}{\partial t} - \epsilon^{rnp}\Omega_p v_n = -g^{rm}\frac{\partial}{\partial x^m}\left(\frac{p}{\varrho} + \phi + \frac{v^2}{2}\right)$$
$$\text{or} \qquad \frac{\partial v_i}{\partial t} - \epsilon_{ijk}v^j\Omega^k = -\frac{\partial}{\partial x^i}\left(\frac{p}{\varrho} + \phi + \frac{v^2}{2}\right)$$

► **10.** In terms of physical components, show that in generalized orthogonal coordinates, for $i \neq j$, the rate of deformation tensor D_{ij} can be written $D(ij) = \dfrac{1}{2}\left[\dfrac{h_i}{h_j}\dfrac{\partial}{\partial x^j}\left(\dfrac{v(i)}{h_i}\right) + \dfrac{h_j}{h_i}\dfrac{\partial}{\partial x^i}\left(\dfrac{v(j)}{h_j}\right)\right]$, no summations and for $i = j$ there results $D(ii) = \dfrac{1}{h_i}\dfrac{\partial v(i)}{\partial x^i} - \dfrac{v(i)}{h_i^2}\dfrac{\partial h_i}{\partial x^i} + \displaystyle\sum_{k=1}^{3}\dfrac{1}{h_i h_k}v(k)\dfrac{\partial h_i}{\partial x^k}$, no summations. (Hint: See Problem 17 Exercise 2.1.)

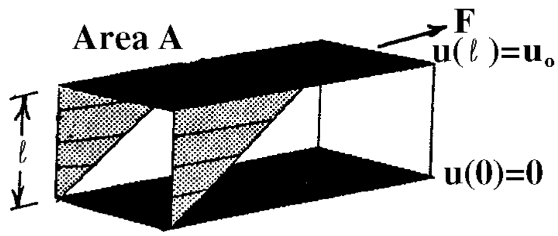

Figure 2.5-6. Plane Couette flow

▶ **11.** Find the physical components of the rate of deformation tensor D_{ij} in Cartesian coordinates. (Hint: See problem 10.)

▶ **12.** Find the physical components of the rate of deformation tensor in cylindrical coordinates. (Hint: See problem 10.)

▶ **13.** (Plane Couette flow) Assume a viscous fluid with constant density is between two plates as illustrated in the figure 2.5-6.

(a) Define $\nu = \frac{\mu^*}{\varrho}$ as the kinematic viscosity and show the equations of fluid motion can be written

$$\frac{\partial v^i}{\partial t} + v^i_{,s} v^s = -\frac{1}{\varrho} g^{im} p_{,m} + \nu g^{jm} v^i_{,mj} + g^{ij} b_j, \quad i = 1, 2, 3$$

(b) Let $\vec{v} = (u, v, w)$ denote the physical components of the fluid flow and make the following assumptions

- $u = u(y)$, $v = w = 0$
- Steady state flow exists
- The top plate, with area A, is a distance ℓ above the bottom plate. The bottom plate is fixed and a constant force F is applied to the top plate to keep it moving with a velocity $u_0 = u(\ell)$.
- p and ϱ are constants
- The body force components are zero.

Find the velocity $u = u(y)$

(c) Show the tangential stress exerted by the moving fluid is $\dfrac{F}{A} = \sigma_{21} = \sigma_{xy} = \sigma_{yx} = \mu^* \dfrac{u_0}{\ell}$. This example illustrates that the stress is proportional to u_0 and inversely proportional to ℓ.

▶ **14.** In the continuity equation make the change of variables

$$\bar{t} = \frac{t}{\tau}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_0}, \quad \bar{\vec{v}} = \frac{\vec{v}}{v_0}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}$$

and write the continuity equation in terms of the barred variables and the Strouhal parameter.

▶ **15.** (Plane Poiseuille flow) Consider two flat plates parallel to one another as illustrated in the figure 2.5-7. One plate is at $y = 0$ and the other plate is at $y = 2\ell$. Let $\vec{v} = (u, v, w)$ denote the physical components of the fluid velocity and make the following assumptions concerning the flow The body forces are zero. The derivative $\dfrac{\partial p}{\partial x} = -p_0$ is a constant and $\dfrac{\partial p}{\partial y} = \dfrac{\partial p}{\partial z} = 0$. The velocity in the x-direction is a function of y only

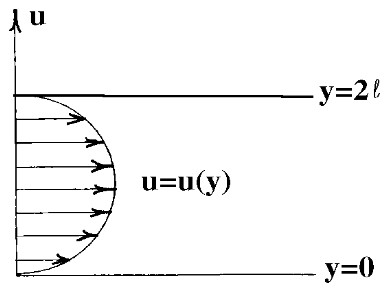

Figure 2.5-7. Plane Poiseuille flow

with $u = u(y)$ and $v = w = 0$ with boundary values $u(0) = u(2\ell) = 0$. The density is constant and $\nu = \mu^*/\varrho$ is the kinematic viscosity.

(a) Show the equation of fluid motion is $\nu \dfrac{d^2u}{dy^2} + \dfrac{p_0}{\varrho} = 0, \qquad u(0) = u(2\ell) = 0$

(b) Find the velocity $u = u(y)$ and find the maximum velocity in the x-direction. (c) Let M denote the mass flow rate across the plane $x = x_0 = constant$, , where $0 \le y \le 2\ell$, and $0 \le z \le 1$.

Show that $M = \dfrac{2}{3\mu^*}\varrho p_0 \ell^3$. Note that as μ^* increases, M decreases.

► **16.** The heat equation (or diffusion equation) can be expressed $\operatorname{div}(\, k \operatorname{grad} u) + H = \dfrac{\partial(\delta c u)}{\partial t}$, where c is the specific heat [cal/gm C], δ is the volume density [gm/cm^3], H is the rate of heat generation [cal/sec cm^3], u is the temperature [C], k is the thermal conductivity [cal/sec cm C]. Assume constant thermal conductivity, volume density and specific heat and express the boundary value problem

$$k\frac{\partial^2 u}{\partial x^2} = \delta c \frac{\partial u}{\partial t}, \quad 0 < x < L$$
$$u(0,t) = 0, \qquad u(L,t) = u_1, \qquad u(x,0) = f(x)$$

in a form where all the variables are dimensionless. Assume u_1 is constant.

► **17.** Simplify the Navier-Stokes-Duhem equations using the assumption that there is incompressible flow.

► **18.** (Rayleigh impulsive flow) The figure 2.5-8 illustrates fluid motion in the plane where $y > 0$ above a plate located along the axis where $y = 0$. The plate along $y = 0$ has zero velocity for all negative time and at time $t = 0$ the plate is given an instantaneous velocity u_0 in the positive x-direction. Assume the physical components of the velocity are $\vec{v} = (u, v, w)$ which satisfy $u = u(y,t)$, $v = w = 0$. Assume that the density of the fluid is constant, the gradient of the pressure is zero, and the body forces are zero. (a) Show that the velocity in the x-direction is governed by the differential equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \qquad \text{with} \qquad \nu = \frac{\mu^*}{\varrho}.$$

Assume u satisfies the initial condition $u(0,t) = u_0 H(t)$ where H is the Heaviside step function. Also assume there exist a condition at infinity $\lim_{y\to\infty} u(y,t)$. This latter condition requires a bounded velocity at infinity. (b) Use any method to show the velocity is

$$u(y,t) = u_0 - u_0 \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) = u_0 \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

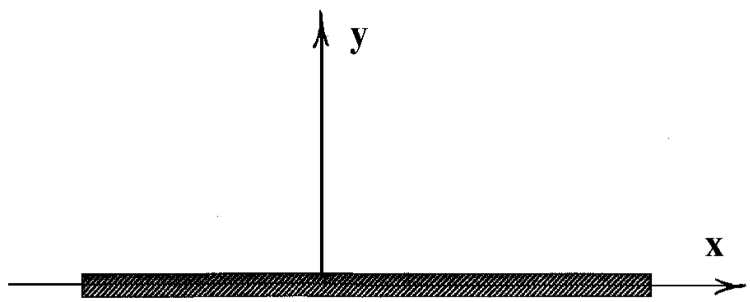

Figure 2.5-8. Rayleigh impulsive flow

where erf and erfc are the error function and complimentary error function respectively. Pick a point on the line $y = y_0 = 2\sqrt{\nu}$ and plot the velocity as a function of time. How does the viscosity effect the velocity of the fluid along the line $y = y_0$?

► **19.** Simplify the Navier-Stokes-Duhem equations using the assumption that there is incompressible and irrotational flow.

► **20.** Let $\zeta = \lambda^* + \frac{2}{3}\mu^*$ and show the constitutive equations (2.5.21) for fluid motion can be written in the form

$$\sigma_{ij} = -p\delta_{ij} + \mu^* \left[v_{i,j} + v_{j,i} - \frac{2}{3}\delta_{ij} v_{k,k} \right] + \zeta \delta_{ij} v_{k,k}.$$

► **21.** (a) Write out the Navier-Stokes-Duhem equation for two dimensional flow in the x-y direction under the assumptions that

- $\lambda^* + \frac{2}{3}\mu^* = 0$ (This condition is referred to as Stoke's flow.)
- The fluid is incompressible
- There is a gravitational force $\vec{b} = -g\nabla h$ Hint: Express your answer as two scalar equations involving the variables $v_1, v_2, h, g, \varrho, p, t, \mu^*$ plus the continuity equation. (b) In part (a) eliminate the pressure and body force terms by cross differentiation and subtraction. (i.e. take the derivative of one equation with respect to x and take the derivative of the other equation with respect to y and then eliminate any common terms.) (c) Assume that $\vec{\omega} = \omega\, \hat{e}_3$ where $\omega = \frac{1}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$ and derive the vorticity-transport equation

$$\frac{d\omega}{dt} = \nu \nabla^2 \omega \qquad \text{where} \qquad \frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + v_1 \frac{\partial \omega}{\partial x} + v_2 \frac{\partial \omega}{\partial y}.$$

Hint: The continuity equation makes certain terms zero. (d) Define a stream function $\psi = \psi(x, y)$ satisfying $v_1 = \frac{\partial \psi}{\partial y}$ and $v_2 = -\frac{\partial \psi}{\partial x}$ and show the continuity equation is identically satisfied. Show also that $\omega = -\frac{1}{2}\nabla^2\psi$ and that

$$\nabla^4 \psi = \frac{1}{\nu}\left[\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} \right].$$

If ν is very large, show that $\nabla^4 \psi \approx 0$.

▶ **22.** In generalized orthogonal coordinates, show that the physical components of the rate of deformation stress can be written, for $i \neq j$

$$\sigma(ij) = \mu^* \left[\frac{h_i}{h_j} \frac{\partial}{\partial x^j} \left(\frac{v(i)}{h_i} \right) + \frac{h_j}{h_i} \frac{\partial}{\partial x^i} \left(\frac{v(j)}{h_j} \right) \right], \quad \text{no summation,}$$

and for $i \neq j \neq k$

$$\begin{aligned} \sigma(ii) = -p + 2\mu^* &\left[\frac{1}{h_i} \frac{\partial v(i)}{\partial x^i} + \frac{1}{h_i h_j} v(j) \frac{\partial h_i}{\partial x^j} + \frac{1}{h_i h_k} v(k) \frac{\partial h_i}{\partial x^k} \right] \\ &+ \frac{\lambda^*}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x^1} \{h_2 h_3 v(1)\} + \frac{\partial}{\partial x^2} \{h_1 h_3 v(2)\} + \frac{\partial}{\partial x^3} \{h_1 h_2 v(3)\} \right], \quad \text{no summation} \end{aligned}$$

▶ **23.** Find the physical components for the rate of deformation stress in Cartesian coordinates. Hint: See problem 22.

▶ **24.** Find the physical components for the rate of deformations stress in cylindrical coordinates. Hint: See problem 22.

▶ **25.** Verify the Navier-Stokes equations for an incompressible fluid can be written $\dot{v}_i = -\frac{1}{\varrho} p_{,i} + \nu v_{i,mm} + b_i$ where $\nu = \frac{\mu^*}{\varrho}$ is called the kinematic viscosity.

▶ **26.** Verify the Navier-Stokes equations for a compressible fluid with zero bulk viscosity can be written $\dot{v}_i = -\frac{1}{\varrho} p_{,i} + \frac{\nu}{3} v_{m,mi} + \nu v_{i,mm} + b_i$ with $\nu = \frac{\mu^*}{\varrho}$ the kinematic viscosity.

▶ **27.** The constitutive equation for a certain non-Newtonian Stokesian fluid is $\sigma_{ij} = -p\delta_{ij} + \beta D_{ij} + \gamma D_{ik} D_{kj}$. Assume that β and γ are constants (a) Verify that $\sigma_{ij,j} = -p_{,i} + \beta D_{ij,j} + \gamma (D_{ik} D_{kj,j} + D_{ik,j} D_{kj})$
(b) Write out the Cauchy equations of motion in Cartesian coordinates. (See page 236).

▶ **28.** Let the constitutive equations relating stress and strain for a solid material take into account thermal stresses due to a temperature T. The constitutive equations have the form $e_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \, \delta_{ij} + \alpha \, T \, \delta_{ij}$ where α is a coefficient of linear expansion for the material and T is the absolute temperature. Solve for the stress in terms of strains.

▶ **29.** Derive equation (2.5.53) and then show that when the bulk coefficient of viscosity is zero, the Navier-Stokes equations, in Cartesian coordinates, can be written in the conservation form

$$\begin{aligned} \frac{\partial(\varrho u)}{\partial t} + \frac{\partial(\varrho u^2 + p - \tau_{xx})}{\partial x} + \frac{\partial(\varrho uv - \tau_{xy})}{\partial y} + \frac{\partial(\varrho uw - \tau_{xz})}{\partial z} &= \varrho b_x \\ \frac{\partial(\varrho v)}{\partial t} + \frac{\partial(\varrho uv - \tau_{xy})}{\partial x} + \frac{\partial(\varrho v^2 + p - \tau_{yy})}{\partial y} + \frac{\partial(\varrho vw - \tau_{yz})}{\partial z} &= \varrho b_y \\ \frac{\partial(\varrho w)}{\partial t} + \frac{\partial(\varrho uw - \tau_{xz})}{\partial x} + \frac{\partial(\varrho vw - \tau_{yz})}{\partial y} + \frac{\partial(\varrho w^2 + p - \tau_{zz})}{\partial z} &= \varrho b_z \end{aligned}$$

where $v_1 = u, v_2 = v, v_3 = w$ and $\tau_{ij} = \mu^*(v_{i,j} + v_{j,i} - \frac{2}{3}\delta_{ij} v_{k,k})$. Hint: Alternatively, consider 2.5.29 and use the continuity equation.

► **30.** Show that for a perfect gas, where $\lambda^* = -\frac{2}{3}\mu^*$ and $\eta = \mu^*$ is a function of position, the vector form of equation (2.5.25) is

$$\varrho \frac{D\vec{v}}{Dt} = \varrho\vec{b} - \nabla p + \frac{4}{3}\nabla(\eta\nabla\cdot\vec{v}) + \nabla(\vec{v}\cdot\nabla\eta) - \vec{v}\,\nabla^2\eta + (\nabla\eta)\times(\nabla\times\vec{v}) - (\nabla\cdot\vec{v})\nabla\eta - \nabla\times(\nabla\times(\eta\vec{v}))$$

► **31.** Derive the energy equation $\varrho\frac{D\,h}{Dt} = \frac{D\,p}{Dt} + \frac{\partial Q}{\partial t} - \nabla\cdot\vec{q} + \Phi$. Hint: Use the continuity equation.

► **32.** Show that in Cartesian coordinates the Navier-Stokes equations of motion for a compressible fluid can be written

$$\begin{aligned}
\rho\frac{Du}{Dt} =& \rho b_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu^*\frac{\partial u}{\partial x} + \lambda^*\nabla\cdot\vec{V}\right) + \frac{\partial}{\partial y}\left(\mu^*(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right) + \frac{\partial}{\partial z}\left(\mu^*(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})\right)\\
\rho\frac{Dv}{Dt} =& \rho b_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left(2\mu^*\frac{\partial v}{\partial y} + \lambda^*\nabla\cdot\vec{V}\right) + \frac{\partial}{\partial z}\left(\mu^*(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right) + \frac{\partial}{\partial x}\left(\mu^*(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x})\right)\\
\rho\frac{Dv}{Dt} =& \rho b_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left(2\mu^*\frac{\partial w}{\partial z} + \lambda^*\nabla\cdot\vec{V}\right) + \frac{\partial}{\partial x}\left(\mu^*(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})\right) + \frac{\partial}{\partial y}\left(\mu^*(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right)
\end{aligned}$$

where $(V_x, V_y, V_z) = (u, v, w)$.

► **33.** Show that in cylindrical coordinates the Navier-Stokes equations of motion for a compressible fluid can be written

$$\begin{aligned}
\varrho\left(\frac{DV_r}{Dt} - \frac{V_\theta^2}{r}\right) =& \varrho b_r - \frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(2\mu^*\frac{\partial V_r}{\partial r} + \lambda^*\nabla\cdot\vec{V}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\mu^*(\frac{1}{r}\frac{\partial V_r}{\partial\theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r})\right)\\
&+ \frac{\partial}{\partial z}\left(\mu^*(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r})\right) + \frac{2\mu^*}{r}(\frac{\partial V_r}{\partial r} - \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} - \frac{V_r}{r})\\
\varrho\left(\frac{DV_\theta}{Dt} + \frac{V_rV_\theta}{r}\right) =& \varrho b_\theta - \frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(2\mu^*(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r}) + \lambda^*\nabla\cdot\vec{V}\right) + \frac{\partial}{\partial z}\left(\mu^*(\frac{1}{r}\frac{\partial V_z}{\partial\theta} + \frac{\partial V_\theta}{\partial z})\right)\\
&+ \frac{\partial}{\partial r}\left(\mu^*(\frac{1}{r}\frac{\partial V_r}{\partial\theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r})\right) + \frac{2\mu^*}{r}(\frac{1}{r}\frac{\partial V_r}{\partial\theta} + \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r})\\
\varrho\frac{DV_z}{Dt} =& \varrho b_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left(2\mu^*\frac{\partial V_z}{\partial z} + \lambda^*\nabla\cdot\vec{V}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\mu^* r(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r})\right)\\
&+ \frac{1}{r}\frac{\partial}{\partial\theta}\left(\mu^*(\frac{1}{r}\frac{\partial V_z}{\partial\theta} + \frac{\partial V_\theta}{\partial z})\right)
\end{aligned}$$

► **34.** Show that the dissipation function Φ can be written as $\Phi = 2\mu^* D_{ij}D_{ij} + \lambda^*\Theta^2$.

► **35.** Verify the identities:

$$(a) \qquad \varrho\frac{D}{Dt}(e_t/\varrho) = \frac{\partial e_t}{\partial t} + \nabla\cdot(e_t\vec{V}) \qquad (b) \quad \varrho\frac{D}{Dt}(e_t/\varrho) = \varrho\frac{De}{Dt} + \varrho\frac{D}{Dt}\left(V^2/2\right).$$

► **36.** Show that the conservation law for heat flow is given by

$$\frac{\partial T}{\partial t} + \nabla\cdot(T\vec{v} - \kappa\nabla T) = S_Q$$

where κ is the thermal conductivity of the material, T is the temperature, $\vec{J}_{advection} = T\vec{v}$, $\vec{J}_{conduction} = -\kappa\nabla T$ and S_Q is a source term. Note that in a solid material there is no flow and so $\vec{v} = 0$ and the above equation reduces to the heat equation. Assign units of measurements to each term in the above equation and make sure the equation is dimensionally homogeneous.

▶ **37.** Show that in spherical coordinates the Navier-Stokes equations of motion for a compressible fluid can be written

$$\begin{aligned}
\varrho(\frac{DV_\rho}{Dt} - \frac{V_\theta^2+V_\phi^2}{\rho}) =& \varrho b_\rho - \frac{\partial p}{\partial \rho} + \frac{\partial}{\partial \rho}\left(2\mu^* \frac{\partial V_\rho}{\partial \rho} + \lambda^* \nabla\cdot\vec{V}\right) + \frac{1}{\rho}\frac{\partial}{\partial \theta}\left(\mu^*(\rho \frac{\partial}{\partial \rho}(V_\theta/\rho) + \frac{1}{\rho}\frac{\partial V_\rho}{\partial \theta})\right)\\
&+ \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}\left(\mu^*(\frac{1}{\rho\sin\theta}\frac{\partial V_\rho}{\partial \phi} + \rho\frac{\partial}{\partial\rho}(V_\phi/\rho))\right)\\
&+ \frac{\mu^*}{\rho}(4\frac{\partial V_\rho}{\partial\rho} - \frac{2}{\rho}\frac{\partial V_\theta}{\partial\theta} - \frac{4V_\rho}{\rho} - \frac{2}{\rho\sin\theta}\frac{\partial V_\phi}{\partial\phi} - \frac{2V_\theta\cot\theta}{\rho} + \rho\cot\theta\frac{\partial}{\partial\rho}(V_\theta/\rho) + \frac{\cot\theta}{\rho}\frac{\partial V_\rho}{\partial\theta})\\
\varrho(\frac{DV_\theta}{Dt} + \frac{V_\rho V_\theta}{\rho} - \frac{V_\phi^2\cot\theta}{\rho}) =& \varrho b_\theta - \frac{1}{\rho}\frac{\partial p}{\partial\theta} + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{2\mu^*}{\rho}(\frac{\partial V_\theta}{\partial\theta} + V_\rho) + \lambda^*\nabla\cdot\vec{V}\right)\\
&+ \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}\left(\mu^*(\frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}(V_\phi/\sin\theta) + \frac{1}{\rho\sin\theta}\frac{\partial V_\theta}{\partial\phi})\right) + \frac{\partial}{\partial\rho}\left(\mu^*(\rho\frac{\partial}{\partial\rho}(V_\theta/\rho) + \frac{1}{\rho}\frac{\partial V_\rho}{\partial\theta})\right)\\
&+ \frac{\mu^*}{\rho}\left[2\left(\frac{1}{\rho}\frac{\partial V_\theta}{\partial\theta} - \frac{1}{\rho\sin\theta}\frac{\partial V_\phi}{\partial\phi} - \frac{V_\theta\cot\theta}{\rho}\right)\cot\theta + 3\left(\rho\frac{\partial}{\partial\rho}(V_\theta/\rho) + \frac{1}{\rho}\frac{\partial V_\rho}{\partial\theta}\right)\right]\\
\varrho\left(\frac{DV_\phi}{Dt} + \frac{V_\phi V_\rho}{\rho} + \frac{V_\theta V_\phi\cot\theta}{\rho}\right) =& \varrho b_\phi - \frac{1}{\rho\sin\theta}\frac{\partial p}{\partial\phi} + \frac{\partial}{\partial\rho}\left(\mu^*\left(\frac{1}{\rho\sin\theta}\frac{\partial V_\rho}{\partial\phi} + \rho\frac{\partial}{\partial\rho}(V_\phi/\rho)\right)\right)\\
&+ \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{2\mu^*}{\rho}\left(\frac{1}{\sin\theta}\frac{\partial V_\phi}{\partial\phi} + V_\rho + V_\theta\cot\theta\right) + \lambda^*\nabla\cdot\vec{V}\right)\\
&+ \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\mu^*\left(\frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}(V_\phi/\sin\theta) + \frac{1}{\rho\sin\theta}\frac{\partial V_\theta}{\partial\phi}\right)\right)\\
&+ \frac{\mu^*}{\rho}\left[3\left(\frac{1}{\rho\sin\theta}\frac{\partial V_\rho}{\partial\phi} + \rho\frac{\partial}{\partial\rho}(V_\phi/\rho)\right) + 2\cot\theta\left(\frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}(V_\phi/\sin\theta) + \frac{1}{\rho\sin\theta}\frac{\partial V_\theta}{\partial\phi}\right)\right]
\end{aligned}$$

▶ **38.** Verify all the equations (2.5.28).

▶ **39.** Use the conservation of energy equation (2.5.47) together with the momentum equation (2.5.25) to derive the equation (2.5.48).

▶ **40.** Verify the equation (2.5.55).

▶ **41.** Consider nonviscous flow and write the 3 linear momentum equations and the continuity equation and make the following assumptions: (i) The density ϱ is constant. (ii) Body forces are zero. (iii) Steady state flow only. (iv) Consider only two dimensional flow with non-zero velocity components $u = u(x,y)$ and $v = v(x,y)$. Show that there results the system of equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\varrho}\frac{\partial P}{\partial x} = 0, \qquad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\varrho}\frac{\partial P}{\partial y} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Recognize that the last equation in the above set as one of the Cauchy-Riemann equations that $f(z) = u - iv$ be an analytic function of a complex variable. Further assume that the fluid flow is irrotational so that $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$. Show that this implies that $\frac{1}{2}\left(u^2+v^2\right) + \frac{P}{\varrho} =$ Constant. If in addition u and v are derivable from a potential function $\phi(x,y)$, such that $u = \frac{\partial\phi}{\partial x}$ and $v = \frac{\partial\phi}{\partial y}$, then show that ϕ is a harmonic function. By constructing the conjugate harmonic function $\psi(x,y)$ the complex potential $F(z) = \phi(x,y) + i\psi(x,y)$ is such that $F'(z) = u(x,y) - iv(x,y)$ and $\overline{F'(z)}$ gives the velocity. The family of curves $\phi(x,y)$ =constant are called equipotential curves and the family of curves $\psi(x,y)$ = constant are called streamlines. Show that these families are an orthogonal family of curves.