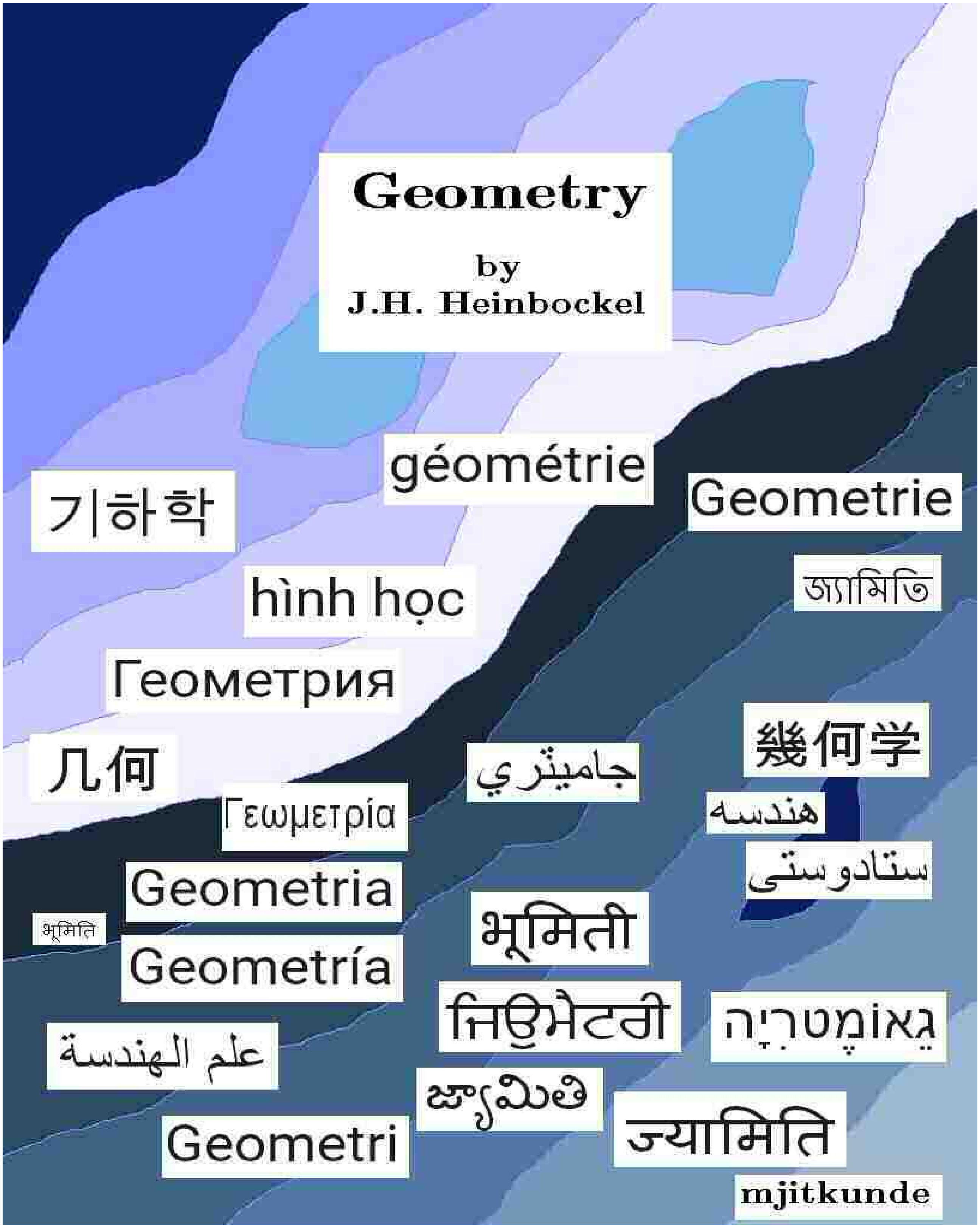


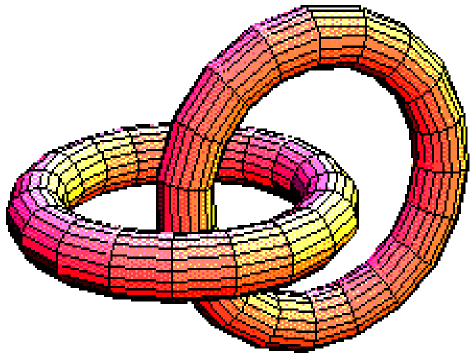
Geometry
by
J.H. Heinbockel
géométrie
기하학
Geometrie
জ্যামিতি
hình học
Геометрия
几何
جاميتري
幾何学
Γεωμετρία
هندسه
Geometria
भूमिति
भूमिती
Geometría
ਜਿਓਮੈਟਰੀ
גֵאוֹמֶטְרִיָה
علم الهندسة
జ్యామితి
Geometri
ज्यामिति
mjitkunde

The Cover

The word "geometry" is written in the following languages-

English, Chinese, Spanish, Arabic, Bengali, Hindi, Russian, Portuguese, Japanese, German, Javanese, Korean, French, Italian, Greek, Vietnamese, Punjabi, Frisian, Farsi, Urdu, teluga, Marthi, Sindhi and Sanskrit.

There are more than 7000 languages in the world. Euclid, who gave his "Elements" to the world, had an influence on almost all these languages.



Geometry

by J.H. Heinbockel
Emeritus Professor of Mathematics
Old Dominion University

Preface

Elementary geometry is the stepping stone to more advance mathematical subjects. It is needed to understand many components of physics, chemistry and engineering. Even the biological sciences require an understanding of geometry, especially biochemistry. In fact, more and more scientific disciplines have come to rely on mathematics of some kind which usually involves some component of geometry.

This book is written so that students and instructors studying geometry can have a reference book presenting a more detailed investigation of selected advanced topics from both plane and solid geometry. The material presented begins with elementary concepts and ideas and progresses rapidly to more advanced concepts. It introduces trigonometry and vectors and shows how these topics can aid in the study of geometry. It offers the presentation of geometry fundamentals from a variety of viewpoints. Many examples are given to help clarify new ideas.

In depth investigations into Projective Geometry, Differential Geometry, Non-Euclidean Geometry, Stochastic Geometry, Combinatorial Geometry, Riemannian Geometry have been omitted from this text. Students wishing to pursue an understanding of these more advanced topics are encouraged to do so. Geometry does not end after taking just one course.

J.H. Heinbockel
July 2017

Geometry

Geometry

Chapter 1

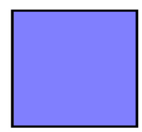
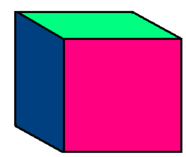
Preliminaries

Euclid of Alexandria, Egypt (325-265) BCE is well known for his mathematical investigations into geometry. His surviving work called *Elements* consists of 13 books on geometry. These books have been around for over 20 centuries and have been translated into all the major languages of the world. The thirteen books from the *Elements* can be found on the internet. The first four books and the sixth book investigate properties of lines, figures and shapes in a two-dimensional plane. The fifth and sixth books are concerned with the theory of proportions. The books seven, eight and nine are numerical in nature. The tenth book deals with irrational quantities. The remaining three books deal with solid three-dimensional geometry.

It is not known if Euclid of Alexandria wrote all of the thirteen books on geometry. It has been suggested that a team of mathematicians wrote the thirteen books under Euclid's direction. The matter is an open question because not much is known about Euclids life. Below is a 17th century painting entitled Euclid by the painter Antonio Cifrondi. The picture is courtesy of Wikimedia Commons.

Figure 1-1. Euclid, 17th century painting by Antonio Cifrondi.

In his study of geometry Euclid presented certain fundamental propositions, which are assumed to always be true. He defined concepts about points, lines, planes, angles, plane figures, parallel lines, solids and other quantities associated with geometry. Starting with these fundamental propositions, Euclid used various reasoning techniques to establish additional basic truths associated with points, lines, plane figures and solid figures. Euclid's book the *Elements* was a mathematics book that introduce critical thinking and reasoning applied to geometric objects and set a standard for future mathematical analysis.

			
point	*line segment*	*plane section*	*cube*
no dimension	*one dimension*	*two dimension*	*three dimension*

A **point is a location**. A point has **no dimension.** If the point is moved along a shortest distance to another point, **a line segment is generated**. This is an example of a **one dimensional object.** One can think of the line segment as a continuous connection of points from a point P_1 to another point P_2. One can then denote this line segment using the notation $\overline{P_1P_2}$. The line segment can be extended from either end to lengthen the line segment.[1]

A directed line segment or vector, which is positive, is written $\overrightarrow{P_1P_2}$ has both a length and a direction in moving from point P_1 to point P_2. If $\overrightarrow{P_1P_2}$ is a directed line segment, then $\overrightarrow{P_2P_1} = -\overrightarrow{P_1P_2}$ because the direction from P_2 to P_1 is opposite that of moving from P_1 to P_2.

If the line segment created is moved up or down in a flat surface to create a square or rectangle, then **a plane section** is generated by the motion of the line. This is an example of a **two dimensional figure**. You can think of the plane section generated as defining a rectangle on a flat surface. The edges of the flat surface can be extended without any bounds to form an infinite flat two-dimensional surface

[1] For illustrative purposes the point and line illustrated above have a thickness and are not like the ideal point and line.

called an infinite plane. If the two-dimensional plane section is moved up or down perpendicular to the plane, then a **three-dimensional** figure is generated. If the plane section moved was a square, with side x, and it is moved upward a distance x, then the three dimensional object would be a cube.

Geometry is the investigation of the properties associated with points, lines, planes, and various shaped objects in one, two and three dimensions. Plane geometry restricts itself to two-dimensional objects and their properties, while solid geometry investigates objects in a three-dimensional space. In more advanced geometry courses one can investigate the goings on in four-dimensional and higher dimensional spaces. The point, line, square and cube are simple concepts used to start our study of plane geometry.

The Cartesian method

A Frenchman by the name of Rene′ Descartes (1546-1650) wrote a book called *LaGéométrie* which approached geometry using algebra and equations to represent various geometric concepts. The resulting new approach to geometry gave a geometric meaning to algebraic equations as well as an algebraic representation to geometric ideas. This new geometry is now called Cartesian geometry or analytic geometry. Descartes accomplished this change to geometry by introducing a rectangular system of coordinates which could be used to provide an algebraic approach to geometry.

Figure 1-2.
Rene′ Descartes
from Wikimedia Commons

You will find that analytical geometry or Cartesian geometry in its simplicity and ease of presentation is far superior to Euclid's method of writing things out and making things very long and wordy. We will refer to the Descartes method of doing things quite often in order to demonstrate the power of his representation of geometry. The Cartesian method is sometimes referred to as the **start of modern mathematics.**

One example illustrating the difference between the approach to geometry taken by Euclid and Descartes is the following. Euclid defines "point in the plane" as "that of which there is no part." Rene′ Descartes defined a point by constructing two straight lines which were perpendicular to one another. He created a coordinate system by labeling the horizontal line **the x-axis** and the vertical line **the y-axis**. He then labeled the point of intersection of the lines as **the origin** and placed number scales on the two perpendicular lines in order to measure distance from the origin. A point in this coordinate system is **a location** with respect to the origin and is labeled as a number pair[2] (x_0, y_0), called the **coordinates of the point** with respect to the origin. Here x_0 is called **the abscissa or distance from the y-axis** and y_0 is called **the ordinate or distance from the x-axis.** The resulting coordinate system used to define points is called **a Cartesian[3] coordinate system**, illustrated in the figure 1-3. Note that a point in either system is the representation of a location.

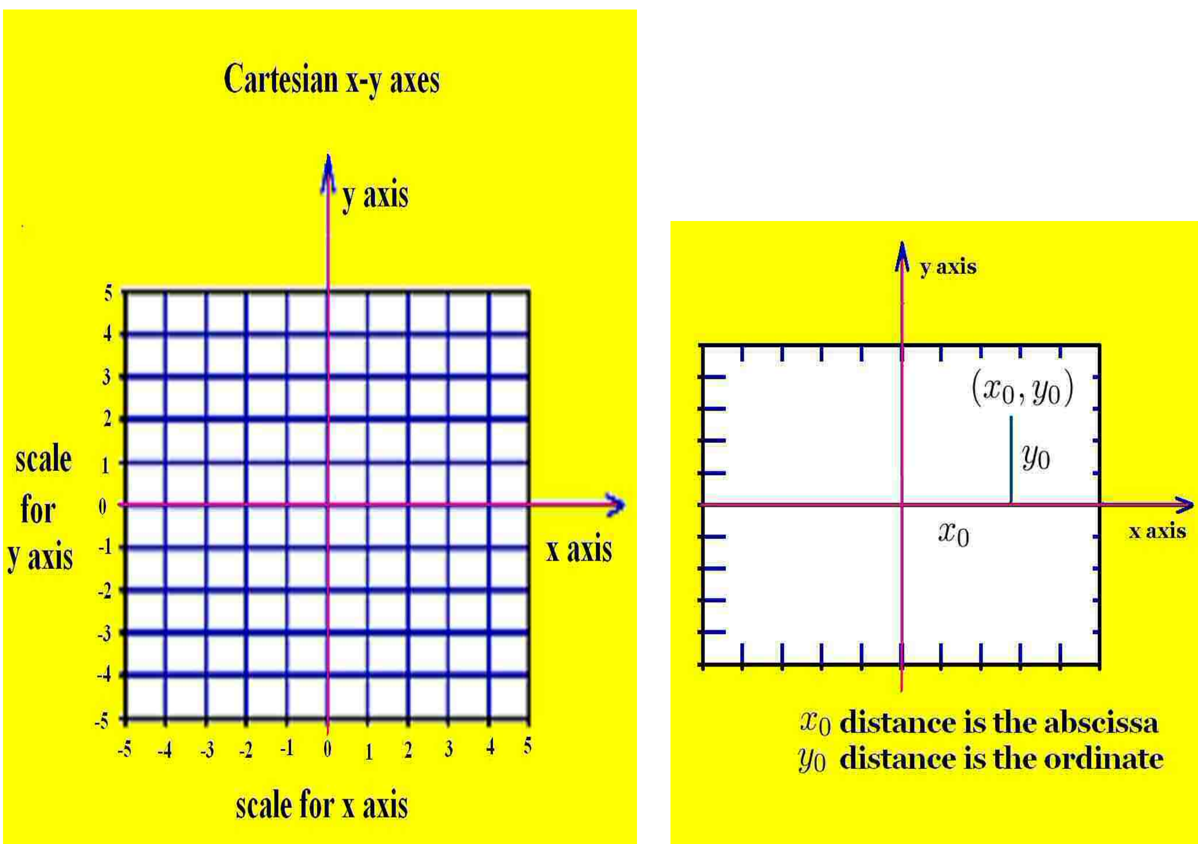


Figure 1-3. Cartesian coordinate system

[2] Subscripts placed upon letters are used to denote constant values.
[3] Named after Rene′ Descartes. Note that Cartesian is always spelled with a capital C.

In both the Euclid and Cartesian systems a point has **no size** only a **location**. **For illustrative purposes a point is sometimes represented as a dot (·) on a planar surface to emphasize where the location is.**

As another example of the difference between the Euclid and Descartes approach to geometry consider the following. Euclid defines a line as a length without breadth on a plane. Two distinct points P_1 and P_2 define a line on a planar surface. One can think of part of the line as a line segment obtain by point P_1 moving along the shortest distance to the point P_2. The points P_1 and P_2 are then called the endpoints of the line segment. The line segment thus created can then be extended beyond its endpoints to whatever length line you desire. For illustrative purposes a line can be sketched using a straight edged ruler with the line passing through two given points. The lines as shown in figure 1-4 have breadth or a line thickness and is for illustrative purposes only and differs from the ideal line constructed using an ideal point moving in a direction and having no breadth.

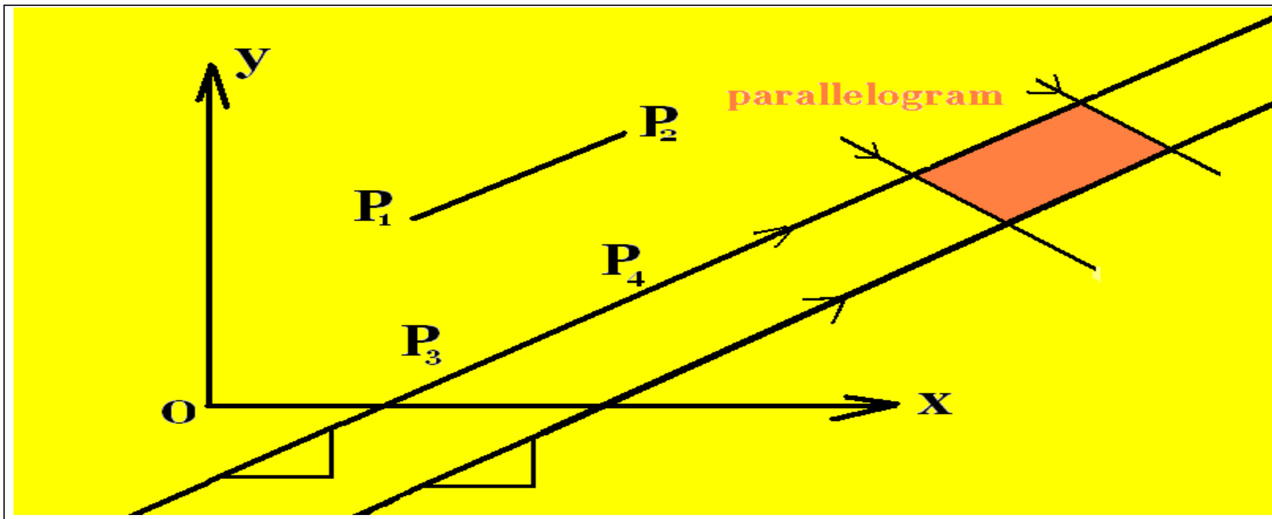


Figure 1-4. A line segment and parallel lines in Cartesian coordinates

Equation of line

The Descartes system allows one to replace the line by an **algebraic equation**. This is accomplished by defining the **slope** of the line passing through two given points having coordinates (x_1, y_1) and (x_2, y_2) as

$$Slope = m = \frac{change\ in\ y\ values}{change\ in\ x\ values} = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.1}$$

as illustrated in the figure 1-5. Note the slope of a line never changes. If (x, y) denotes **a variable point** which moves anywhere on the line, then it can be used together with the first point (x_1, y_1) to calculate the slope. One gets the equation

$$Slope = m = \frac{y - y_1}{x - x_1} = \frac{change\ in\ y\ values}{change\ in\ x\ values} \tag{1.2}$$

because **the slope of a line never changes,** no matter where the point (x, y) lies on the line. If the variable point (x, y) is used together with the second point on the line (x_2, y_2), one gets the equation

$$Slope = m = \frac{y - y_2}{x - x_2} = \frac{change\ in\ y\ values}{change\ in\ x\ values} \tag{1.3}$$

The equations (1.2) and (1.3) are known as the **point-slope formulas for the equation of a line through two given points.**

$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad y - y_2 = m(x - x_2), \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.4}$$

Also make note in figure 1-4 there is a set of **parallel lines**. In Cartesian coordinates lines are called **parallel** whenever their **slopes are the same** and **the perpendicular distance between the parallel lines remains constant**. Sometimes arrows are placed upon parallel lines to emphasize that they are parallel. Whenever two sets of parallel lines intersect **a parallelogram**[4] results.

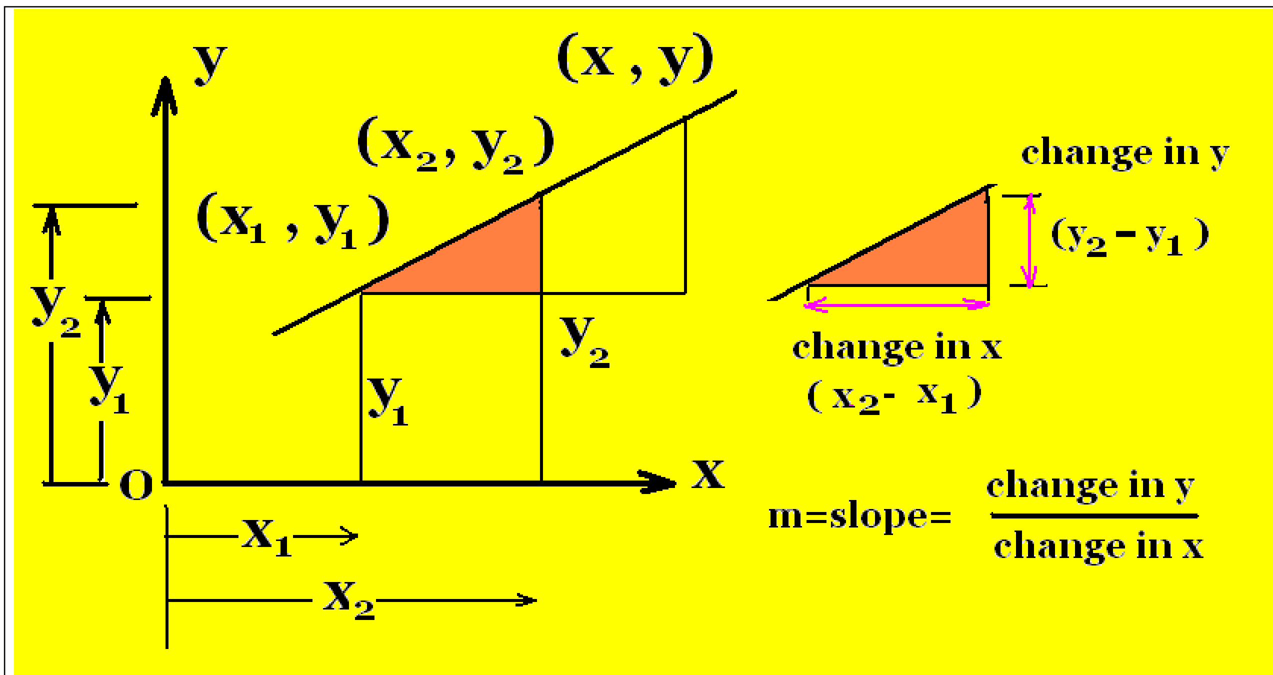


Figure 1-5. Slope of line through two given points (x_1, y_1) and (x_2, y_2)

[4] A parallelogram is a four sided closed figure where opposite sides are parallel and are of equal length.

It doesn't matter which point-slope formula you select in order to represent the line. The equations (1.4) are representations for the straight line through the given points (x_1, y_1) and (x_2, y_2). You now have an equation where you can substitute for x any numerical value and then solve for y which represents the height of the straight line above the x-axis for the x-value selected.

Example 1-1. Show that it does not matter which of the equations (1.4) are used to represent the equation for a line. For example, find the equation of the line which passes through the two points $(2, 1)$ and $(4, 5)$. Then determine if the points $(7, 11), (18, 33), (123, 243)$ and $(52, 100)$ are on the line.

Solution

The slope of the line through the given points is $m = \dfrac{\text{change in y-values}}{\text{change in x-values}} = \frac{5-1}{4-2} = 2$ The point-slope formula for the equation of the line is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is any point on the line. Selecting the point $(2, 1)$ as (x_1, y_1), the equation of the line is

$$y - 1 = 2(x - 2) \tag{1.5}$$

and selecting (x_2, y_2) as the point $(4, 5)$ the equation of the line is expressed in the form

$$y - 5 = 2(x - 4) \tag{1.6}$$

The equations (1.5) and (1.6) look different, but you can do some algebra and see they are really the same equation. As a check that you have the correct answer for the equation of the line you can substitute the point $(4, 5)$ for (x, y) into equation (1.5) and verify $x = 4$ and $y = 5$ **satisfies the derived equation (1.5).** One finds $(5 - 1) = 2(4 - 2)$ is true and so the point $(4, 5)$ is on the line.

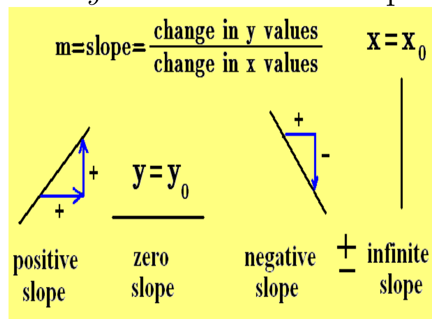
To test the other points that are given to see if they are on the line one must substitute into equation (1.5) the values (x, y) associated with the given points. If the equation (1.5) **is satisfied**, then an identity will result if **the point is on the line.** If the equation **does not produce an identity** when a set of x, y-values are substituted into the equation, then the equation (1.5) **is not satisfied**. In this case **the point is off the line**. Show that $x = 52,\ y = 100$ when substituted into equation (1.5) **does not satisfy the equation** and so the point $(52, 100)$ is off the line. The other given points all produce an identity when their values are substituted into equation (1.5) and when this happens the equation (1.5) is said to be satisfied and the point is said to be on the line.

■

Slopes of straight lines

Observe that in calculating the slope of a straight line **it is customary to always assume that the change in the x-value is always positive.** Pick a point on the line and move in the positive x–direction and stop. Next move in the y–direction either up for + or down for – in order to return to the line.

Here an examination of the change in y-values is then either (a) positive (b) zero or (c) negative. These changes give rise to lines with either positive, zero or negative slopes as illustrated in the figure on the right.



Straight lines passing through the point (x_0, y_0)

(a) with positive slopes have the form $\quad y - y_0 = +m_1(x - x_0)$

(b) with negative slopes have the form $\quad y - y_0 = -m_2(x - x_0)$

where m_1 and m_2 are non zero positive constants. Straight lines with zero slope are given by equations having the form $y = y_0 = a\ constant$. Vertical lines have a $\pm$ infinite slope and are represented by equations having the form $x = x_0 = a\ constant$.

Other forms for a straight line

Equations having one of the forms

$$
\begin{aligned}
&(a) \quad Ax + By = C \qquad A, B, C \text{ constants}\\
&(b) \quad y - y_0 = m(x - x_0) \qquad m, x_0, y_0 \text{ constants}\\
&(c) \quad y = mx + b \qquad m, b \text{ constants}\\
&(d) \quad \frac{x}{a} + \frac{y}{b} = 1 \qquad a, b \text{ constants}\\
&(e) \quad x = x_0 + a_1 t, \quad y = y_0 + a_2 t, \quad t_0 < t < t_1 \quad t_0, t_1, x_0, y_0, a_1, a_2 \text{ constants}
\end{aligned}
\tag{1.7}
$$

are all algebraic representations for a straight line or a segment of a straight line in Cartesian coordinates.

Equations having the form (a) above are referred to as the **general form** for a straight line. Equations having the form (b) above are known as the **point-slope form** for the equation of a line. The form (c) above is known as the **slope-intercept form** and the form (d) is known as the **intercept form** for a straight line. The form (e), with parameter t, is called **a parametric representation** of a straight line. If the parameter t varies only between certain values, say $t_0 \le t \le t_1$, then only **a segment of the line** is represented. Also make note of the fact that the **parametric form for a**

line is not unique. For example, the line $3x+4y=7$ can be represented in parametric form just about any way you want. Three possible forms are (i) $x=7t, y=\frac{1}{4}(7-21t)$ (ii) $x=3t-1, y=\frac{1}{4}(10-9t)$, (iii) $x=f(t),\ y=\frac{1}{4}(7-3f(t))$ where $f(t)$ is any function of t you want it to be.

In order to use the form (b) above you have to calculate the slope m of the line and know a point (x_0,y_0) on the line. Note that in the form (c) when $x=0$, then $y=b$ which is called the y-intercept, hence the name for the equation. Setting $x=0$ in the form (d) one finds $y=b$, the y-intercept and then setting $y=0$ one finds $x=a$, the x-intercept, hence the name associated with the equation. If one uses algebra and eliminates the parameter t in form (e) one obtains the point slope equation of a line. For example,

$$\frac{x-x_0}{a_1}=t=\frac{y-y_0}{a_2} \quad \text{or} \quad y-y_0=m(x-x_0) \quad \text{with slope } m=\frac{a_2}{a_1}$$

Also observe that the general form for a straight line $Ax+By=C$, with A,B,C constants, can be converted to any of the other forms using a little algebra. For example,

$$Ax+By=C \;\Rightarrow\; \frac{A}{C}x+\frac{B}{C}y=1 \;\Rightarrow\; \frac{x}{\left(\frac{C}{A}\right)}+\frac{y}{\left(\frac{C}{B}\right)}=1 \text{ intercept form}$$

$$Ax+By=C \;\Rightarrow\; y=\frac{-A}{B}x+\frac{C}{B} \text{ slope-intercept form}$$

If (x_0,y_0) is a point on the line, then (x_0,y_0) are coordinates which satisfies the equation of the line or $Ax_0+By_0-C=0$. One can then write

$$Ax+By=C \;\Rightarrow\; A(x-x_0)+B(y-y_0)=C-Ax_0-By_0=0 \;\Rightarrow\; y-y_0=\frac{-A}{B}(x-x_0) \text{ point-slope form}$$

Substitute the value $x=x_0+a_1t$ into the general equation of a line to obtain the equation $A(x_0+a_1t)+By=C$ and then solve for y to obtain

$$y=\frac{-a_1A}{B}t+\left(\frac{C-Ax_0}{B}\right) \quad \Rightarrow\; y=a_2t-y_0,\ \text{where } a_2=\frac{-a_1A}{B} \text{ and } y_0=\frac{Ax_0-C}{B}$$

This is one of many parametric forms for the line.

Example 1-2. Find the equation of the line which passes through the points $(0,5)$ and $(7,0)$.

Solution

The given points represent the y-intercept and x-intercepts for the equation of the line. Using the intercept form one can write the equation of the line as

$$\frac{x}{7}+\frac{y}{5}=1$$

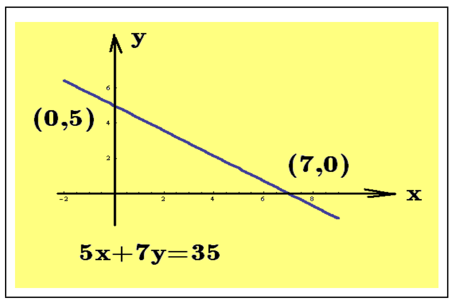
Using algebra one can write

$$\frac{x}{7} + \frac{y}{5} = 1 \;\Rightarrow\; 5x + 7y = 35$$

as the general form for representing the line. Alternative representations are

$$5x + 7y = 35 \;\Rightarrow\; \frac{y}{5} = 1 - \frac{x}{7} \;\Rightarrow\; y = \left(\frac{-5}{7}\right) x + 5$$

which is the slope-intercept form with slope $m = -5/7$ and y-intercept $y = 5$.



Using the slope $m = -5/7$ and point $(0,5)$ the point-slope formula gives $y - 5 = (-5/7)(x - 0)$ or using the slope $m = -5/7$ and point $(7,0)$, the point-slope formula gives $y - 0 = (-5/7)(x - 7)$. It is left as an exercise to verify that the last two equations are equivalent to one another.

■

Example 1-3. Let the endpoints of the line segment $\overline{\text{AB}}$ be $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Find the parametric equation representing this line segment.

Solution:

One can express the points on the line segment $\overline{\text{AB}}$ as a set of values defined by

$$\overline{\text{AB}} = \{\, (x,y) \;\mid\; x = (1-t)x_1 + t\,x_2,\;\; y = (1-t)y_1 + t\,y_2,\;\; 0 \le t \le 1 \,\}$$

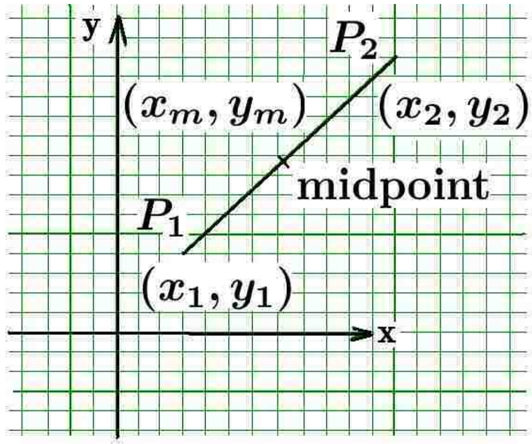
■

The midpoint of a line segment

In Cartesian coordinates consider two points P_1 and P_2 having coordinates (x_1, y_1) and (x_2, y_2) respectively. Draw a straight line connecting these points to form the line segment $\overline{P_1P_2}$. The midpoint (x_m, y_m) of the line segment $\overline{P_1P_2}$ is obtained by taking **the average** of the x-values and the average of the y-values to obtain

$$x_m = \frac{1}{2}(x_1 + x_2), \qquad y_m = \frac{1}{2}(y_1 + y_2)$$

This gives the midpoint of the line segment $\overline{P_1P_2}$ which is the point halfway between the points P_1 and P_2.



Example 1-4.

Find the midpoint of the line segment connecting the points $(-3, 5)$ and $(5, -1)$

Solution

Let (x_m, y_m) denote the midpoint of the line segment connecting the given points. The midpoint is obtained by taking the average of the x-values and y-values to obtain

$$x_m = \frac{1}{2}(-3 + 5) = 1 \quad \text{and} \quad y_m = \frac{1}{2}(5 - 1) = 2$$

so that $(x_m, y_m) = (1, 2)$ is the midpoint of the line segment. ■

The Greek alphabet

Letters from the Greek alphabet are used quite often in representing mathematical concepts and are listed below for future reference.

Greek Alphabet

A	α	Alpha	H	η	Eta	N	ν	Nu	T	τ	Tau
B	β	Beta	Θ	θ	Theta	Ξ	ξ	Xi	Υ	υ	Upsilon
Γ	γ	Gamma	I	ι	Iota	O	o	Omicron	Φ	ϕ	Phi
Δ	δ	Delta	K	κ	Kappa	Π	π	Pi	X	χ	Chi
E	ϵ	Epsilon	Λ	λ	Lambda	P	ρ	Rho	Ψ	ψ	Psi
Z	ζ	Zeta	M	μ	Mu	Σ	σ	Sigma	Ω	ω	Omega

Proportion

Any equation having the form $\frac{a}{b} = \frac{c}{d}$, which states that two ratios are equal, is called a proportion. Many times in solving a proportion the **cross product property**

$ad = bc$ is often used. The above proportion can also be written in the form $a : b = c : d$. One can write that if

$$\frac{a}{b} = \frac{c}{d} \quad \text{then} \quad ad = bc \tag{1.8}$$

A proportion such as the one given in equation (1.8) must sometimes be altered. This can be accomplished **by selecting non zero constants** λ, μ and forming the new ratio

$$\frac{\lambda a + \mu c}{\lambda b + \mu d} = \frac{a}{b} = \frac{c}{d} \tag{1.9}$$

This new ratio equals the old ratios because of the cross product property. That is

$$\frac{\lambda a + \mu c}{\lambda b + \mu d} = \frac{a}{b} \quad \Rightarrow \quad (\lambda a + \mu c) \cdot b = (\lambda b + \mu d) \cdot a \quad \text{is an identity by algebra}$$
$$\lambda ab + \mu cb = \lambda ba + \mu da = \lambda ab + \mu cb$$

because the cross product $da = cb$ holds from equation (1.8).

Example 1-5. Prove that if the following ratios are true $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$, then

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \frac{a+b+c}{A+B+C}$$

Solution

If $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = r$, then

$$a = rA, \quad b = rB, \quad c = rC$$

so by addition

$$(a+b+c) = r(A+B+C) \Rightarrow \quad r = \frac{a+b+c}{A+B+C} = \frac{a}{A} = \frac{b}{B} = \frac{c}{C}$$

Alternatively, if $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$, then use equation (1.9) with $\lambda = 1$ and $\mu = 1$ to show

$$\frac{\lambda a + \mu b}{\lambda A + \mu B} = \frac{a}{A} \quad \Rightarrow \quad \frac{a+b}{A+B} = \frac{a}{A} = \frac{b}{B} = \frac{c}{C} \tag{1.10}$$

Now repeat what has just been done using the equation (1.9) with the new ratio (1.10). Write

$$\frac{\lambda(a+b) + \mu c}{\lambda(A+B) + \mu C} = \frac{a}{A} = \frac{b}{B} = \frac{c}{C}$$

with $\lambda = 1$ and $\mu = 1$ to obtain the desired result.

■

Mean proportion

The **mean proportion** of two quantities a and b, also called the **geometric mean** of two quantities, is defined

$$\text{geometric mean} = \text{mean proportion} = \sqrt{ab}$$

The quantity x is called the **mean proportion** of a and b if $\frac{a}{x} = \frac{x}{b}$ or $x^2 = ab$ or $x = \sqrt{ab}$.

The **geometric mean or mean proportion** can be generalized.

Geometric mean of three numbers x_1, x_2, x_3 is $\sqrt[3]{x_1 x_2 x_3}$

Geometric mean of four numbers x_1, x_2, x_3, x_4 is $\sqrt[4]{x_1 x_2 x_3 x_4}$

Geometric mean of five numbers x_1, x_2, x_3, x_4, x_5 is $\sqrt[5]{x_1 x_2 x_3 x_4 x_5}$

and the pattern continues as the number of products increase.

The **arithmetic mean of n numbers** is obtained by **adding the n numbers** and then **dividing the sum by n** giving

$$\text{Arithmetic mean } = \bar{x}_a = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \tag{1.11}$$

The arithmetic mean answers the question, "If all n numbers had the same value, what would this value be to **obtain the same sum?**"

The geometric mean of n numbers is obtained by **multiplying the n numbers** and then taking the n**th root of the product** giving

$$\text{Geometric mean} = \bar{x}_g = \sqrt[n]{x_1 x_2 x_3 \cdots x_n} \tag{1.12}$$

The geometric mean is the answer to the question, "If all n numbers had the same value, what would this value be to obtain **the same product?** "

Numbers

Various special types of numbers are employed in the study of geometry. Many advanced number theoretic concepts assume you know about these special types of numbers. Remember that the ancient Greeks and Egyptians had number systems which were not ideal systems for performing mathematical calculations and consequently many geometric methods resulted for solving mathematical problems. It wasn't until about the 15th century that our current number system began to be accepted by Europeans.

Integers

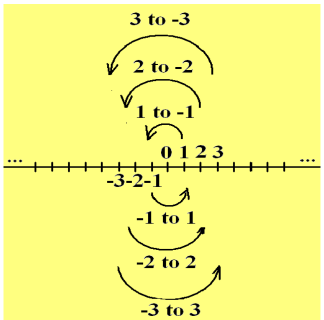


Figure 1-6.
Number axes illustrating multiplication by -1 effect on numbers

Everyone is familiar with the positive integers $0, 1, 2, 3, \ldots$. The notation $\ldots$ after the number 3 is called **an ellipsis** and its use is to mean that the representation is to be continued with $4, 5, 6,$ *etc* following the 3. The ellipsis is a mathematical notation indicating continuation. The positive numbers can be written on a line and the line can be called the positive number axes. One can define multiplication of these numbers by negative 1 as the **counterclockwise rotation** of the positive number axis as half of a full rotation of the axis about the origin (0). This produces a negative number axes with the origin 0 separating the negative numbers from the positive numbers.

That is, each time a number is multiplied by -1, **a counterclockwise rotation occurs equal to half of a full rotation**. This is illustrated in figure 1-6. Note that if a negative number is multiplied by -1, one must perform a counterclockwise half of a full rotation. Thus $(-1)(-1) = 1$.

Prime numbers

A prime number is divisible only by itself and the number 1. The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, \ldots$. Euclid showed that there is no largest prime number so an infinite number of primes exist. Goldbach's[5] conjecture is that all positive even integers greater than two can be expressed as the sum of two primes. This conjecture remains unproven and waits for someone to prove it. Computers have verified the conjecture up to very large numbers. However, this is not a proof for all even numbers greater than two.

Rational numbers

Let m, n denote positive integers with n different from zero, then numbers having the form of a ratio $\frac{m}{n}$ are known as **rational numbers.** One can construct a table of rational numbers as illustrated in the figure 1 - 7. Note the ellipsis symbols denoting

[5] Christian Goldbach (1690-1764) a German mathematician.

that the given patterns are to be continued. The set of rational numbers have the following properties.

(1) An **integer** can result. For example $5/1$ is simplified to 5

(2) A **terminating decimal** can result as $1/4 = 0.25$

(3) An **infinite recurring decimal** can result as $7/11 = 0.6363636363\ldots$

One mathematical notation for representing a recurring decimal is to place a line over the repeating part. This gives $7/11 = 0.\overline{63}$

n \ m	1	2	3	4	5	6	7	8	9	10	11	12	13	...
1	1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1	9/1	10/1	11/1	12/1	13/1	...
2	1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	9/2	10/2	11/2	12/2	13/2	...
3	1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3	9/3	10/3	11/3	12/3	13/3	...
4	1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4	9/4	10/4	11/4	12/4	13/4	...
5	1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5	9/5	10/5	11/5	12/5	13/5	...
6	1/6	2/6	3/6	4/6	5/6	6/6	7/6	8/6	9/6	10/6	11/6	12/6	13/6	...
7	1/7	2/7	3/7	4/7	5/7	6/7	7/7	8/7	9/7	10/7	11/7	12/7	13/7	...
8	1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8	9/8	10/8	11/8	12/8	13/8	...
9	1/9	2/9	3/9	4/9	5/9	6/9	7/9	8/9	9/9	10/9	11/9	12/9	13/9	...
10	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	10/10	11/10	12/10	13/10	...
11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11	11/11	12/11	13/11	...
12	1/12	2/12	3/12	4/12	5/12	6/12	7/12	8/12	9/12	10/12	11/12	12/12	13/12	...
13	1/13	2/13	3/13	4/13	5/13	6/13	7/13	8/13	9/13	10/13	11/13	12/13	13/13	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Figure 1-7. Partial listing of rational numbers m/n

Given a number represented by an infinite **repeating decimal** one can employ basic mathematics to express the number in the form of a ratio m/n, where m and n are integers.

Example 1-6. Find the rational number R having the decimal expansion

$$R = 0.5123123123123\ldots \text{ or } R = 0.5\overline{123}$$

Solution Just use multiplication and subtraction on the given number to convert it into the form of a ratio of two integers. Multiply R by 10 to get

$$10\,R = 5.123123123\ldots = 5 + 0.\overline{123} \tag{1.13}$$

Next multiply equation (1.13) by $10^3 = 1000$ and find

$$10^4\,R = 5123.123123123\ldots = 5123 + 0.\overline{123} \tag{1.14}$$

The decimal part of the numbers in equations (1.13) and (1.14) are now the same and so can be removed by using elementary algebra. One finds

$$10^4R - 10R = 5123 - 5 \quad \text{or} \quad R = \frac{5118}{10^4 - 10} = \frac{5118}{9990} = \frac{853}{1665}$$

■

Irrational numbers

An example of an irrational number that arises quite often can be found by studying circles. A circle is defined as the set of points which are equidistant from a fixed point called the **center** of the circle. This set of points forms a curve around the center called the **circumference** or **boundary** of the circle. The distance from the center of the circle to the boundary of the circle is called the **radius** of the circle. The plural of radius is radii and one can say that all the radii of the same circle are equal. A line from a boundary point through the center of the circle to an opposite boundary point is called the **diameter** of the circle. Opposite points at the ends of a diameter are called **antipodal points**. **Concyclic** points are points all lying on the same circumference of the circle.

An **irrational number** is a real number that **cannot be expressed as the ratio of two numbers.** Properties of irrational numbers are

(a) They have an **infinite number of decimal digits.**

(b) There are **no repetitious or repeating patterns** in the representation of the decimal digits

Examine the figure 1-8 where two circles of different sizes are displayed. Imagine the left circle being magnified to produce the right circle. All the dimensions of the left circle will be increased proportionally and one can say the circles are similar in that the dimensions are proportional.

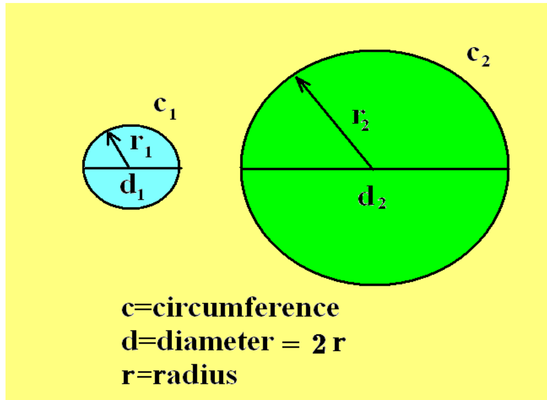


Figure 1-8.
Two circles are similar with proportional dimensions

If c_1, c_2 denote the circumferences of the two circles, and d_1, d_2 denote their diameters, then one can say that the circles are related in that the ratios of their respective parts are the same because of **scaling.** One finds the ratios

$$\frac{d_1}{d_2} = \frac{c_1}{c_2} \quad \text{or} \quad \frac{c_1}{d_1} = \frac{c_2}{d_2} = \frac{\text{circumference}}{\text{diameter}} = \text{constant}$$

This tells us the ratio of circumference divided by diameter of every circle will be a constant. This constant value turns out to be an **irrational number** which is represented by the Greek letter π and is found[6] to have the approximate value

$$\pi = 3.14159265369\ldots \tag{1.15}$$

A calculation for the number π will be given in chapter 6.

No finite number of digits can represent π exactly. The ellipsis ... is used to emphasize the **number of decimal digits continue without end and there are no repeating patterns in the representation of the digits**.

Note that the diameter (d) of the circle is twice the radius (r), so that one can write $d = 2r$. The ratio $\frac{\text{circumference of circle}}{\text{diameter of circle}} = \pi$ implies $c/2r = \pi$ or the

[6] Modern day mathematicians have devised a method to determine the nth digit of π for any integer n without having to know any of the previous digits. The technique is known as the Bailey-Borwein-Plouffe formula for calculating the nth binary digit of π using base 16 mathematics.

circumference (c) of any circle is given by the relation $c = 2\pi r$. In summary, a circle with radius r has the following properties.

$$\begin{aligned} &\textbf{diameter is twice the radius} \quad d = 2r \\ &\textbf{circumference is } 2\pi \textbf{ times the radius} \quad c = 2\pi r \\ &\text{or } \textbf{circumference is } \pi \textbf{ times the diameter} \quad c = \pi d \end{aligned} \tag{1.16}$$

Square root of 2 is an irrational number

To demonstrate that the square root of 2 is an irrational number one can proceed as follows. Assume that the square root of 2 is a rational number and then show that this assumption is not true. This type of proof is known as **Reductio ad absurdum** which comes from the Latin and translates to the phrase–reduction to absurdity.

Assume the square root of 2 is a rational number and write $\sqrt{2} = \frac{p}{q}$, where p and q are **coprime numbers**. This means that the integers p and q have **no common factors** and the ratio $\frac{p}{q}$ is irreducible. One can write

$$\begin{aligned} \sqrt{2} =& \frac{p}{q} \\ \text{square both sides} \qquad 2 =& \frac{p^2}{q^2} \\ \text{multiply both sides by } q^2 \qquad 2q^2 =& p^2 \end{aligned} \tag{1.17}$$

Now even integers all have the form $2n$ where n is an integer and all odd integers have the form $(2m+1)$ where m is an integer. This implies that for all integers m and n, an even number times an even number results in an even number since $(2n)(2m) = 2(2mn)$ is even. Also an odd number times an odd number will always results in an odd number since, $(2n+1)(2m+1) = (4mn + 2(n+m) + 1)$ represents an even number plus one, which is an odd number. The equation (1.17) implies p^2 is an even number and therefore p must be an even number and have the form $p = 2m$, where m is some integer. The equation (1.17) can therefore be written

$$2q^2 = p^2 = (2m)^2 = 4m^2 \quad \text{or} \quad q^2 = 2m^2, \quad q^2 \text{ is even and so } q \text{ is even}$$

This gives the ratio $\frac{p}{q} = \frac{even}{even}$ so that the ratio $\frac{p}{q}$ has a common factor which **contradicts the original assumption that the ratio had no common factor**. Therefore, the original assumption is false and $\sqrt{2}$ cannot be a rational number. Consequently, it is an irrational number.

One finds the **approximate decimal form** for the square root of 2 can be represented $\sqrt{2} = 1.41421356\ldots$ or $\sqrt{2} \approx 1.414$ where again the ellipsis is used to signify that the **decimal part goes on forever** and the symbol $\approx$ represents the symbol for approximately. In dealing with irrational numbers like the square root of 2, always use $\sqrt{2}$ rather than $1.4142\ldots$ or the approximate value 1.414, because $\sqrt{2}$ is a symbol representing the exact value, whereas the truncated decimal $1.4142\ldots$ or 1.414 representations only give an approximate value for the square root. The square root of 2 symbol $\sqrt{2}$ is to be treated as a symbol in the same sense as the previous symbol π is used. That is, always use the symbol to represent the exact value of the irrational number because irrational numbers are always associated with **never ending decimals**.

The golden ratio

Two positive quantities x and y with $x > y$, are said to be in a **golden ratio** ϕ if

$$\phi = \frac{x}{y} = \frac{x+y}{x} \tag{1.18}$$

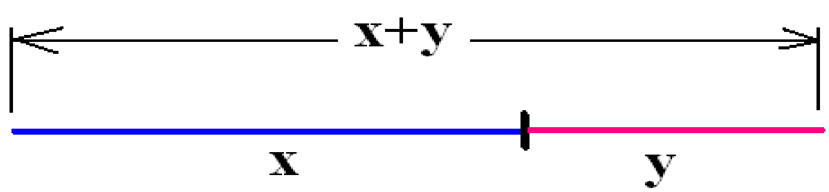


Figure 1-9. Golden ratio for line segments

That is, the ratio of the sum of the two numbers $(x+y)$ to the larger number x is in the same ratio as the larger quantity x over the smaller quantity y. Observe that,

$$\phi = \frac{x}{y} = \frac{x+y}{x} = 1 + \frac{y}{x} = 1 + \frac{1}{\phi} \tag{1.19}$$

which requires that ϕ satisfy the quadratic equation

$$\phi^2 - \phi - 1 = 0 \tag{1.20}$$

Solving the quadratic equation (1.20) for the **positive root**, one finds that the golden ratio ϕ is an irrational number having the value

$$\phi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots \tag{1.21}$$

The golden ratio is found in many ancient texts under various alternative names such as golden rectangle[7], golden mean, golden number, golden proportion, divine proportion, golden section, and divine section. The golden ratio is an irrational number and can be found in many areas of mathematics, art, architecture, music, nature and geometry. It has even been used to predict highs and lows of stock prices.

Numbers with a different base

The base 10 number system, which we use in everyday applications of mathematics, means that the numbers we use are **all simplifications** of numbers having the form of **a summation of terms involving powers of ten.** This summation has the form

$$\cdots + \_\_(10)^3 + \_\_(10)^2 + \_\_(10)^1 + \_\_(10)^0 + \_\_(10)^{-1} + \_\_(10)^{-2} + \_\_(10)^{-3} + \cdots \tag{1.22}$$

where you fill in each of the blank spaces ($\_\_$) using one of the integers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. It is to be understood that **all leading and trailing zeros are not written down and the powers of 10 are not written down.** Also note that $(10)^0 = 1$. For example, the base ten number 872.13 is represented base 10 as

$$872.13 = 872.13_{10} = 872.13]_{10} = 8(10)^2 + 7(10)^1 + 2(10)^0 + 1(10)^{-1} + 3(10)^{-2} \tag{1.23}$$

where the notation $]_{10}$ is used to emphasize it is a base 10 number. Sometimes a subscript 10 is placed after the number, as 872.13_{10}, to emphasize the number is base 10. Most of the time the notation $]_{10}$ or subscript 10 after the number, is dropped because everyone knows the base 10 is being used. When dealing with numbers with different bases the bracket or subscript notation is preferred.

A **base 2 number system**, also called a **binary system**, occurs when all the tens in equation (1.22) are **replaced by the number 2**. One then obtains a summation of terms involving powers of two having the form

$$\cdots + \_\_(2)^3 + \_\_(2)^2 + \_\_(2)^1 + \_\_(2)^0 + \_\_(2)^{-1} + \_\_(2)^{-2} + \_\_(2)^{-3} + \cdots \tag{1.24}$$

and you fill in the blanks ($\_\_$) using only one of the digits 0 or 1 and $2^0 = 1$. Note a base 2 number system uses only two symbols 0 and 1. Again **all leading and trailing zeros and the powers of 2 are not written down.** Below are some base 2 number representations along with their base 10 equivalents.

[7] The golden rectangle is any rectangle where the ratio of length divided by width equals $\phi = \frac{1+\sqrt{5}}{2}$.

$$
\begin{aligned}
1]_2 =& 1_2 = 1(2)^0 = 1\\
10]_2 =& 10_2 = 1(2)^1 + 0(2)^0 = 2\\
11]_2 =& 11_2 = 1(2)^1 + 1(2)^0 = 3\\
100]_2 =& 100_2 = 1(2)^2 + 0(2)^1 + 0(2)^0 = 4\\
101]_2 =& 101_2 = 1(2)^2 + 0(2)^1 + 1(2)^0 = 5\\
1111]_2 =& 1111_2 = 1(2)^3 + 1(2)^2 + 1(2)^1 + 1(2)^0 = 15\\
10101]_2 =& 10101_2 = 1(2)^4 + 1(2)^2 + 1(2)^0 = 21
\end{aligned}
\qquad
\begin{aligned}
1000]_2 =& 1000_2 = 1(2)^3 = 8\\
10000]_2 =& 10000_2 = 1(2)^4 = 16\\
100110100100_2 =& 2468\\
10101001111_2 =& 1359\\
11.1010101110000101_2 =& 3.67\\
1001.001_2 =& 9.125\\
1001.01_2 =& 9.25\\
1001.1_2 =& 9.5
\end{aligned}
$$

A base 8 or **octal number system** involves powers of 8 and has the form

$$
\cdots + \_\_(8)^3 + \_\_(8)^2 + \_\_(8)^1 + \_\_(8)^0 + \_\_(8)^{-1} + \_\_(8)^{-2} + \_\_(8)^{-3} + \cdots \tag{1.25}
$$

where the integers going into the blank ($\_\_$) spaces are restricted to the eight digits $0, 1, 2, 3, 4, 5, 6, 7$. Some sample numbers from the base eight number system are the following.

$$
\begin{aligned}
1_8 =& 1\\
2_8 =& 2\\
3_8 =& 3\\
4_8 =& 4\\
5_8 =& 5
\end{aligned}
\qquad
\begin{aligned}
6_8 =& 6\\
7_8 =& 7\\
10_8 =& 8\\
11_8 =& 9\\
12_8 =& 10
\end{aligned}
\qquad
\begin{aligned}
24_8 =& 20\\
144_8 =& 1(8)^2 + 4(8)^1 + 4(8)^0 = 100\\
175.1_8 =& 1(8)^2 + 7(8)^1 + 5(8)^0 + 1(8)^{-1} = 125.125\\
175.01_8 =& 125.015625\\
11.11_8 =& 9 + .125 + .015625 = 9.140625
\end{aligned}
$$

In general, **a base b number system**, b an integer, occurs when you replace all the tens in equation (1.22) with the integer b everywhere to obtain

$$
\cdots + \_\_(b)^3 + \_\_(b)^2 + \_\_(b)^1 + \_\_(b)^0 + \_\_(b)^{-1} + \_\_(b)^{-2} + \_\_(b)^{-3} + \cdots \tag{1.26}
$$

and **restrict the integers going into the blank** ($\_\_$) **spaces to the b-digits**

$$
0, 1, 2, 3, 4, \ldots, 8, 9, \ldots, (b-1)
$$

Note that **a base b number system needs b-digits** 0 through $b-1$. This means that if 0 through 9 are the first 10 digits, then **you have to make up symbols to represent the digits** 10 **through** $(b-1)$**.** Consider the base 16 number system, often called a **hexadecimal number system**. This system needs sixteen symbols to represent the digits. It is customary to define the 16 digits required as 0,1,2,3,4,5,6,7,8,9 and then define the symbols $A = 10,\ B = 11,\ C = 12,\ D = 13,\ E = 14,\ F = 15.$ In equation (1.22) replace all the tens by 16 to obtain

$$\cdots + \_\_(16)^3 + \_\_(16)^2 + \_\_(16)^1 + \_\_(16)^0 + \_\_(16)^{-1} + \_\_(16)^{-2} + \_\_(16)^{-3} + \cdots \quad (1.27)$$

and use the digits for the blank ($\_\_$) spaces from the set of 16-digits

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$$

and again make note of the fact that all leading and trailing zeros and powers of 16 are not written down. Some example number representations in the hexadecimal system are given below.

$$18A_{16} = 1(16)^2 + 8(16)^1 + 10(16)^0 = 394 \qquad DE_{16} = 13(16)^1 + 14(16)^0 = 222$$

$$EA_{16} = 14(16)^1 + 10(16)^0 = 234 \qquad 45DB_{16} = 4(16)^3 + 5(16)^2 + 13(16)^1 + 11(16)^0 = 17883$$

$$17_{16} = 1(16)^1 + 7(16)^0 = 23 \qquad B.999A_{16} = 11.6$$

$$33_{16} = 51 \qquad B.1_{16} = 11.0625$$

$$8F_{16} = 143 \qquad B.01_{16} = 11.00390625$$

Example 1-7.

Convert the numbers 431 and 432 to a hexadecimal representation.

Solution

Divide the given numbers by 16 and keep track of the remainders R as illustrated below.

$$16\,\overline{)431}^{\;26}, \quad R = 15 = F \qquad\qquad 16\,\overline{)432}^{\;27}, \quad R = 0$$

$$16\,\overline{)26}^{\;1}, \quad R = 10 = A \qquad\qquad 16\,\overline{)27}^{\;1}, \quad R = 11 = B$$

$$1AF = 1(16)^2 + 10(16)^1 + 15(16)^0 = 431 \qquad 1B0 = 1(16)^2 + 11(16)^1 + 0(16)^0 = 432$$

Here one takes the last quotient together with the remainders to form the base 16 number. The technique behind this is to list the powers of 16 and then determine the highest power of 16 that you can subtract from the given number. Repeat this process after subtraction. Here one substitutes zero if you skip a power of 16.

■

The representation of numbers in different base number systems can vary in the symbols used and the way the symbols are interpreted. For example, the ancient Summerians[8] used a **sexagesimal number system.** Their representation of numbers was much different from our present day representation of numbers.

The Mayan's used a base 20 number system. Computers were developed using binary mathematics. Mathematicians have even experimented using non integer base numbers like π and square root of 2. A base 12 number system is said to be much more interesting than a base 10 number system because 12 has more divisors than 10.

The following is a list of names associated with various **number base systems.**

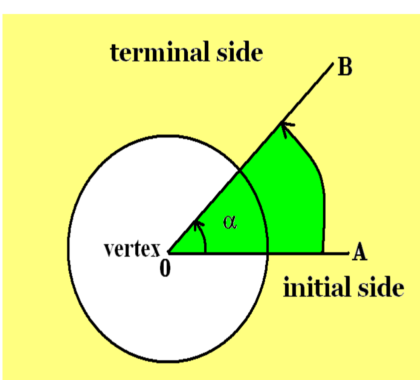
base b	Name of number base
2	binary
3	trinary or ternary
4	quaternary
5	quinary
6	senary
7	septenary
8	octal or octonary
9	nonary
10	decimal

base b	Name of number base
11	undenary
12	duodecimal
13	tridecimal
14	quattuordecimal
15	quindecimal
16	hexadecimal or sexadecimal
17	septendecimal
18	octodecimal
19	nonadecimal
20	vigesimal

base b	Name of number base
30	trigesimal
40	quadragesimal
50	quinquagesimal
60	sexagesimal
70	septagesimal
80	octagesimal
90	nonagesimal
100	centimal or centesimal
200	bicentimal
300	tercentimal
400	quatrrocentimal
500	quincentimal

List of names for various number base systems

Angular measure



Any line having only one endpoint and extending forever, is called **a ray.** Whenever **two rays have the same endpoint**, then an angle is formed between the rays and the common endpoint of the rays is called **a vertex.** Draw a circle and then draw a line from the center of the circle horizontally to the right to create a ray called the initial side of the angle. Now rotate the ray **counterclockwise** about the center point of the circle and stop and call the rotated ray the terminal side of the angle.

[8] The Summerian civilization developed in the Tigris-Euphrates valley and existed from 5000 BCE to approximately 1700 BCE. The Summerian notation for the number 64 was 1.4 representing one sixty plus four.

The rotated ray sweeps out and defines **an angle of rotation about the center of the circle.** This angle of rotation is illustrated in the figure above and labeled using the Greek symbol alpha (α). The angle of rotation can also be represented using the notation $\angle A0B$ which represents the angle obtained in moving from points A to 0 to B. The center point 0 of the rotation is called the vertex of the angle and the terminal side and initial side rays are called **the sides of the angle.** The measure of an angle α is denoted $m\angle\alpha$. For example, $m\angle\alpha = 30^\circ$, which is read, "The measure of angle α is 30 degrees, where the unit of degrees is defined as follows.

The degree (deg)

The ancient Summerians from about 2500 BCE considered the special case of the angle created when **the initial side made one complete rotation about the center point.** This created a situation where the terminal side lay atop the initial side. The resulting angle was then divided into 360 parts[9] where each part was called one degree. The degree is denoted by the mathematical symbol $^\circ$ and there are 360 degrees in a circle or one revolution equals 360°. The figure 1-10 illustrates several selected **positive angles of rotation in the positive counterclockwise direction.** Also illustrated are selected **negative angles of rotation in the clockwise direction.** One complete rotation of the initial side about the circle center denotes the angle $\pm 360^\circ$ depending upon the direction of the rotation.

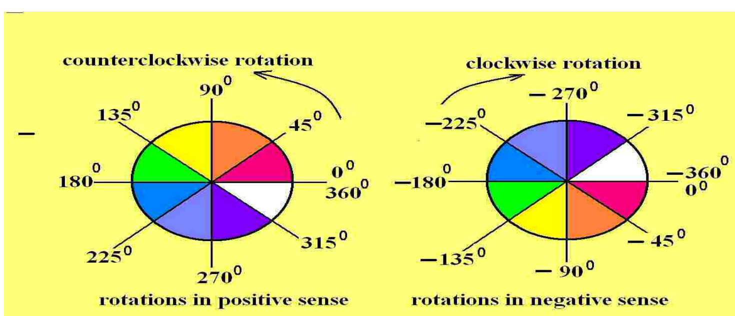


Figure 1-10. Selected angles of rotation about center of circle

The degree is further subdivided into units called "minutes" and "seconds" where

$$60 \text{ minutes} = 1 \text{ degree} \quad \text{and} \quad 60 \text{ seconds} = 1 \text{ minute} \tag{1.28}$$

9 It is not known why the number 360 was selected for the circular division. It is speculated that the Summerians thought there were 360 days in a whole complete year. Another speculation is based upon the fact that the Sumerian culture had a base 60 or sexagesimal number system.

The mathematical notation for minutes is $'$ and the notation for seconds is $''$. Thus $34^\circ\,24'\,18''$ is read as 34 degrees, 24 minutes and 18 seconds. Note that these units of measurements have survived for over 2000 years as we still have 60 seconds in a minute and 60 minutes in an hour in our time system of measurements.

Example 1-8.

Convert the **decimal degrees** 133.742° into **degrees, minutes and seconds.**

Solution

(a) Subtract the integer part of the decimal degrees to get 133°

(b) Convert the decimal part of the degree $(.742^\circ)$ to minutes using the conversion factor $1^\circ = 60'$ to obtain

$$.742^\circ \frac{60'}{1^\circ} = 44.52'$$

The integer part of this answer gives $44'$ as the number of minutes.

(c) Convert the decimal part of the minutes to seconds using the conversion $1' = 60''$. One finds

$$.52' \frac{60''}{1'} = 31.8''$$

The final answer is

$$133.742^\circ = 133^\circ\ 44'\ 31.8''$$

which is read, 133.742 degrees equals 133 degrees, 44 minutes and 31.8 seconds.

∎

Example 1-9.

Convert the **degrees, minutes, second** $17^\circ\,24'\,45''$ into its **decimal degree** equivalent.

Solution

(a) Convert the second to minutes using the conversion factor $60'' = 1'$. This gives

$$45'' \frac{1'}{60''} = \frac{45}{60}' = 0.75'$$

This gives $(24 + 0.75)' = 24.75'$ in the given angle.

(b) Convert the minutes to degrees using the conversion factor $60' = 1^\circ$ to obtain

$$24.75' \frac{1^\circ}{60'} = 0.4125^\circ$$

The final answer is obtained by addition giving

$$17^\circ\ 24'\ 45'' = 17.4125^\circ$$

∎

The gradian (grad)

One complete rotation of the circle radius about the center of the circle can be divided into any number of parts. The **gradian**, abbreviated **grad**, unit of angular measurement, divides one complete rotation of the radius into 400 parts called grads (sometimes referred to as gons or grades). This allows one to set up a table of angular equivalents between grads and degrees.

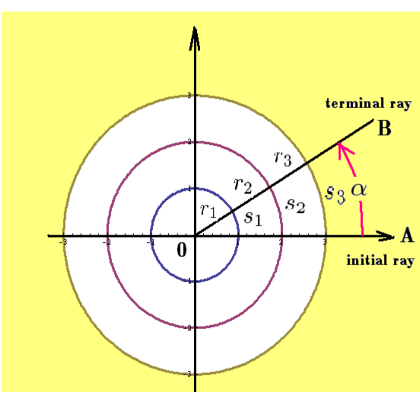
$0^\circ = 0\ grad$	$180^\circ = 200\ grad$
$45^\circ = 50\ grad$	$225^\circ = 250\ grad$
$90^\circ = 100\ grad$	$270^\circ = 300\ grad$
$135^\circ = 150\ grad$	$315^\circ = 350\ grad$
$180^\circ = 200\ grad$	$360^\circ = 400\ grad$

The grad unit of angular measurement is sometimes used in surveying, military calculations and in developing instrumentation for specialized equipment.

The conversion factor between degrees and gradians is

$$360^\circ = 400\ grad \qquad \text{or} \qquad 1^\circ = \frac{400}{360}\ grad = \frac{10}{9}\ grad$$

The radian (rad)



Let s denote **the arc length** which is produced on the circumference of any circle as the radius r of the circle rotates from an initial position to a final terminal position. Let $\alpha = \angle A0B$ denote the angle associated with the above rotation, then the **radian measure** used to describe the resulting angle α is defined as **the ratio of arc length s produced divided by the radius r of the circle or $\alpha = \frac{s}{r}$ radians.** For the circles illustrated $\alpha = \frac{s_1}{r_1} = \frac{s_2}{r_2} = \frac{s_3}{r_3}$.

A special case occurs when the radius of the circle makes one complete rotation about the center of the circle. The arc length s becomes the circumference c of the circle and we know this circumference is given by $c = 2\pi r$.

After one rotation about the center of the circle one finds $\alpha = 360^\circ$ and the arc length is $s = c = 2\pi r$. Therefore, the radian measure of the angle α is related to the degree measurement of α by the equation

$$\begin{aligned} \alpha = 360^\circ = \frac{s}{r} \text{ radians} = \frac{2\pi r}{r} = 2\pi \text{ radians} \\ \text{or } 1 \text{ rad } = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi}{180} \text{ rad} \end{aligned} \tag{1.29}$$

The **radian** is abbreviated **rad.** It is a **dimension-less** quantity because both the numerator and denominator in the definition of radian measure have units representing dimensions of length which divide out.

One can verify from the conversion factors in equation (1.29), that 1 radian is approximately 57.296°. The above conversion factors enable one to produce the following equivalence relations from degrees to radians.

$$\begin{array}{llll} 0^\circ = 0 & 90^\circ = \frac{\pi}{2} & 270^\circ = \frac{3\pi}{2} & 1^\circ = \frac{\pi}{180} \\ 30^\circ = \frac{\pi}{6} & 135^\circ = \frac{3\pi}{4} & 315^\circ = \frac{7\pi}{4} & x^\circ = x\left(\frac{\pi}{180}\right) \\ 45^\circ = \frac{\pi}{4} & 180^\circ = \pi & 360^\circ = 2\pi & 180^\circ = \pi \\ 60^\circ = \frac{\pi}{3} & 225^\circ = \frac{5\pi}{4} & & \end{array}$$

Note that the term rad does not have to be written after the value because radians are dimensionless. To convert degrees to radians multiply the degrees by $\frac{\pi}{180}$ and to convert radians to degrees multiply the radian measure by $\frac{180}{\pi}$. The French *Systéme International d'unités*, designated SI in all languages, is the official international system for units of measurements throughout the world. In this system the radian measure is used as the official angular unit of measurement.

Note that on most hand-held calculators you have to select between degrees (deg), radians (rad) and gradians (grad) before doing any calculations. In using a calculator or computer for **scientific calculations** be sure to use **radian measure** at all times, unless told otherwise. Also note that in the absence of an angular unit of measurement it is to be understood that one must use radian measure.

Special angles

An angle between 0° and 90° $(0 \text{ and } \frac{\pi}{2})$ is called **an acute angle.** An angle between 90° and 180° $(\frac{\pi}{2} \text{ and } \pi)$ is called **an obtuse angle**. An angle at 90° $(\frac{\pi}{2})$ is called a **right angle**. An angle at 180° (π) is called a straight angle.

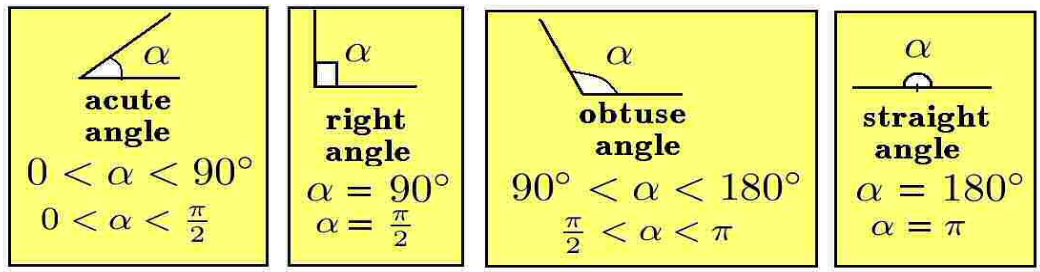


Figure 1-11. Special angles

The protractor

The protractor is a semicircular device for measuring and/or constructing angles. There are usually several sets of markings on the arc of the protractor. The marking on the lower scale indicate angular measure in degrees from 0° to 180° reading right to left. The markings on the upper scale are also in degrees and go from 0° to 180° but reads from left to right. There is also a marking for the center of the protractor. These scales are illustrated in the figure below.You can purchase a special protractor which uses radian measure on its markings.

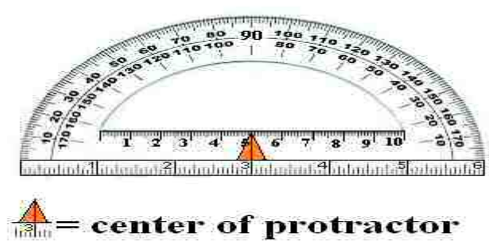


Figure 1-12. Protractor for measuring and plotting angles

To measure an angle place the center of the protractor at the vertex of the angle. The line through the center of the protractor is then aligned with one side of the angle. The other line defining the angle then intersects the protractor scale at the angle measurement in degrees. Below is an illustration of a 50 degree angle being measured.

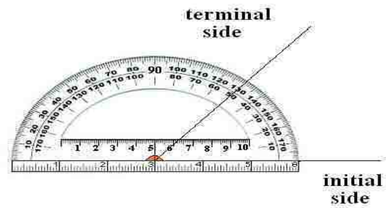


Figure 1-13. Protractor measuring angle of 50°

The right angle

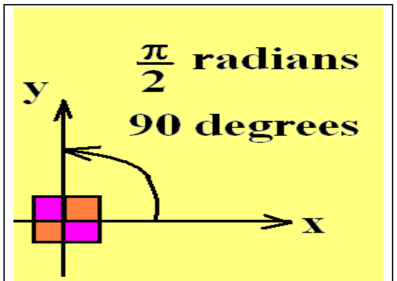
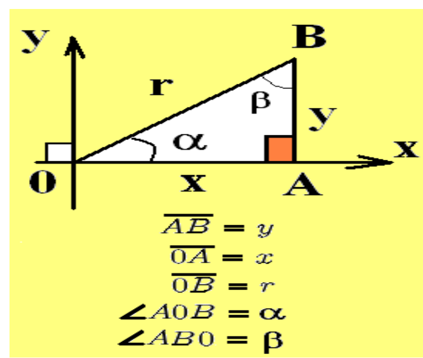


Figure 1-14.
The 90 degree angle.

Make special note of the 90 degree angle ($\frac{\pi}{2}$ radians). This angle plays an important part in many geometric drawings and is referred to as **a right angle**. Two lines which intersect have angles associated with the point of intersection. See the intersection of the x and y axes in figure 1-14.



The intersection of two lines is called **perpendicular if all the angles about the point of intersection are right angles.** The special angle $\frac{\pi}{2}$ is usually emphasized by placing a box in the corner where a vertical line is intersecting perpendicular to a horizontal line as illustrated in the figure 1-14.

A figure constructed using three connected straight line segments forming a closed loop is called **a triangle**. See for example the connected line segments $\overline{0B}$, $\overline{BA}$ and $\overline{0A}$ in the figure above. The points where the line segments meet are called **vertices.** The rotation of one side of a triangle to another side creates the **interior angles associated with a triangle.** If one of the angles of a triangle is a 90° angle ($\frac{\pi}{2}$ radians), then the triangle is called **a right triangle.** The 90° angle along with the angles α and β are the interior angles of the above triangle. Observe that in a right triangle, the two shorter legs of the triangle intersect in a 90 degree angle and the side opposite the 90° angle, **called the hypotenuse**, is the longest side of the right triangle.

In any triangle a line segment through a vertex which is perpendicular to the opposite side (or extended opposite side) is called **an altitude of the triangle**. Every triangle has three altitudes. In the figure above the line segment $\overline{\text{BA}}$ is the altitude through vertex B.

Assumption

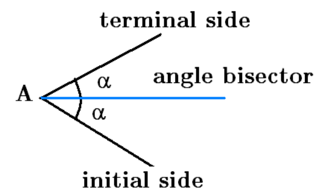
All right angles are equal to one another.

Notation

(i) $\angle A0B$ is a notation used to represent the angle formed by sides $A0$ and $0B$ with the middle letter 0 denoting the vertex of the angle.
(ii) $\overline{AB}$ is the line segment from A to B, treated as a positive length.
(iii) $\perp$ is the symbol for perpendicular. For example $\overline{\text{0A}} \perp \overline{\text{AB}}$
(iv) $\angle A0B = \alpha$ Sometimes Greek letters are used to represent angles.
(v) $m\angle A0B = 72^\circ$ is a notation used for denoting the measure of angle $\angle A0B$
(vi) If angle $\angle ABC$ is a vertex angle of a triangle, then the notation $\angle B$ is sometimes used to represent the vertex angle.

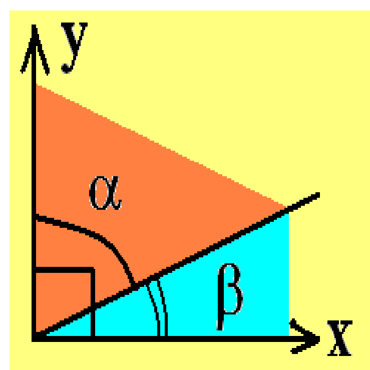
Angle bisector

An angle bisector is a ray that divides an angle into two equal angles. In the figure given $m\angle A = 2\alpha$.



Complementary angles

Two angles α and β, in degrees, which add up to 90 degrees are called **complementary angles.** Two angles α and β, in radians, that add up to $\frac{\pi}{2}$ are also called complementary angles. That is, the angles α, β are called **complementary angles** if one of the equations



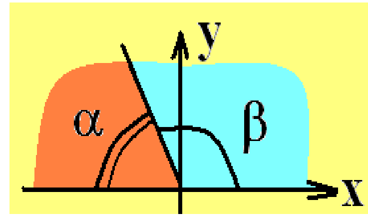
$$\alpha + \beta = 90^\circ \quad \text{or} \quad \alpha + \beta = \frac{\pi}{2}$$

holds true.
The equation used for testing depends upon the units used to measure angles. In scientific computing the radian measure is used most often.

Equations must be **homogeneous in the units selected for use**. This means both sides of an equation must have the same units of measurement. If two or more angles add to 90°, then this group of angles is called a complementary set.

Supplementary angles

Two angles α and β, measured in degrees, which add up to $180°$, are called **supplementary angles. If the angles are measured in radians, then the sum $\alpha + \beta = \pi$ is required to be satisfied for the angles α and β to be called supplementary.**



Remember that both sides of an equation must have the same units of measurement. If two or more angles add to $180°$, then the group of angles is called a supplementary set.

Some mathematical symbols

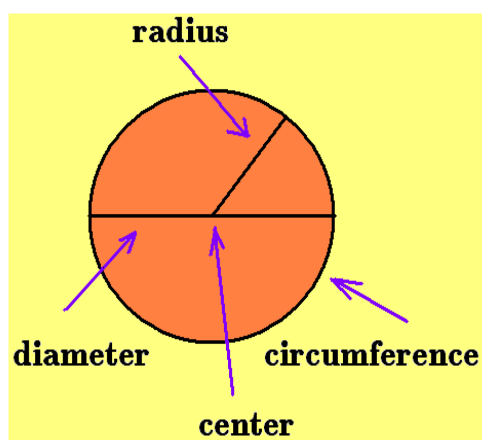
$=$ equal	$>$ greater than	$\angle$ angle
$\neq$ not equal	$\geq$ greater than or equal	$\ldots$ continuation
$\approx$ approximately	$^\circ$ degrees	$m\angle B =$ measure of angle B
$<$ less than	$'$ minutes	$\overset{\frown}{AB} =$ arc AB on circle
$\leq$ less than or equal	$''$ seconds	$\overline{AB} =$ line segment A to B

Loci

Many geometric problems require one to (i) find the motion of a point satisfying prescribed conditions or (ii) locate all points which satisfy given prescribed conditions. For example, find the locus of points in the plane which are equidistant from a fixed point. The solution is a circle and the fixed distance is called a radius and the fixed point is called the center of the circle. The collection of all points satisfying the prescribed condition are all points on the boundary or circumference of the circle.

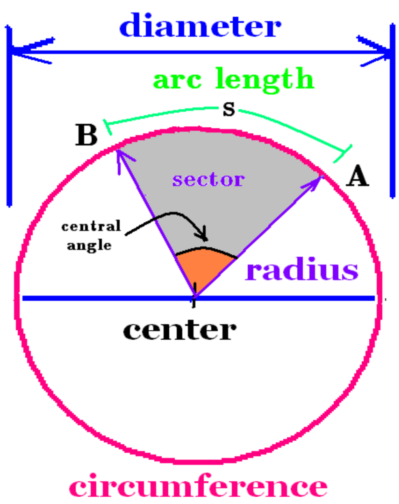
The circle

A **circle** is defined as the set of points which are equidistant from a fixed point called the **center** of the circle. The **radius** of the circle is the straight line distance from the center to any point on the circle boundary. The circle boundary is called the **circumference of the circle.** A line segment from a point on the circumference which passes through the center of the circle to a point on the other side of the circle is called the **diameter** of the circle.



The **circumference** of a circle is also known as the perimeter of the circle, distance around the circle or boundary of the circle. Observe that the diameter is twice the radius of any circle. For future reference also make note of the fact that the rotation of a circle about its diameter will produce the 3-dimensional image called a sphere.

Circle sector and arc length We have previously dealt with the circle in order to introduce the irrational number π representing the ratio of circumference divided by the diameter. A point moving around a circle can be described by an angle changing as the point moves. A point moving on the circumference of a circle moves along a curved path called the circumference of the circle and the motion of the point creates a path called the arc length along the circumference. This path length is denoted by the symbol $\widehat{\text{AB}}$ and the arc length is associated with some distance s.



Observe that as the position radius changes from point A to B it sweeps out an angle called a **central angle** with vertex at the center point of the circle.

The **radius (plural radii) of the circle** is a straight line distance from the center of the circle to an edge point of the circle. If A and B are two points on the circumference of the circle and one constructs the radii lines from the center 0 to each point A and B, then the area bounded by the two radii and the arc $\widehat{\text{AB}}$ on the circumference of the circle is called a **sector of the circle.** The angle θ between the two radii is called the **central angle** associated with the sector. If the central angle θ is less than 180°, the arc $\widehat{\text{AB}}$ on the circumference is called a **minor arc** and if θ is greater than 180° the arc is called a **major arc** of the circle connecting the points A and B on the circumference.

To find the arc length s of the minor arc $\widehat{\text{AB}}$ one can use proportions. Express the ratio of length of minor arc s to the total length of circumference $2\pi r$ as being in the same ratio as the central angle θ to one complete revolution or 2π. One obtains the proportion

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi} \qquad \text{or} \qquad s = r\theta \tag{1.30}$$

which states that the arc length between two points on a circle is determine by the product of the radius times the central angle θ, **where θ must be expressed in radians.**

Definitions

The Merriam Webster dictionary provides the following for the word 'definition'.

definition a transitive verb

a: to determine or identify the essential qualities or meaning of

b: to discover and set forth the meaning of

A definition describes a quantity or quantities by assigning a 'name' and describing in what context the name is being used. It gives a meaning to the terminology being used. The **definitions are the most important things to study in any mathematics course**.

Using definitions one can build up a vocabulary to communicate more complicated concepts. For example, knowing that for two lines to be 'parallel' they must

(a) both lie in the same plane and

(b) remain a constant distance apart and not intersect

enables one to understand concepts where parallel lines are used. Parallel lines can also be defined as two nonintersecting lines with the same slopes.

Definitions give us some starting points from which one can begin a study of plane geometry. Definitions are very important, you must study them, understand them and recognize them when they occur in problems. The following are some basic concepts which can be used to start our introduction to geometry.

Postulates

A postulate is a statement about something that is intuitively true and is to be accepted without any kind of proof. Some examples of postulates occurring in geometry are:

▸ A straight line segment can be extended beyond its endpoints.

▸ A geometric figure can be moved from one location to another in a plane without changing its size or shape.

▸ An angle can be bisected by a line through the vertex.

▸ All the radii of a circle, or of equal circles, have the same length.

▸ Two lines in a plane either intersect or they are parallel.

▸ Given a point not on a given line, then one can construct another line through the point which is perpendicular to the given line.

▸ Given a point not on a given line, then one can construct another line through the point which is parallel to the given line.

Definitions and postulates

▶ A **point** is a location and has no dimension.

▶ A **line segment** between two points is a collection of points defining the shortest distance between the two given points. The line segment can be extended from either end point in a constant direction to form an infinite line. A line can be thought of as the motion of a point in a constant direction. A line segment is one dimensional and has length.

▶ A line is determined by two different points. These points define **the slope** of the line as $m = \dfrac{\text{change in y-values}}{\text{change in x-values}}$.

▶ The motion of a line segment, ends not extended, in 2-dimensional space will usually create a surface. The surface is called **a plane** or flat surface if a straight line joining any two points on the surface will lie only on the surface. The ideal planar surface has no thickness and extends without end. The plane is two dimensional and has both length and width. A plane is determined by three different points.

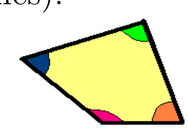
▶ The **intersection** of two quantities consists of a point or points that are common to both quantities. The intersection of two lines is a single point. The intersection of two planes is a line. A line, not in the plane, intersects a plane in a single point.

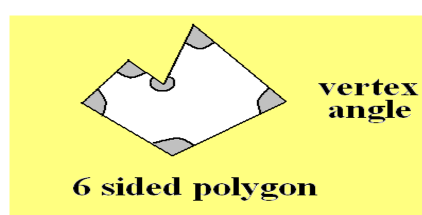
▶ A set of points that lie on the same line are said to be **collinear.**

▶ A set of points is called **noncollinear** if at least one of the points is not on the line determined by the other points.

▶ A set of points and/or lines are called **coplanar** if they all lie on the same plane.

▶ A set of points and/or lines are called **noncoplanar** if one of the points (or lines) does not lie in the given plane determine by the other points (or lines).

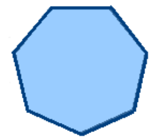


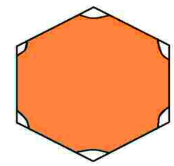


▶ An **irregular polygon** is a two dimensional plane figure having three or more sides. The sides are straight line segments which form a closed loop. A point where two line segments of the polygon meet forms **an interior vertex angle** of the polygon. An **irregular polygon** is one where each side and each vertex angle may be different.

Polygons are given names based upon the number of line segments used in their construction. Three sides for a **triangle**, four sides a **quadrilateral**, five sides a **pentagon**, six sides a **hexagon**, seven sides a **heptagon**, eiqht sides an **octagon**, etc.

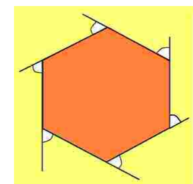
▶ A **regular polygon** is a polygon with all vertex angles equal and all sides are of equal length. A regular polygon is said to be equilateral and equiangular.



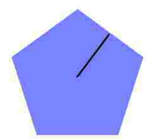


▶ The **interior angles** of a polygon occur at each vertex and inside of the polygon.

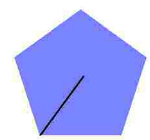
▶ An **exterior angle** of a polygon occurs when **a side of the polygon is extended. The extended line and side of polygon meet at a vertex of the polygon and form the exterior angle.**



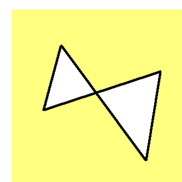
▶ The **apothem** of a regular polygon is a line from the center of the polygon to the midpoint of a side.



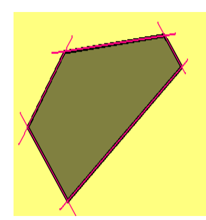
▶ The **radius** of a regular polygon is a line from the center of the polygon to any vertex point.



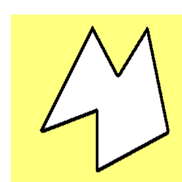
▶ A **self-intersecting or crossed polygon** occurs whenever one or more sides crosses back over another side. A **simple polygon** is one that is **non self-intersecting.** Self-intersecting polygons do not have simple properties like regular or simple polygons. They are best studied by **breaking them up into many simple polygons.**



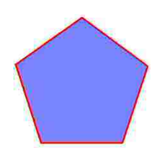
▶ A polygon is called **convex** if none of the extended sides pass through the center of the polygon. A polygon is convex if none of the interior angles are greater than π radians. Regular polygons are always convex.

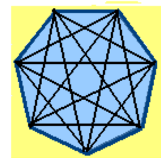


▶ A **concave** polygon has one or more interior angles greater than 180°. Concave polygons are easily recognized whenever any vertex points inward toward the center of the polygon. Polygons which are not concave are called convex.

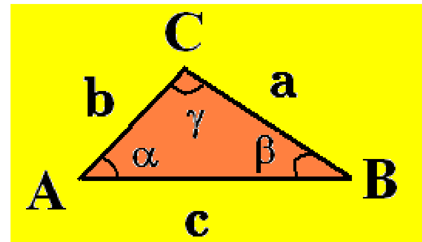


▶ The **perimeter** of a simple polygon is the distance around the polygon. It is the sum of the lengths of the line segments which make up the sides of the polygon.





► A **diagonal line of a polygon** is any line segment connecting two nonadjacent vertices.



► A polygon with **three sides** is called a **triangle**. A triangle can be expressed using the symbol △ followed by three letters denoting the vertices of the triangle. For example, the notation $\triangle ABC$ is used to represent the triangle having vertices ABC.

Angles of a triangle can be added by placing the vertex angles together at a common point.

Note the notation of lower case a opposite vertex A to denote the length of the side opposite vertex A and angle α as an alternative notation for vertex angle A ($\angle A$ or $\angle CAB$). This notation is also applied to the other angles and sides whenever convenient. The names of triangles depend upon their interior angles.

An angle is called acute if it is less than 90°, it is called a right angle if it equals 90° and called obtuse if it is greater than 90°. A triangle is called

acute if all of the angles in the triangle are acute

right if one of the angles of the triangle is a right angle.

obtuse if one of the angles of the triangle is obtuse.

A triangle has six parts. There are three sides and three vertex angles. Some problems associated with a given triangle are:

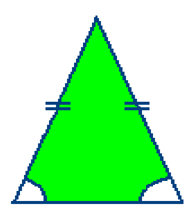
(i) Determine unknown parts from a set of given parts.

(ii) Determine the area of the triangle.

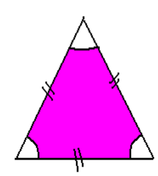
(iii) Determine the midpoints of the sides.

(iv) Constructing special lines from a vertex angle to the opposite side.

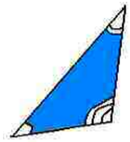
(v) What conditions must the sides and angles satisfy in order for a triangle to exist?



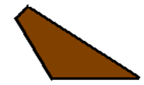
► A triangle with **two equal sides is called isosceles.** Equal sides are highlighted by making identical slash marks on those sides.



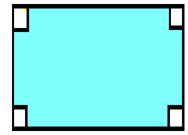
► A triangle with **three equal sides is called an equilateral triangle.**

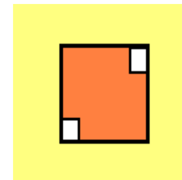


▶ A **scalene triangle** is a polygon with three unequal sides and three unequal angles.



▶ A **quadrilateral** is a four sided polygon.

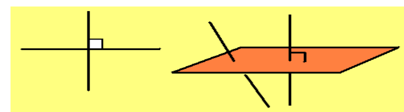




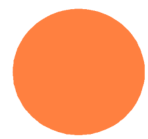
▶ A **rectangle** is a four sided polygon with opposite sides parallel and all vertex angles equal to 90° ($\frac{\pi}{2}$ *radians*).

▶ A **square** is a rectangle having all sides of equal length.

▶ A line which intersects another line or plane at an angle of 90° (or $\frac{\pi}{2}$ radians) is called a **perpendicular** line. Any other line produces an oblique intersection.

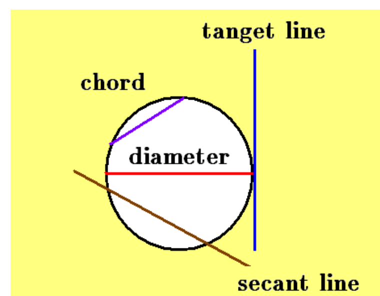


▶ A **circle** is a plane figure defined by a closed curve having all of its points equidistant from an interior point called the center of the circle.

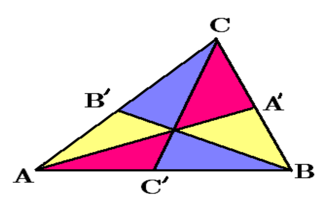


▶ Two **semicircles** result when a line is drawn through the center of a circle.

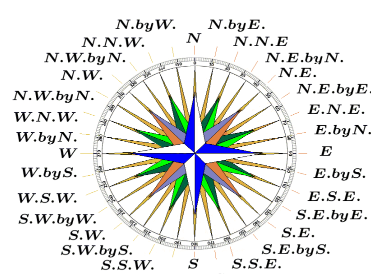
▶ The **chord** of a circle is any straight line which joins two points on the circumference. A **tangent** line to a circle touches the circle at only one point and is perpendicular to the circle diameter through that point. A **secant line** is any straight line passing through the circle.



▶ The **medians** of a triangle are line segments from each vertex of a triangle to the midpoint of the opposite side. Every triangle has three medians. In the figure on the right the line segments $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are the medians of triangle $\triangle ABC$.



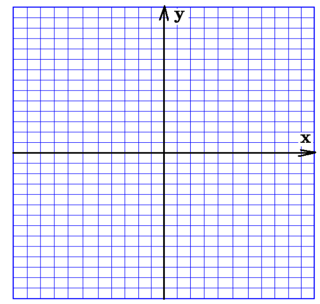
▶ Directions or bearings on a Mariner's compass make reference to the cardinal points north, east, south, west. For example, northeast direction is referenced as north 45 degrees east with measurement from the north direction. The southeast direction is referenced as south 45 degrees east.



Exercises

▶ **1-1.**
In Cartesian coordinates plot the points

$$(1,1),(5,4),(-5,3),(-1,1),(-3,4),(2,-3),(4,-2)$$



▶ **1-2.** Find the slope of the line segment connecting the given points

(a) $(1,1),(5,3)$ $\quad$ (c) $(1,1),(4,4)$

(b) $(1,1),(-5,2)$ $\quad$ (d) $(1,1),(5,-3)$

▶ **1-3.** Find the point-slope formula representing the line through the given points

(a) $(1,1),(5,3)$ $\quad$ (c) $(1,1),(4,4)$

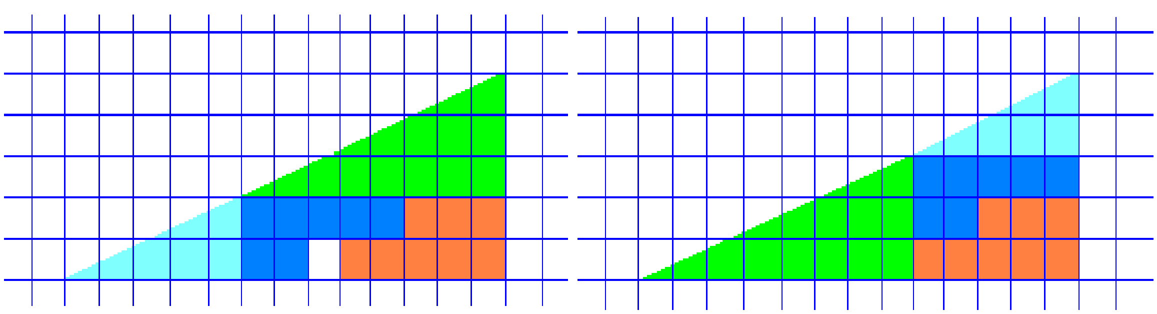
(b) $(1,1),(-5,2)$ $\quad$ (d) $(1,1),(5,-3)$

▶ **1-4.** Find the slope of the given line and sketch the line.

(a) $\dfrac{x}{4}+\dfrac{y}{5}=1$ $\quad$ (c) $x=3+4t,\ y=7-3t$

(b) $3x+4y=5$ $\quad$ (d) $y-7=\left(\dfrac{8}{3}\right)(x-4)$

▶ **1-5.** (The triangle puzzle) Explain the hole in the following diagram.



Hint: What is the definition of slope.

▶ **1-6.** Are the given points collinear or noncollinear?

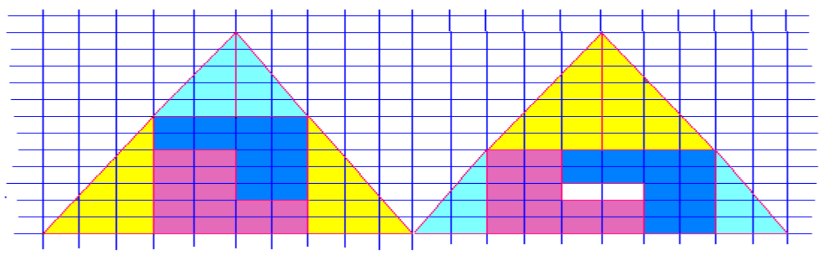
(a) $(2,1),(10,25),(100,295)$ $\quad$ (c) $(-7,-7),(-4,-3),(2,5)$

(b) $(3,2),(16,15),(9,10)$ $\quad$ (d) $(-8,-4),(-6,-1),(0,8)$

▶ **1-7.** If $\frac{a}{b} = \frac{c}{d}$, show that

$$(a)\ \frac{b}{a} = \frac{d}{c} \qquad (b)\ \frac{a+b}{b} = \frac{c+d}{d} \qquad (c)\ \frac{a-b}{b} = \frac{c-d}{d} \qquad (d)\ \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

▶ **1-8.** A similar puzzle problem. Explain what is happening.



▶ **1-9.** Find the midpoint of the line segment $\overline{AB}$ if

(a) $A = (-2, 5),\ B = (4, 8)$ $\quad (c)$ $A = (0, 0),\ B = (-5, 9)$

(b) $A = (4, 7),\ B = (-1, 1)$ $\quad (d)$ $A = (0, 0),\ B = (7, 8)$

▶ **1-10.** Express the line $8x - 7y = 5$ into the form specified.

(a) slope-intercept form $\quad (c)$ parametric form

(b) intercept form $\quad (d)$ point-slope form

▶ **1-11.** Find the arithmetic mean of the given numbers.

(a) $1, 2, 3, 4, 5$ $\quad (c)$ $9, 10, 11$

(b) $7, 5, 3$ $\quad (d)$ $68, 78, 88$

▶ **1-12.** Find the geometric mean of the given numbers.

(a) $4, 9$ $\quad (c)$ $1, 3, 9$

(b) $4, 18$ $\quad (d)$ $2, 4, 4, 8$

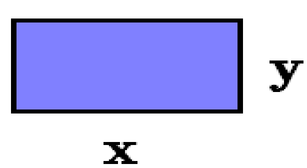
▶ **1-13.** Express the decimal number as a rational number.

(a) $1.2\overline{34}$ $\quad (c)$ $7.4\overline{21}$

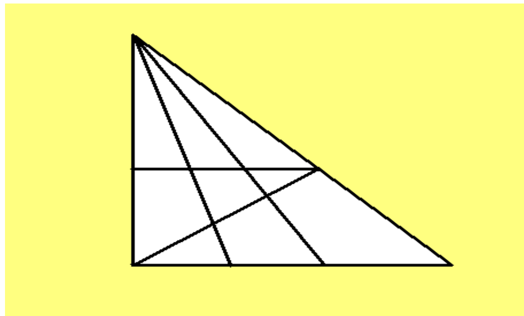
(b) $5.\overline{631}$ $\quad (d)$ $34.\overline{3205}$

▶ **1-14.**

An artist wants to paint a picture inside a rectangle with height y and base x. If $x = 64$ inches, what is the approximate height y such that x/y is a golden ratio?



▶ **1-15.** How many triangles can you find in the given figure.



Hint: There are more than 20. Can you sketch each one?

▶ **1-16.** Represent the given base 10 numbers as binary numbers.

(a) 11 (c) 12.5

(b) 5.25 (d) 7.75

▶ **1-17.** Represent the given base 10 numbers as octal numbers.

(a) 11 (c) 32

(b) 12.125 (d) 17

▶ **1-18.** Represent the given base 10 numbers as hexadecimal numbers.

(a) 1.0625 (c) 65

(b) 32 (d) 127

▶ **1-19.** Find in degrees the angular measure for

(a) one counterclockwise revolution.

(b) one-half of a counterclockwise revolution.

(c) one-third of a clockwise revolution.

(d) one-fourth of a counterclockwise revolution.

(e) one-fifth of a clockwise revolution.

▶ **1-20.** Find in radians the angular measure for

(a) one counterclockwise revolution.

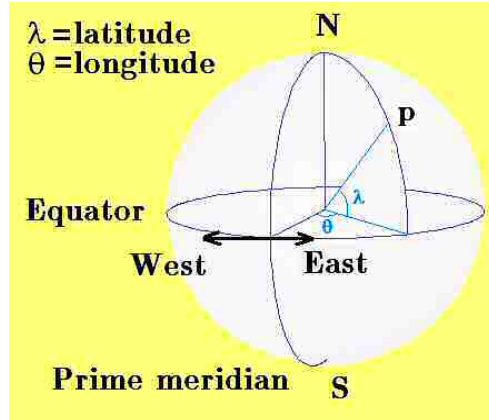
(b) one-half of a counterclockwise revolution.

(c) one-third of a clockwise revolution.

(d) one-fourth of a counterclockwise revolution.

(e) one-fifth of a clockwise revolution.

► **1-21. Latitude-Longitude**



Assume the earth is a sphere and construct a line from a point P on the earths surface and the center of the earth. This line forms an angle λ with the earths equatorial plane and is known as the latitude of the point P. The angle λ varies from 0° to $90^\circ\, N$ and 0° to $90^\circ\, S$.

The angle θ east or west of the prime meridian is called the longitude of a point on the earths surface. The angle θ varies from 0° to $180^\circ\, E$ and 0° to $180^\circ\, W$.

Express the given decimal angles in units of degree, minutes and seconds

(a) $66.5^\circ\, N$ Arctic circle latitude

(b) $23.26^\circ\, N$ Tropic of cancer latitude

(c) $51.5074^\circ\, N,\ 0.1278^\circ\, W$ London

(d) $40.7128^\circ\, N,\ 74.0059^\circ\, W$ New York City

► **1-22.** Express the angular measure in decimal form

(a) $66^\circ\, 33'\, 46.5''\, N$ Arctic circle latitude

(b) $23^\circ\, 26'\, 13.5''\, N$ Tropic of cancer latitude

(c) $55^\circ\, 45'\, N,\ 37^\circ\, 36'\, E$ Moscow

(d) $35^\circ\, 40'\, N,\ 139^\circ\, 45'\, E$ Tokyo

► **1-23.** Plot the parametric equations

(a)
$$x = 3\left(\frac{1-t^2}{1+t^2}\right)$$
$$y = 2\left(\frac{1-t^2}{1+t^2}\right) + 3$$
$$-1 \le t \le 1$$
Show $y = \frac{2}{3}x + 3$
$$0 \le x \le 3$$

(b)
$$x = 3t + 4$$
$$y = 4t + 5$$
$$0 \le t \le 1$$
Show $y = \frac{4}{3}x - \frac{1}{3}$
$$4 \le x \le 7$$

(c)
$$x = \frac{2}{t} + 3$$
$$y = \frac{4}{t} - 1$$
$$1 \le t \le 2$$
Show $y = 2x - 7$
$$4 \le x \le 5$$

t	x	y
0		
1		

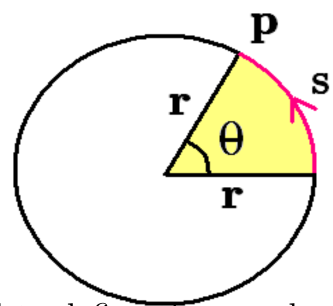
t	x	y
0		
1		

(d) How many points are needed to draw a straight line segment?

(e) Find the midpoints of the line segments in parts (a),(b),(c).

▶ **1-24.**

A particle P rotates in a circular motion about the center of a circle with radius r. The arc length distance s swept out during a time interval t is used to define the linear speed v of the particle as



$$\text{linear speed} = v = \frac{s}{t} = \frac{\text{distance}}{\text{time change}}$$

As P rotates the central angle changes with time and is used to define the angular speed ω of the particle as

$$\text{angular speed} = \omega = \frac{\theta}{t} = \frac{\text{angular change}}{\text{time change}}$$

We know $s = r\theta$, therefore $\frac{s}{t} = r\frac{\theta}{t}$ gives the linear speed of the particle as $v = r\omega$.

(a) A wheel rotates at 3 revolutions per second. Express the angular speed in units of radians per second.

(b) The Earth is approximately 93 million miles from the Sun and rotates in approximately a circle during a period of approximately 365 days.

(i) Find the linear speed of the Earth as it orbits the Sun. Express your answer in units of miles per hour, then convert answer to units of kilometers per hour.

(ii) Find the angular speed of the earth in units of radians per second.

(c) Assume the earth spins about an axis with angular speed of 1 revolution per day at the equator. The radius of the earth is approximately 3960 miles and the circumference at the equator is approximately 25,000 miles.

(i) What is the linear speed of a point on the equator?

(ii) What is the linear speed of a point at the poles?

(iii) Find the earth's angular speed in units of radians per second.

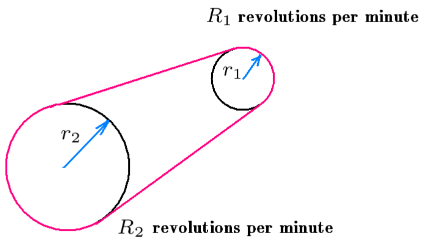
▶ **1-25.** Construct the following figures.

(a) An isosceles triangle
(b) An equilateral triangle
(c) A scalene triangle
(d) An acute triangle
(e) An obtuse triangle
(f) A right triangle

▶ **1-26.**

For each triangle of the previous problem (a) construct the altitudes (b) construct the medians (c) construct the angle bisectors. Note in some cases it is necessary to extend the side of a triangle.

▶ **1-27.**



If distance s equals velocity v times time t, then one can write $s = vt$. If the motion is on the circumference of a circle having radius r, then distance traveled is $s = r\theta$. Therefore, $v = \frac{s}{t} = \frac{r\theta}{t} = r\omega$, where $\omega = \frac{\theta}{t}$ is called the angular velocity in units of *radians/sec.*

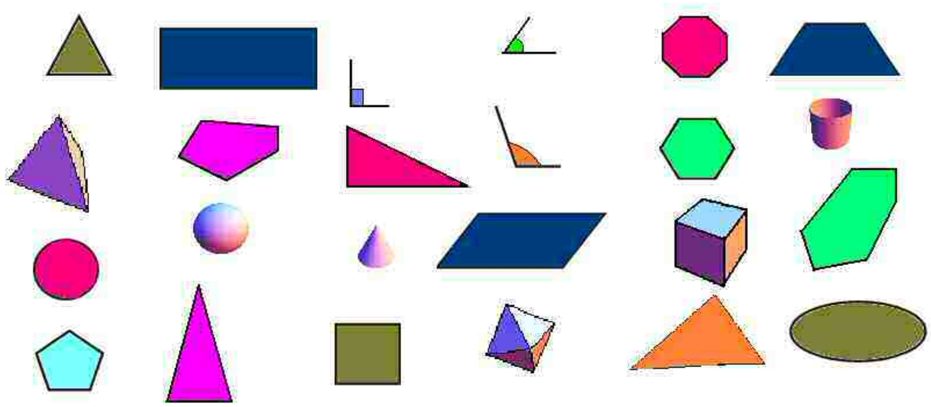
Two wheels with radii r_1 and r_2 are connected with a belt around the wheels. Let R_1 and R_2 denote the number of revolutions per minute of each wheel. The linear speed of a point p on the belt must be constant so one can say that the linear speed of a point on wheel 1 must equal the linear speed of the belt on wheel 2.

(i) Find the angular speed ω_1 of wheel 1 and angular speed ω_2 of wheel 2 in units of radians /sec.

(ii) Find the linear speed v_1 of a point on wheel 1 and the linear speed v_2 of a point on wheel 2.

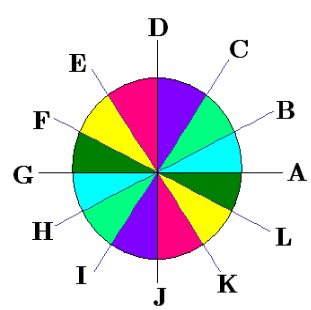
(iii) Equate v_1 to v_2 and show $\frac{R_1}{R_2} = \frac{r_2}{r_1}$

▶ **1-28.** Describe the geometric representations below.



▶ **1-29.**

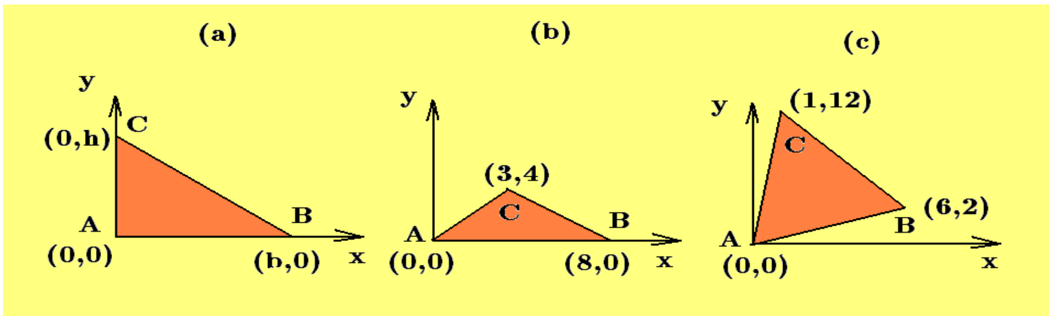
Let 0 denote the center of the circle illustrated where $360°$ is divided by 12 so that each angle is $30°$. Starting at A move counterclockwise about the circle and label each angle in radians. For example $\angle A0D = \frac{\pi}{2}$ and $\angle A0B = \frac{\pi}{6}$



▶ **1-30.**

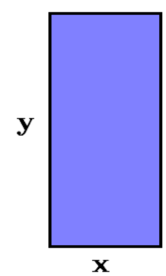
(a) Under what conditions will all three medians of a triangle have the same length? (b) Under what conditions will two medians of a triangle have the same length?

▶ **1-31.**



Find the Cartesian equation for the lines $\overline{AB}, \overline{BC}, \overline{CA}$ which represent the sides associated with each of the above triangles.

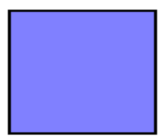
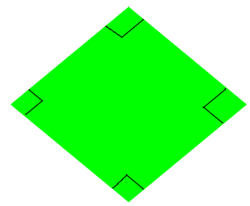
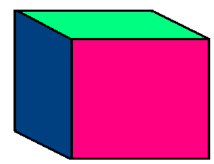
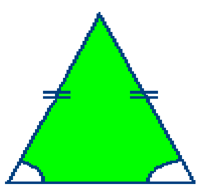
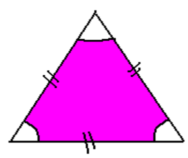
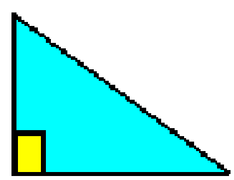
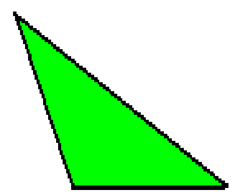
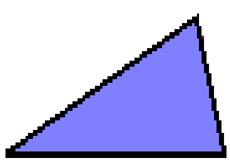
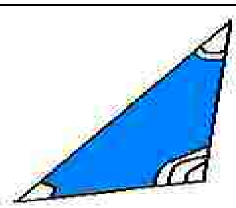
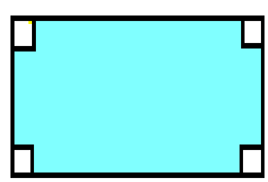
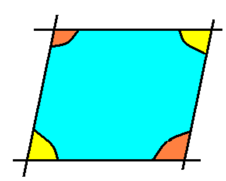
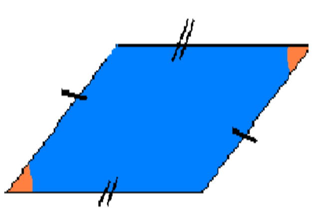
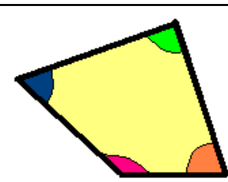
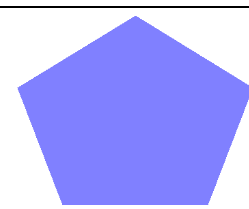
▶ **1-32.**

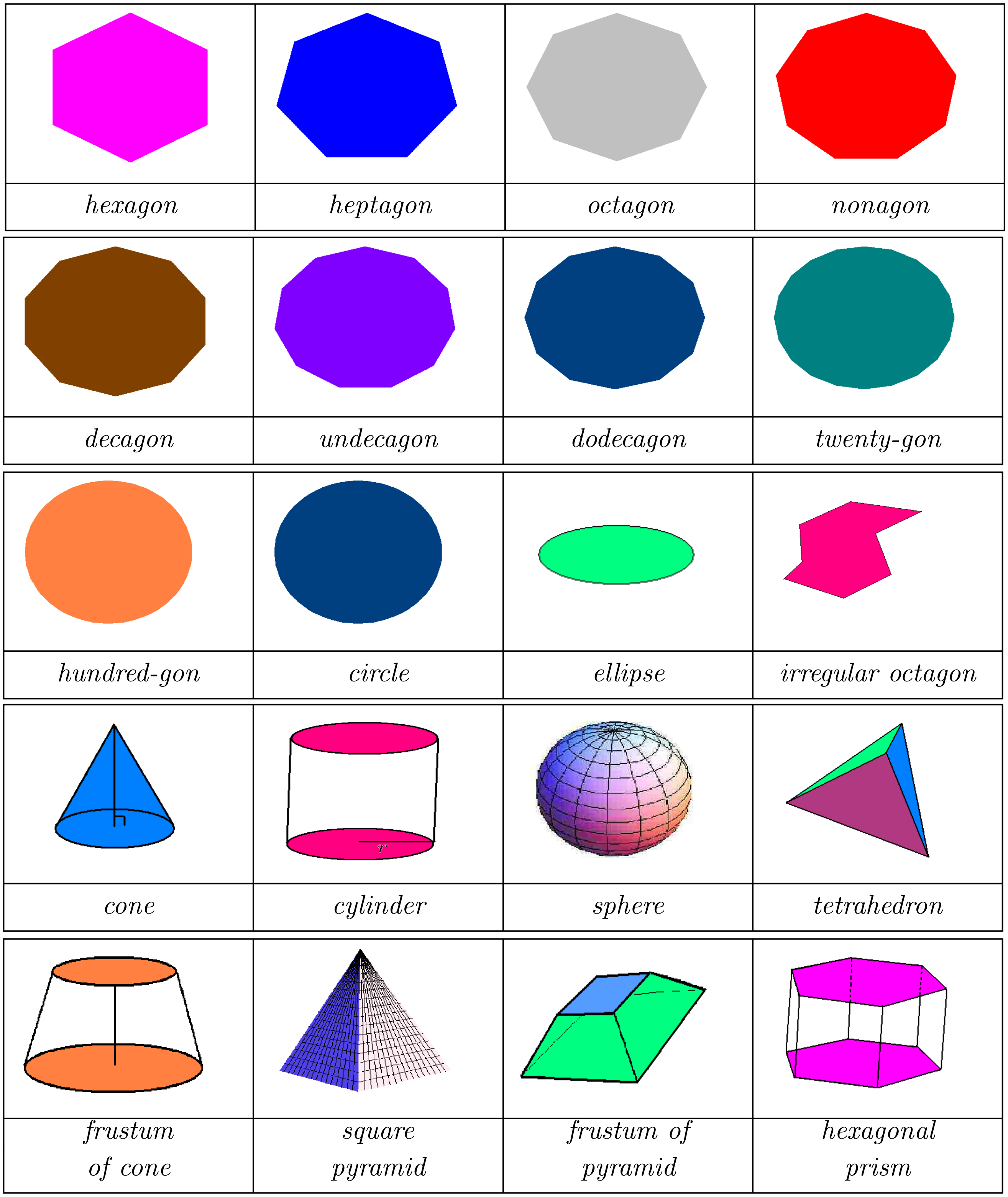


Any rectangle with sides in a golden ratio is called a golden rectangle.
(a) Show that one can divide a golden rectangle into two parts with part one a square and part two a golden rectangle.
(b) The golden rectangle is the only rectangle that can be divided into a square plus golden rectangle. Write out in words what happens if you divide the original golden rectangle into these two parts and then divide the smaller golden rectangle into two parts followed by dividing the third smaller golden rectangle into two parts, etc. The final result is called "the eye of god".

Geometry
Chapter 2
Basic Geometric Concepts

The following are some basic geometric shapes that we will be dealing with in this chapter and later chapters. Finding properties associated with various shapes is an important part of geometry.

			
line	*plane*	*square*	*cube*
			
isosceles triangle	*equilateral triangle*	*right triangle*	*obtuse triangle*
			
acute triangle	*scalene triangle*	*rectangle*	*rhombus*
			
parallelogram	*trapezoid*	*quadrilateral*	*pentagon*



Most of the polygons illustrated above are regular polygons. In general, a regular polygon with n-sides is called a n-gon. Note that there is hardly a difference between the 100-sided regular polygon inscribed within a circle and the circle itself. The early Greek geometers made use of this fact by observing that the apothem of a regular n-gon inscribed inside a circle approached the radius of the circle as the number of polygon sides n increased in size.

Also note that the names given to the polygons are closely related to the Greek and Latin prefixes associated with the numbers three through twelve.

Definitions

A picture is worth a thousand words. However it is nice to have a formal definition associated with the things you are about to get involved with.

▶ A **parallelogram** is a quadrilateral with two pairs of parallel sides.

▶ A **rhombus** is a parallelogram with four equal sides.

▶ A **trapezoid** is a quadrilateral having only one pair of parallel sides.

▶ An **isosceles trapezoid** has the nonparallel sides equal to one another.

▶ A **pentagon** is a five sided polygon and a regular pentagon has 5 equal sides and vertex angles.

▶ A **hexagon** is a six sided polygon and a regular hexagon has 6 equal sides and vertex angles.

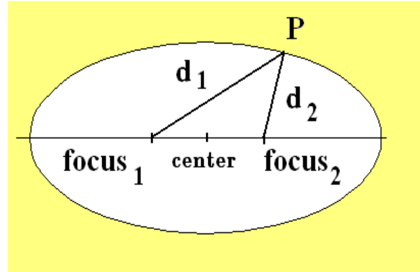
▶ A **heptagon** is a seven sided polygon and a regular heptagon has 7 equal sides and vertex angles.

▶ A **curved line** has no portion which is a straight line.

The ellipse

An **ellipse** can be constructed by taking a string of fixed length and then anchoring the end points of the string at two different points on a given line. The two points selected for the end points of the string are called foci and each point is called a focus. Take a pencil point and pull the string taut and then move the pencil point around the foci keeping the string taut. The resulting curve is an ellipse. A more formal definition of an ellipse is that it is the locus of points P such that the sum of the distances d_1 and d_2 from two fixed points (foci) remains constant. That is

$d_1 + d_2$ =constant=length of string



The center of the ellipse is halfway between the foci.

One of the properties of an ellipse is that if the elliptic boundary is a reflecting surface, then sound coming from one focus can easily be heard at the other focus. Many buildings throughout the world have elliptic shape chambers which make use of this property.

The postulate

An axiom or postulate is a statement about something which is assumed to be true and no proof is required for it to be true. Axioms and postulates are usually associated with everyday concepts assumed to be true and not needing any proof. For example, in Euclid's *Elements*, there is the axiom, "Things that coincide with one another are equal to one another". This is a statement defining one meaning of equality. It is a common sense statement that if two figures coincide in every way, then the two figures are identical. The mathematical symbol $\cong$ is used to represent **congruence** indicating two things are identical in every way.

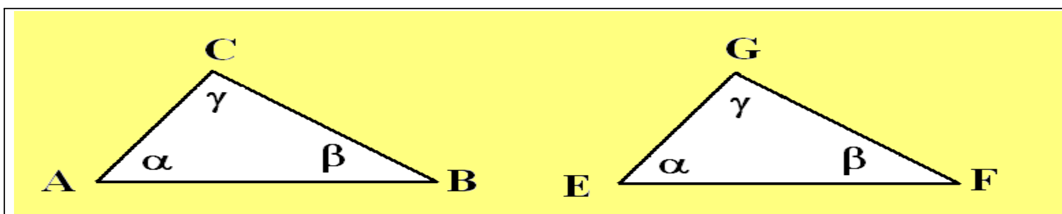


Figure 2-1.
Triangle ABC congruent to triangle EFG written as $\triangle ABC \cong \triangle EFG$

Examine the figure 2-1 and imagine that you can pick up triangle $\triangle ABC$ and place it on top of triangle $\triangle EFG$ and the two figures coincide in every way– meaning all corresponding angles are equal and all corresponding sides are equal, then we can say that **triangle ABC is congruent to triangle EFG** and write $\triangle ABC \cong \triangle EFG$.

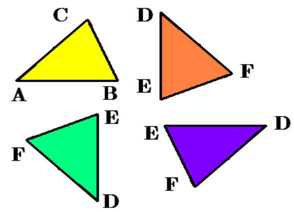
Notation

The statement $\triangle ABC \cong \triangle DEF$ is the notation for representing triangle $\triangle ABC$ being congruent to triangle $\triangle DEF$. This notation represents 6 statements

(i) $\angle A = \angle D$ $\qquad$ (iv) $\overline{\text{AB}} = \overline{\text{DE}}$

(ii) $\angle B = \angle E$ $\qquad$ (v) $\overline{\text{BC}} = \overline{\text{EF}}$

(iii) $\angle C = \angle F$ $\qquad$ (vi) $\overline{\text{CA}} = \overline{\text{FD}}$



Note that the statement $\triangle ABC \cong \triangle EFD$ has a completely different meaning.

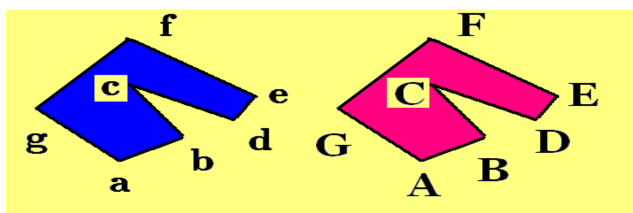


Figure 2-2. Two polygons with equal sides, equal angles and equal area.

Congruence can be used and applied to any shaped objects. Two line segments are congruent if they have the same length. Two circles are congruent if they have the same diameter. In general two objects are said to be congruent or equal to one another if they have **the same size and shape.** That is, corresponding parts of congruent figures are equal. For example, if the two polygons illustrated in the figure 2-2 are congruent, then one can write $abcdefg \cong ABCDEFG$ indicating corresponding angles and sides are equal.

In the study of geometry, certain postulates, definitions and notations have to be made in order to get started. The **postulate**, as used in geometry, is something that cannot be proved and must be accepted as being true.

Throughout this textbook note the emphasis on

(a) **Definitions**

(b) **The introduction of symbols for notation**

(c) **The introduction of axioms and postulates**

More postulates and definitions

▶ **Equality**

Things equal to the same thing, or to equal things, are equal to each other.

▶ **Addition**

Equals added to equals, the results are equal.

Equals added to unequals, the sums are unequal in the same order.

If unequals are added to unequals, then

(i) less than $(a < b) + (c < d) \Rightarrow (a + c) < (b + d)$

(ii) greater than $(a > b) + (c > d) \Rightarrow (a + c) > (b + d)$

the sums are unequal in the same order.

▶ **Subtraction**

Equals subtracted from equals the results are equal.

Equals subtracted from unequals, the results are unequal in the same order.

If unequals are subtracted from equals, then the results are unequal in the reverse order.

For example, if $(2 = 2)$ and $(3 < 5)$, then $(2 = 2) + (-3 > -5) \Rightarrow -1 > -3$

▶ **Multiplication**

Equals multiplied by equals produce products that are equal.

If unequals are multiplied by equals, then the products will be unequal in the same order.

▶ **Division**

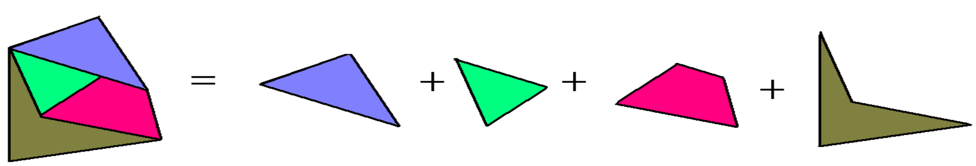
If equals are divided by equals, then the quotients will be equal.

If unequals are divided by equals, then the quotients will be unequal in the same order.

▶ **Transitive property for inequalities**

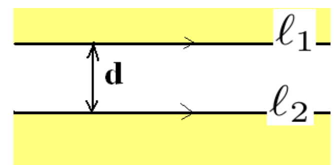
If $a > b$ and $b > c$, then $a > c$

▶ The whole is equal to the sum of its parts.



The whole is greater than any of its parts.

▶ **Parallel lines** are two lines in the same plane that do not intersect. The perpendicular distance between the lines remains constant. This Postulate is also known as Euclids fifth postulate. The math symbol $\|$ is used to denote lines are parallel. One can write $(\ell_1 \;\|\; \ell_2)$ to state that line ℓ_1 is parallel to line ℓ_2. In Cartesian plane geometry, lines are parallel if they are **in the same plane** and have the **same slope.**

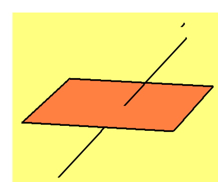


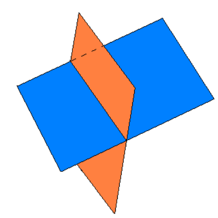
▶ Two points P_1 and P_2 determine **a unique line segment** which in turn can be extended into an infinite line.

▶ Three points P_1, P_2 and P_3, which are **noncollinear**, determine a **plane surface** which can be extended without limits to create an infinite two-dimensional plane surface. A consequence of this is that (a) A plane is determine by a point and a line and (b) A plane is determined by two intersecting lines.

▶ If two lines intersect, they intersect in **a single point common to both lines.**

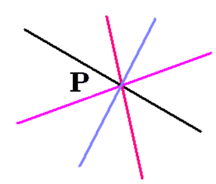
▶ If a three-dimensional line passes through a plane, and the line is not in the plane, then the intersection is **a single point common to the line and plane.**



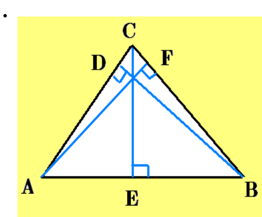


▶ The intersection of two planes results in a straight line.

▶ Three or more lines or curves are called **concurrent** if they all intersect at the same point. The common point of intersection is called the **point of concurrency.** Note that lines in the same plane which are not parallel can be called concurrent because they will intersect somewhere.



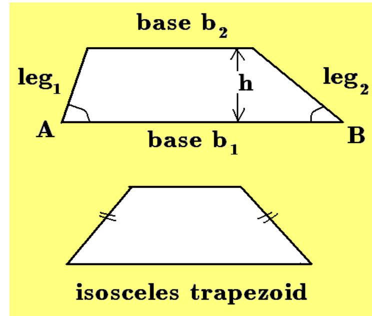
One famous problem in geometry is to determine conditions necessary for lines drawn inside a triangle to meet at a point of concurrency.



▶ A **trapezoid** is a quadrilateral with one pair of opposite sides parallel. The parallel sides are called the bases[1] of the trapezoid and the nonparallel sides are

[1] The side a polygon stands on is sometimes referred to as its **base**. Any side of a polygon can be called its base.

called the legs of the trapezoid. The shortest distance between the parallel lines is called the height of the trapezoid. The angles $\angle A$ and $\angle B$ are called the **base angles** of the trapezoid. If the base angles are congruent and the legs of the trapezoid are congruent, then the trapezoid is called an **isosceles trapezoid.**



Notation Here are some more mathematical symbols.

$\perp$ perpendicular	$\propto$ proportional to	$\sim$ similar	$\therefore$ therefore
$\parallel$ parallel	$\pm$ plus or minus	$\rightarrow$ or $\Rightarrow$ implies	$\triangle ABC$ triangle ABC
$\cong$ congruent	$\triangle$ triangle	$\equiv$ equivalent to	$[ABC]$ Area of $\triangle ABC$

Angles and transversals

A **transversal line** is a line which intersects two or more lines in the same plane. In the figure 2-3 a transversal line t intersects lines ℓ_1 and ℓ_2 to produce the angles 1,2,3,4,5,6,7 and 8.

Two angles are called **corresponding angles** if they occupy corresponding positions. For example, the angles

$$\angle 1 \text{ and } \angle 5, \quad \angle 2 \text{ and } \angle 6, \quad \angle 3 \text{ and } \angle 7, \quad \angle 4 \text{ and } \angle 8$$

are corresponding angles.

Two angles are called **alternate interior angles** if (i) they lie between the lines ℓ_1 and ℓ_2 and (ii) they lie on opposite sides of the transversal line. For example, the angles

$$\angle 3 \text{ and } \angle 5, \qquad \angle 4 \text{ and } \angle 6$$

are alternative interior angles.

Two angles are called **alternate exterior angles** if (i) they lie on opposite sides of the transversal and (ii) they lie outside of the lines ℓ_1 and ℓ_2. For example, the angles

$$\angle 2 \text{ and } \angle 8, \qquad \angle 1 \text{ and } \angle 7$$

are alternate exterior angles.

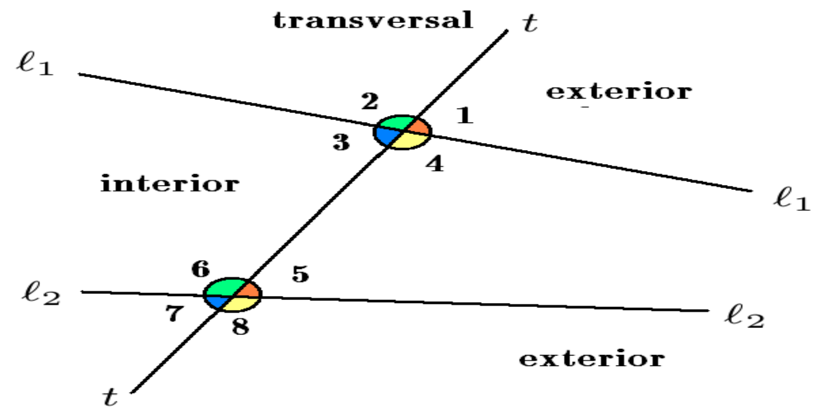


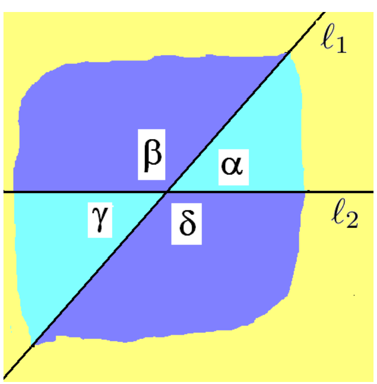
Figure 2-3. Angles created by a transversal line.

Two angles are called same-side interior angles if (i) they lie between lines ℓ_1 and ℓ_2 and (ii) they are on the same side of the transversal. For example, the angles

$$\angle 3 \; and \angle 6, \qquad \angle 4 \; and \; \angle 5$$

are same-side interior angles.

Intersecting lines



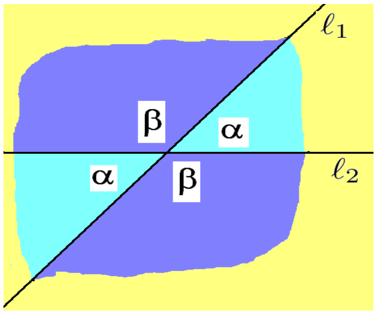
Whenever two lines ℓ_1 and ℓ_2 intersect, there is always created two pairs of **opposite angles**. In the accompanying figure the angles α and γ are opposite angles and the angles β and δ are opposite angles . **Opposite angles are equal.** To show that this is true make the assumption that all the angles $\alpha, \beta, \gamma, \delta$ are different.

Using radian measure, one can write

$$\alpha + \beta = \pi, \quad \beta + \gamma = \pi, \quad \gamma + \delta = \pi, \quad \alpha + \delta = \pi$$

because **the angles listed are supplementary angles.** Using the assumption that **things equal to the same thing are equal to each other** one can equate the representations for π to obtain

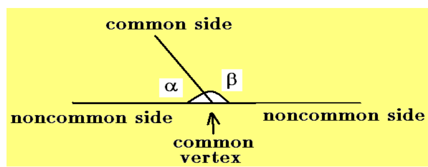
$\alpha+\beta=\beta+\gamma$ which implies $\alpha=\gamma$. In a similar fashion one finds $\beta+\gamma=\gamma+\delta$ which implies $\beta=\delta$ so our original assumption was false.



Therefore, one can say that **opposite angles**, formed by the intersection of two lines, must be equal. **Opposite angles** are also called **vertical angles** and can be defined as a pair of non-adjacent angles formed by the intersection of two straight lines.

Two angles are called adjacent if they have a common vertex.

Two adjacent angles with the same vertex are called a **linear pair** if their non-common sides are on the same line. Supplementary angles forming a straight line is an example of a linear pair.



Parallel lines and transversals

Any line ℓ which intersects two or more **parallel lines** is called a **transversal line.** Note that when a **transversal line** ℓ intersects **two parallel lines** there are created the following angles:

(i) The angles α and β are **alternate interior angles. Alternate interior angles are congruent and therefore equal.**

(ii) The red and yellow angles are **alternate exterior angles.**

(iii) The orange, green, red and blue angles are **corresponding angles.** Corresponding angles are congruent.

In summary, the angles illustrated in the figure 2-4 are such that

$\beta_1=\delta_1,\quad \alpha_1=\gamma_1$ Angles are opposite angles.

$\delta_1=\delta_2,\quad \alpha_1=\alpha_2$ Corresponding angles are equal.

$\beta_2=\delta_2,\quad \alpha_2=\gamma_2$ Angles are opposite angles.

$\beta_1=\beta_2,\quad \gamma_2=\gamma_1$ Corresponding angles are equal.

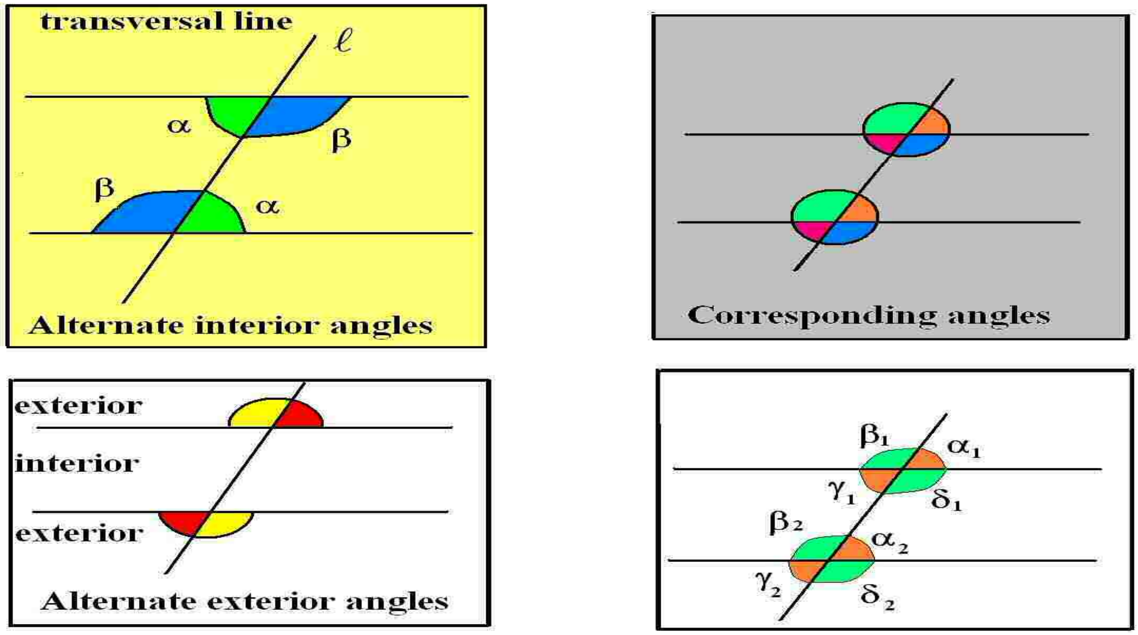


Figure 2-4. Angles created by transversal line

Note that in the **special case of parallel lines** the **corresponding angles are equal to one another** because you can pick up the lower portion of the transversal line intersection and place it on top of the upper portion of the transversal line intersection to show that the corresponding angles are congruent and therefore they must equal one another. The process of picking up one figure and placing it upon another to show the two figures coincide is known as superposition of the figures. Here by superposition $\alpha_1 = \alpha_2 = \gamma_2 = \gamma_1$. Using the same type of reasoning one can verify that $\beta_1 = \delta_1 = \delta_2 = \beta_2$. These results are known as the corresponding angles postulate.

Sum of angles inside a general triangle

The sum of the interior angles of any triangle must sum to $180°$ or π radians.

Given an arbitrary triangle $\triangle ABC$ with vertex angles α, β and γ, as illustrated in the figure 2-5. Construct a line through the vertex C which is parallel to the base $\overline{AB}$. The parallel lines have the transversal lines $\overline{CA}$ and $\overline{CB}$. The alternate interior angles associated with these transversal lines are the angles α and β. The construction of the line through vertex C which is parallel to the base $\overline{AB}$ demonstrates that the angles α, β and γ are supplementary angles so that

$$\alpha + \beta + \gamma = 180° \qquad \text{or} \qquad \alpha + \beta + \gamma = \pi \tag{2.1}$$

depending upon the units used to measure the angles of the general triangle $\triangle ABC$. (Euclid, Book 1, Proposition 32)

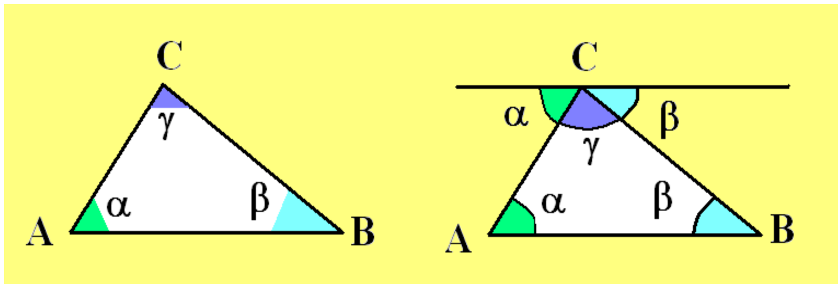


Figure 2-5. General triangles with vertex angles α, β, γ

If degrees are used to measure the angles, then one can say that **the sum of the interior angles of any triangle must sum to** 180°**.** If radians are used to measure the interior angles, then **the sum of the interior angles of any triangle must equal** π **radians**. In scientific computing the radian unit of measure is preferred.

■

Exterior angles of a triangle

In the triangle $\triangle ABC$ of figure 2-6 extend the line segment of any side, then the angle between the extended side and side of the original triangle is called an **exterior angle** of the triangle. For example, extend the side $\overline{AB}$ to the right and create the exterior angle δ as illustrated in the figure 2-6. Also construct a line through the vertex C which is parallel to the base $\overline{AB}$.

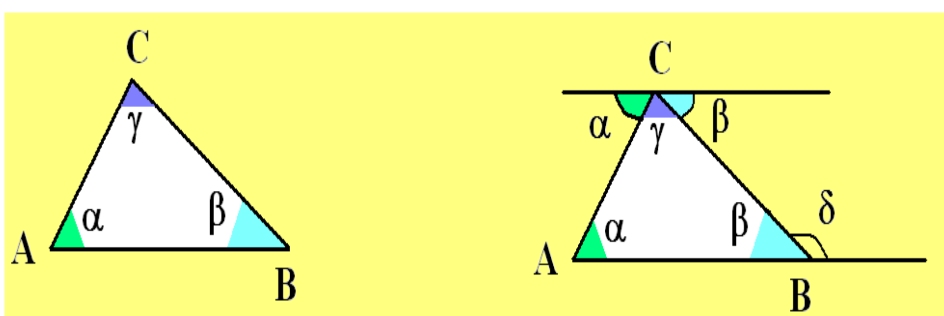


Figure 2-6. Finding exterior angle of a triangle.

Make note that the parallel lines, one through vertex C and the other the line segment $\overline{AB}$, have the transversal lines $\overline{BC}$ and $\overline{AC}$ so that the angles α and β are alternate interior angles which are equal. Also δ and $\alpha + \gamma$ are alternate interior angles which are equal so that one can express the exterior angle as

$$\delta = \alpha + \gamma$$

This demonstrates that **the exterior angle of a triangle is always equal to the sum of the two opposite interior angles.** (Euclid, Book 1, proposition 32)

Parallel lines

In Euclid's book 1, proposition 27, it is demonstrated that if a straight line crosses two lines ℓ_1 and ℓ_2 such that either (a) the corresponding angles are equal or (b) the alternate interior angles are equal, then the lines ℓ_1 and ℓ_2 are parallel lines.

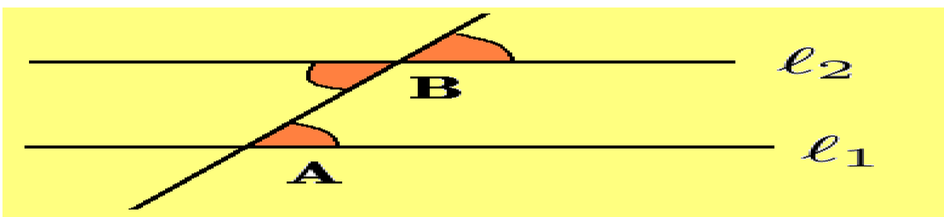


Figure 2-7. Parallel lines

The proof given by Euclid uses the following line of reasoning. If the alternate interior angles are equal, then angle A must equal angle B. If one assumes the lines are **not parallel**, then one can say that line ℓ_1 eventually meets line ℓ_2 at some point, say to the right of line segment $\overline{\text{AB}}$. The lines meeting leads to the construction of a triangle with base line $\overline{\text{AB}}$ and exterior angle B. This leads to a contradiction that the angle A cannot equal angle B because in the previous section we have shown that the exterior angle must equal the sum of the two opposite interior angles. This is how Euclid proved that if the alternate interior angles of the intersection are equal, then one can conclude that the lines must be parallel.

Area

A **square** can be used to define **area associated with a figure**. If each side of a square has length ℓ, then one says the area of the square is ℓ-squared units, written as ℓ^2 units. The length of ℓ can be any length, small or large. If $\ell = 1$ *millimeter*(*mm*), then the area is $1\ mm^2$ (1 *millimeter squared*). If $\ell = 1$ *mile*, then the area of the square is $1\ mi^2$ (1 *square mile*). If ℓ is one foot (ft), then the area of the square is $1\ ft^2$

(one square foot). If $\ell = 1$ $meter(m)$, then the area of the square is $1\ m^2$ (one square meter).

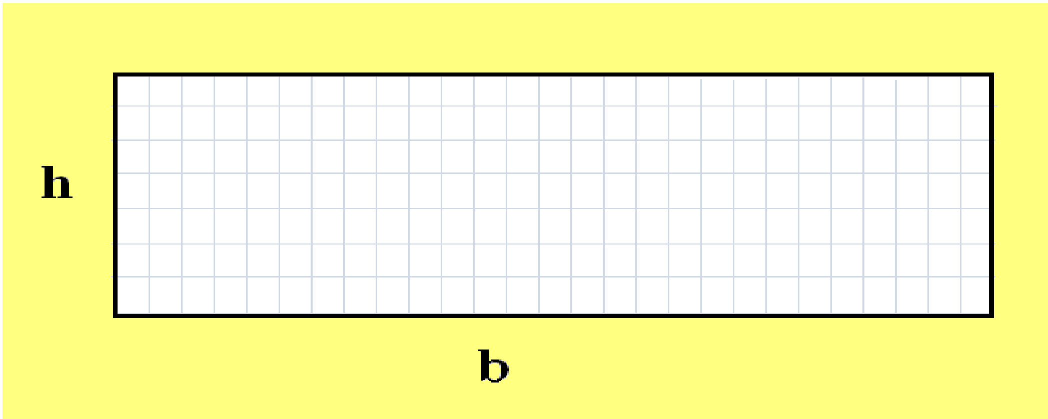


Figure 2-8. Finding the area of a rectangle with base b and height h

Consider the problem of finding the area A of the rectangle illustrated in the figure 2-8, where h is the height and b is the base of the rectangle. One can select a unit of measurement, say 1 centimeter (1 cm), such that there exists integers h and b, where h centimeters represents the height of the rectangle and b centimeters represents the base of the rectangle. Marking off the height and base in centimeters and constructing vertical and horizontal lines at these marks, one can create a grid of square centimeters. One can then count the number of squares within the rectangle to determine the total square area. One finds the total area A of the rectangle can be expressed

$$\text{Area of rectangle} = (\text{base})(\text{height}) \qquad \text{or} \qquad A = bh$$

square units of area. Here the unit of measurement is the centimeter, so the total area would be in units of centimeters squared or cm^2. The **units of measurement** selected for the base b and height h **must be the same** in order to get square units of area. The length of a side of the unit square can be any convenient unit of measurement. The area is not always an integer but depends upon the unit of measurement selected.

In the study of various geometrical shapes one will find that the area of the shape can many times be expressed as being proportional to some length or lengths associated with the shape which are then used in an appropriate manner to determine the area enclosed by the shape.

Area of a right triangle

Consider the rectangle ABCD with base length of b units and a height of h units, as illustrated in the figure below.

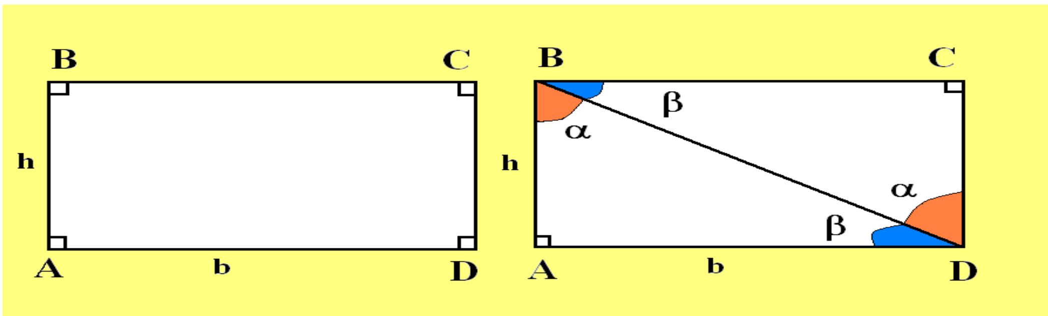


Figure 2-9.
Rectangle and rectangle with diagonal line $\overline{BD}$

The rectangle can be thought of as the perpendicular intersection by two sets of parallel lines giving a right angle at all the corners of the intersection. The area A_r of the rectangle is the base b times the height h or $A_r = bh$. If one draws the diagonal line $\overline{BD}$ and labels the alternate interior angles α and β, then two right triangles are formed. One can now split the rectangle into two right triangles as illustrated in the figure below.

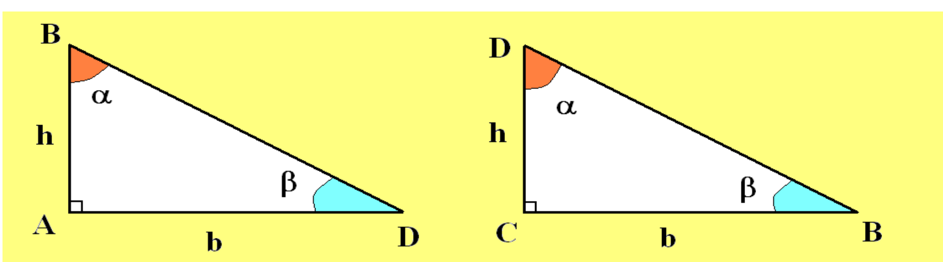


Figure 2-10.
The two right triangles are congruent

Observe that by symmetry the diagonal $\overline{BD}$ has produced two congruent right triangles. The two right triangles formed are exactly equal to one another. Comparing the two triangles formed one can see that all the angles are the same and all the sides are the same. Now the whole is equal to the sum of its parts, so that one can conclude that the area of a right triangle is half the base times the height. In terms of symbols, let $[ABD]$ denote the area of the right triangle $\triangle ABD$ which is the same as the area $[CDB]$ of the right triangle $\triangle CDB$, so that one can write

$$2[ABD] = A_r \quad \text{or} \quad 2[ABD] = bh \quad \text{or} \quad [ABD] = \frac{1}{2}bh \tag{2.2}$$

which shows that

The area of a right triangle is half the base times the height

Area of an arbitrary triangle

Consider the general triangles $\triangle ABC$ illustrated in the figure 2-11, where all the sides and angles of the triangle are different and the triangles are not right triangles. By dropping a perpendicular line from the vertex B to the base $\overline{\text{AC}}$ (or extend base $\overline{\text{AC}}$) **one can construct two right triangles.** To find the area $[ABC]$ of the triangle in figure 2-11 with vertex angle C less than 90° add the areas of the two right triangles. To find the area $[ABC]$ of the triangle in figure 2-11 with vertex angle C greater than 90° subtract the area of the right triangles formed.

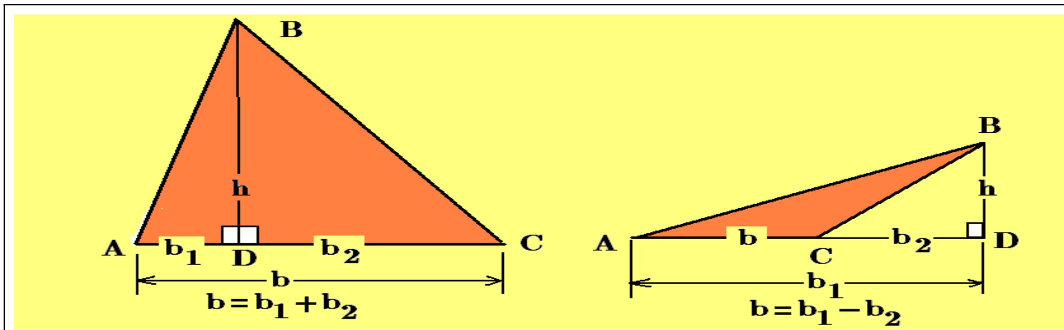


Figure 2-11.

Area of general triangle

Let b denote the base of the triangle and let h denote the height of the triangle as illustrated in the figure 2-11. For the acute triangle in figure 2-11

$$\text{Area } \triangle ABC = \text{ Area } \triangle ABD + \text{ Area } \triangle BDC$$

$$\begin{aligned} \text{or} \quad [ABC] =& [ABD] + [BDC] \\ [ABC] =& \frac{1}{2}b_1 h + \frac{1}{2}b_2 h \\ [ABC] =& \frac{1}{2}(b_1 + b_2)h \\ [ABC] =& \frac{1}{2}bh \end{aligned}$$

because $b = b_1 + b_2$. For the obtuse triangle illustrated in the figure 2-11

$$\begin{aligned}
[ABC] &= \textit{Area } \triangle ABD - \textit{ Area } \triangle BDC = [ABD] - [BDC] \\
[ABC] &= \frac{1}{2}b_1 h - \frac{1}{2}b_2 h \\
[ABC] &= \frac{1}{2}(b_1 - b_2)h \\
[ABC] &= \frac{1}{2}bh
\end{aligned}$$

because $b = b_1 - b_2$ in this case.

This demonstrates

$$[ABC] = \frac{1}{2}bh = \frac{1}{2}(base)(height)$$

In summary, the area of any triangle is equal to half the base times the height

Area of general triangle

Given a general triangle in Cartesian coordinates with vertices $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ as illustrated in the figure 2-12. One can construct a rectangle around the given triangle as illustrated. The area of the rectangle surrounding the given triangle is then

$$\text{Area}_{rectangle} = (x_3 - x_1)(y_2 - y_3) = (base)(height) \tag{2.3}$$

.

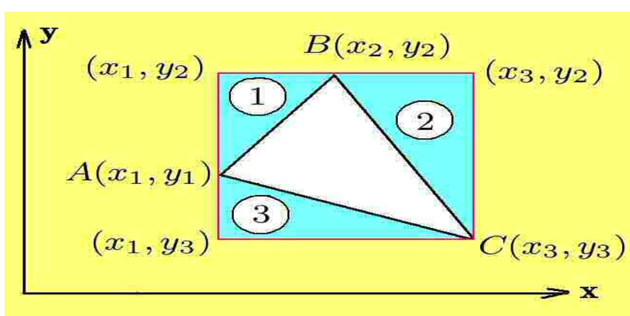


Figure 2-12. General triangle in Cartesian coordinates.

Note that the whole is equal to the sum of its parts, which in this case is the rectangle being composed of four triangles. The given triangle lies inside this rectangle and is surrounded by three other triangles labeled triangle 1, triangle 2 and triangle 3, where the area of each triangle is one-half the base times the height giving

$$\begin{aligned} \text{Area}_{triangle-1} =& \frac{1}{2}(x_2 - x_1)(y_2 - y_1) \\ \text{Area}_{triangle-2} =& \frac{1}{2}(x_3 - x_2)(y_2 - y_3) \\ \text{Area}_{triangle-3} =& \frac{1}{2}(x_3 - x_1)(y_1 - y_3) \end{aligned} \tag{2.4}$$

Therefore one can find the area of the given triangle by writing

$$[\text{ABC}] = Area_{rectangle} - Area_{triangle-1} - Area_{triangle-2} - Area_{triangle-3} \tag{2.5}$$

where [ABC] denotes the area of triangle $\triangle ABC$

or $[\text{ABC}] = (x_2 - x_1)(y_2 - y_3) - \frac{1}{2}(x_2 - x_1)(y_2 - y_1) - \frac{1}{2}(x_3 - x_2)(y_2 - y_3) - \frac{1}{2}(x_3 - x_1)(y_1 - y_3)$

Expand and simplify to show

$$[\text{ABC}] = \frac{1}{2}\left[y_1 x_2 + y_2 x_3 + y_3 x_1 - x_1 y_2 - x_2 y_3 - x_3 y_1\right]$$

The above result is based upon the vertex positions as given by figure 2-12. The area is always considered to be a positive quantity. If the vertex positions change, then the sign associated with the area **may be negative.** To avoid this happening, one uses the **absolute value** of the answer to keep the area positive giving

$$[ABC] = \left|\frac{1}{2}\left[y_1 x_2 + y_2 x_3 + y_3 x_1 - x_1 y_2 - x_2 y_3 - x_3 y_1\right]\right| \tag{2.6}$$

The above result can be remembered using the pattern

$$[ABC] = \frac{1}{2}\left|\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{array}\right|$$

where the arrows show which quantities are to be multiplied. The up arrows receive a + sign and the down arrows receive a - sign and the pattern is enclosed within absolute value symbols.

Euclid's congruence of triangles

In Euclid's *Elements*, one finds the following **three necessary conditions for determining congruence of triangles.**

***SAS* (Side-Angle-Side)** (Euclid, Book 1, Proposition 4)

If two sides, s_1 and s_2 and the include angle α of one triangle are equal respectively to the two sides and included angle of a second triangle, then the two triangles are congruent.

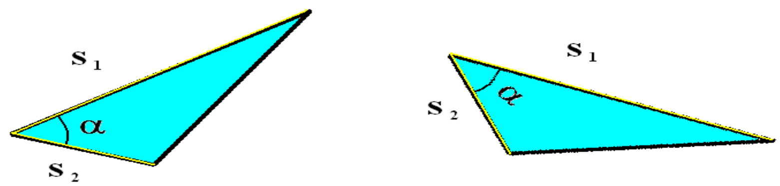


Figure 2-13. Side-Angle-Side required for two triangles to be congruent

***ASA* (Angle-Side-Angle)** (Euclid, Book 1, Proposition 26)

If two angles α and β and the included side s of one triangle are equal respectively to two angles and the include side of another triangle, then the two triangles are congruent.

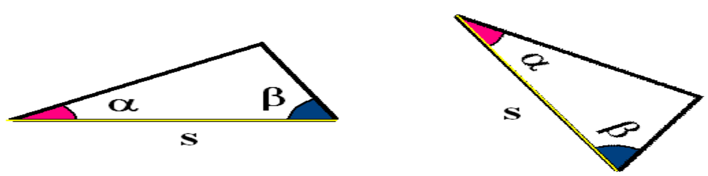


Figure 2-14. Angle-Side-Angle required for two triangles to be congruent

***SSS* (Side-Side-Side)** (Euclid, Book 1, Proposition 8)

If two triangles are such that three sides of one triangle are equal respectively to the three sides of the other triangle, then the two triangles are congruent.

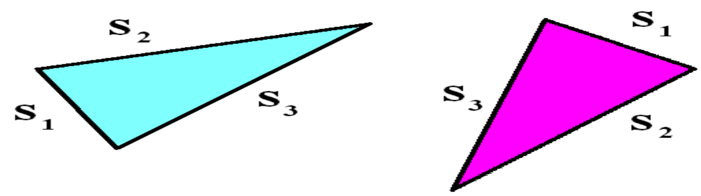


Figure 2-15. Side-Side-Side required for two triangles to be congruent

Euclids proof of SAS and SSS are essentially the observation that triangles will coincide if placed upon one another. Euclid's ASA proof is a proof by contradiction. He assumes the triangles are not congruent with one side different from the other side. He puts in the correct length to make the triangles congruent by use of SAS and then arrives at a contradiction. This shows his original assumption was incorrect and so ASA is valid for establishing congruence of two triangles.

Another test for congruence of triangles

The following test for congruence of triangles is often used. It is known as the Angle-Angle-Side test for congruence. It is not a postulate, but is related to the ASA proposition.

AAS **(Angle-Angle-Side)**

If two triangles are such that **two angles and a non-included side of one triangle** are equal to the corresponding **two angles and non-included side of a second triangle**, then the two triangles are congruent. The situation is illustrated in the figure 2-16.

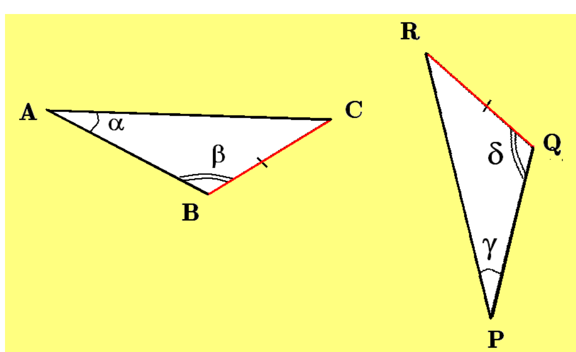


Figure 2-16. Angle-Angle-Side required for two triangles to be congruent

That is, if

$$\begin{array}{ll} \textit{Angle } \alpha \;\cong\; \textit{Angle } \gamma & \textit{Side } \overline{BC} \;\cong\; \textit{Side } \overline{RQ} \\ \textit{Angle } \beta \;\cong\; \textit{Angle } \delta & \text{then } \triangle ABC \;\cong\; \triangle PQR \end{array}$$

Right angle-Hypotenuse-Side (RHS) test for congruence

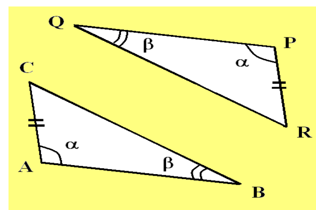
Two right triangles are congruent if the hypotenuse and one side of one triangle equals the hypotenuse and corresponding side of the other triangle.

This is one of the properties of right triangles which will be discussed in the Chapter 4. This test for congruence is related to the SSS proposition.

Example 2-1. Show that the triangles illustrated are congruent.

Solution

Given the two triangles, $\triangle ABC$ and $\triangle PQR$ as illustrated in the accompanying figure. Assume that angles $\angle A = \alpha = \angle P$ and $\angle B = \beta = \angle Q$ are the same and that the non-included sides $\overline{PR}$ and $\overline{AC}$ are congruent so that $\overline{PR} \cong \overline{AC}$.



One can then show $\triangle ABC \cong \triangle PQR$ as follows. We know that there is $180°$ in every triangle and since the angles α and β are known, then $\angle C = \angle R = 180° - (\alpha + \beta)$. One can now use the ASA postulate to show the two triangles $\triangle ABC$ and $\triangle PQR$ are congruent. In other words, the AAS is really equivalent to the ASA postulate.

■

Examination of Euclid's propositions

A heuristic approach to solving a problem is to mentally examine the problem and explore possible ways to attack it without using any sophisticated or rigorous mathematical methods of investigation. Heuristic arguments are sometimes used to suggest how one should proceed toward making a mathematical discovery. Note that heuristic arguments are not used or is any part of formal logical studies and are not accepted in any mathematical proofs.

To illustrate how the four propositions SAS, ASA, SSS, AAS provide enough information in order to study and compare triangles, consider the following heuristic study of each proposition.

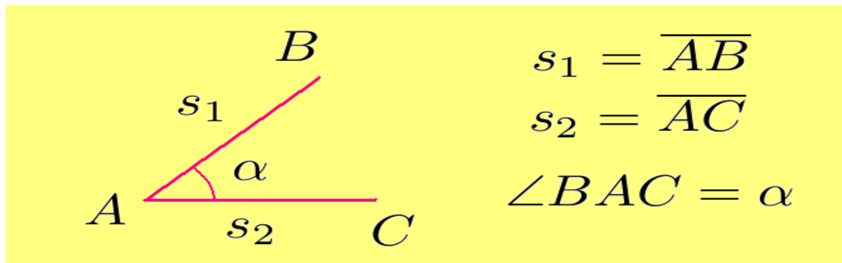


Figure 2-17. Given information for SAS postulate

Let us examine the SAS (Side-Angle-Side) proposition. The given information associated with the SAS proposition is illustrated in the figure 2-17.

If you connect the points B and C by a line segment $\overline{BC}$ you complete the triangle and no other triangle meets all the required conditions. This observation is not a proof of anything but it illustrates how SAS is being employed to establish two triangles being congruent. That is, you don't have to have all the information about a triangle in order to study it and compare it to other triangles. As the above illustration points out, the side of the triangle $\overline{BC}$ is implied and can be constructed from the given information.

Applying the heuristic approach to examine the ASA (Angle-Side-Angle) proposition, make a sketch of the given information as in figure 2-18. Note the rays $\overline{AD}$ and $\overline{BC}$ are implied because the angles are given.

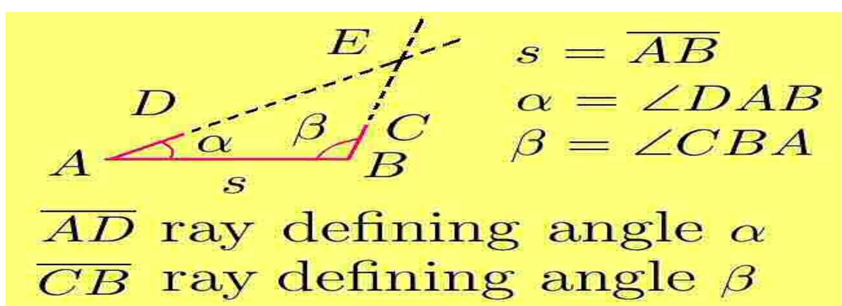


Figure 2-18. Given information for ASA proposition

One can visualize the extensions of the rays $\overline{AD}$ and $\overline{BC}$ intersecting at the point E to form a triangle. All this implies that ASA provides sufficient information for the construction of a triangle and can be used to compare two triangles for congruence.

Examine next the SSS (Side-Side-Side) proposition in a similar manner. Sketch what is given and investigate why this information is enough to create a triangle for further investigation.

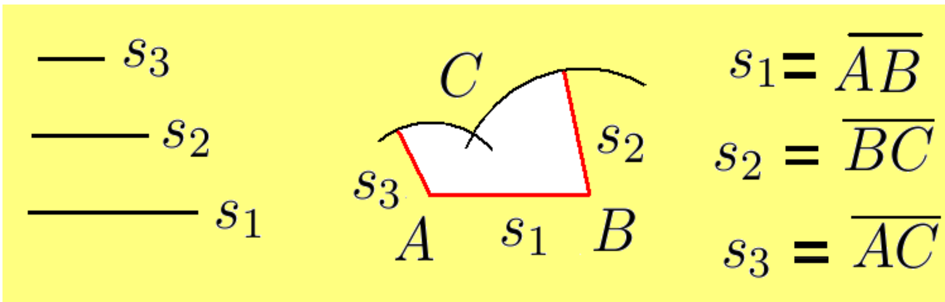


Figure 2-19. Given information for SSS proposition

Examine figure 2-19, and observe that three sides s_1, s_2 and s_3 are given. Attach line s_3 to the left-side of line segment s_1 and attach line s_2 to the right-side of line segment s_1 and label the line segment s_1 as $s_1 = \overline{AB}$. Now rotate side s_3 about the point A and rotate side s_2 about the point B to create arcs of circles. Let C denote the point where the arcs of the circles intersect. If the arcs do not intersect, then no triangle can exist. If the arcs intersect, then the line segments $s_2 = \overline{CB}$ and $s_3 = \overline{CA}$ and $s_1 = \overline{AB}$ create the unique triangle $\triangle ABC$ to be studied. One can see in a heuristic fashion how this information is enough to examine congruence of two triangles.

The above heuristic investigations into the propositions SAS, ASA, SSS, indicate that these propositions provide enough information so that any additional information required to show congruence of triangles can be constructed from the given information.

Still another case to examine

As another combination of angles and sides one can employ for comparison to the Euclid's triangle propositions, examine the figure 2-20 with the given angle α and two given sides s_1 and s_2 . Ask the question, "Can a unique triangle be constructed from (Angle-Side-Side) information?" The answer is no, because there are situations which can occur where the side s_2 is attached to the end of side s_1 and then swings in a circular arc to produce either (a) no intersection with extended line defining the angle α, (b) A unique intersection with extended line, or (c) Two intersections with the extended side of the given angle α. The case (c) produces the two triangles $\triangle ABC$ and triangle $\triangle ABD$ as illustrated in the figure 2-20.

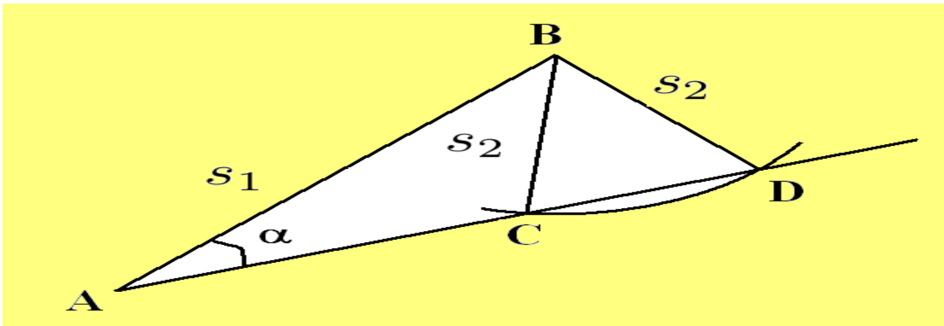


Figure 2-20. Given information for Angle-Side-Side

One can conclude that when given (Angle-Side-Side) data, there are situations where the information does not define a unique triangle and therefore cannot be used to study the congruence of two triangles. This disqualifies the data as not suited for use in a proposition to investigate congruence of triangles.

Terminology

Definitions, axioms and postulates form the starting point for our investigation of geometry. This is followed by theorems or propositions, corollaries and lemmas. A **theorem** is a mathematical statement (proposition) of some kind expressing something being true or false. The mathematical statement presented is usually followed by a demonstration as to how the theorem was arrived at using acceptable mathematical operations. Every theorem has two parts

(i) The **hypothesis** or conditional part and

(ii) The **conclusion** or part to be proved.

Two theorems or propositions are said to be the **converse** of each other whenever

(i) The conclusion of the first theorem becomes the hypothesis of the second theorem and

(ii) The hypothesis of the first theorem becomes the conclusion of the second theorem.

A **corollary** is usually some fact associated with a theorem that is easily proven to be true.

A **lemma** is known as a proved proposition which is used as a prequel to demonstrating a larger more important theorem.

Note that the converse of a true statement may or may not be true. For example, the statement, *All right angles are equal* is a true statement. However, its converse, *All equal angles are right angles* is false.

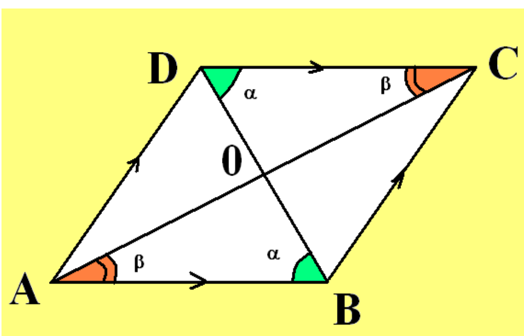
A theorem or proposition is a true statement having been proven true by the use of previous known facts or assumptions. This is known as **deductive reasoning.**

The proof of a mathematical statement should be straight forward without any flaws or contradictions in the reasoning. Theorems are usually statements introducing new ideas different from axioms or postulates. These new ideas are proven using precise language and are presented so that the reader knows exactly how the new idea was arrived at. The following is an example of a theorem and two ways by which the proof of the theorem can be presented.

Example 2-2. Prove the theorem

The diagonals of a rhombus intersect perpendicular and they also bisect each other.

Solution



Start with known facts. Sketch a figure of the rhombus and sketch in the quantities that are being studied. Make use of definitions of a rhombus and observe there are 4 equal sides which are parallel sides. Label the figure with vertices A, B, C, D and then construct the diagonals $\overline{DB}$ and $\overline{AC}$ and label their point of intersection using some symbol like 0.

Examine the figure constructed and note the diagonals are transversal lines associated with the parallel sides. These transversal lines create alternate interior angles which are equal. Label these angles as α and β as illustrated above. We can make the statement side $\overline{DC}$ equals the side $\overline{AB}$ because the sides of a rhombus all have the same length. Therefore, triangle $\triangle A0B$ and triangle $\triangle D0C$ are congruent triangles because of the ASA (Angle-Side-Angle) proposition. Mentally, rotate triangle $\triangle A0B$ counterclockwise about point 0 until it lies on top of triangle $\triangle D0C$ to see that the triangles are congruent and consequently

$$\triangle A0B \cong \triangle C0D \quad \text{so that} \quad \overline{DC} = \overline{AB}, \quad \overline{D0} = \overline{0B} \quad \text{and} \quad \overline{A0} = \overline{0C}$$

This demonstrates that the diagonals of the rhombus bisect one another.

Next examine the triangles $\triangle D0C$ and $\triangle C0B$ and show these triangles are also congruent because of the SSS (Side-Side-Side) proposition. Note side $\overline{0C}$ is common to both triangles and side $\overline{D0}$ equals side $\overline{0B}$, due to diagonals bisecting each other. The sides $\overline{DC}$ and $\overline{BC}$ are equal by definition of a rhombus having 4 equal sides. Since these triangles are congruent, then their corresponding angles are also congruent so that one can write $\angle D0C = \angle C0B$. These angles are also supplementary, so that

$$\angle D0C + \angle B0C = 180^\circ$$

or $2\angle D0C = 180^\circ$, which implies $\angle D0C = \angle B0C = 90^\circ$. This illustrates that the diagonals of the rhombus intersect perpendicularly. ■

Two-column (Statements | Reasons) presentation of proof

Let us examine what we have just done using a symbolic representation of our reasoning by constructing a step-by-step record of our proof.

The first presentation of the proof was given by writing out sentences explaining the step-by-step observations used to prove the theorem. The second presentation of the proof given below is a two-column (Statements | Reasons) approach for presenting proofs. The two-column presentation is a method of illustrating your step-by-step chain of reasoning used to arrive at your conclusions. For each statement given there must be a corresponding reason.

	Statements	Reasons
1.	Sketch rhombus ABCD Sides $\parallel$ and $=$	By definition of rhombus.
2.	$\overline{AB} \parallel \overline{DC}, \alpha, \beta, 0$	Alternate interior angles equal. Labels to figure.
3.	$\overline{DC} = \overline{BC} = \overline{AB} = \overline{AD}$	Sides of rhombus equal.
4.	$\triangle A0B \cong \triangle C0D$	$ASA\ (Angle - Side - Angle)$ $(\beta - \overline{AB} - \alpha)$ $(\beta - \overline{DC} - \alpha)$ where $\overline{AB} = \overline{DC}$
5.	$\overline{CD} = \overline{AB},\ \overline{D0} = \overline{B0},\ \overline{A0} = \overline{C0}$	Sides of congruent $\triangle$'s are equal which shows diagonals bisect one another
6.	$\triangle D0C \cong \triangle C0B$	S S S (Side-Side-Side) $\overline{D0}=\overline{0B},\ \overline{DC}=\overline{BC},\ \overline{0C}=\overline{0C}$
7.	$\angle D0C \cong \angle C0B$	corresponing angles of congruent triangles are equal
8.	$\angle D0C + \angle C0B = 180°$	Angles are supplementary
9.	$2\angle D0C=180°$ $\angle D0C=90°=\angle C0B$	Diagonals intersect $\perp$

Example 2-3. Show the diagonals of a parallelogram bisect one another and the opposite vertex angles are congruent.

Solution

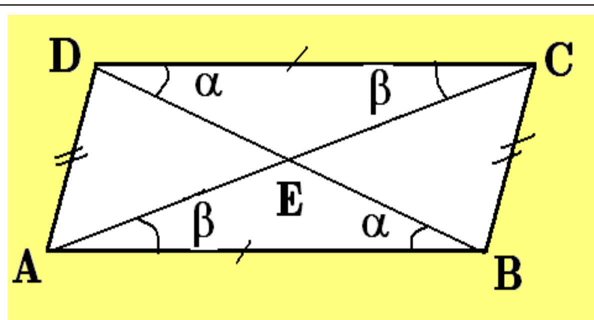


Figure 2-21. Parallelogram with diagonals $\overline{DB}$ and $\overline{AC}$

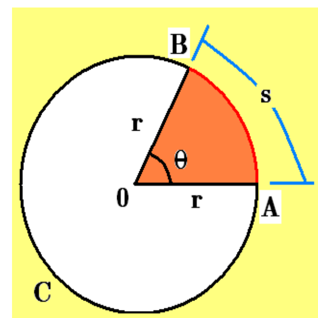
Given the parallelogram in figure 2-21 and note that by definition the opposite sides are congruent with $\overline{\text{AD}} = \overline{\text{BC}}$ and $\overline{\text{DC}} = \overline{\text{AB}}$. Construct the diagonal lines $\overline{\text{DB}}$ and $\overline{\text{AC}}$ and observe the equal angles $\alpha = \angle EDC = \angle EBA$ and $\beta = \angle EAB = \angle ECD$ because alternate interior angles associated with a transversal line are equal. This implies that the triangles $\triangle AEB$ and $\triangle CED$ are congruent because of angle-side-angle. The angles being α and β and the sides being $\overline{DC}$ and $\overline{AB}$, so that one can write $\triangle AEB \cong \triangle CED$. Therefore $\overline{EB} = \overline{ED}$ and $\overline{EA} = \overline{EC}$ because the corresponding sides of congruent triangles are equal. This demonstrates that the diagonals bisect one another.

The triangles $\triangle ADB$ and $\triangle CBD$ are congruent ($\triangle ADB \cong \triangle CBD$) because of SSS (side-side-side) with $\overline{\text{DB}}$ being a common side. Therefore, the opposite vertex angles at A and C are equal ($\angle A = \angle C$). In a similar fashion one can show $\triangle ADC \cong \triangle CBA$ because of SSS, which implies the opposite vertex at D and B are equal ($\angle D = \angle B$).

It is left as an exercise to construct a two-column (Statements | Reasons) proof of the above results.

■

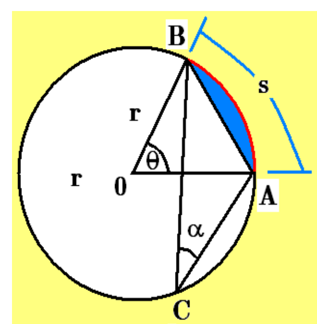
Terminology for the circle



Recall that when using radian measure, the arc length s swept out by the circle radius r rotating counterclockwise about the circle center is given by

$$s = r\theta, \quad \theta \text{ in radians} \tag{2.7}$$

The angle θ is called the central angle associated with the arc length s. Another notation for representing arc length is $\overset{\frown}{\text{AB}}$ for the central angle swept out as point A moves along the circumference to the point B. Note the arc $\overset{\frown}{\text{AB}}$ associated with the smaller central angle is called the **minor arc of the circle** and the arc $\overset{\frown}{\text{BCA}}$ associated with the larger central angle is called the **major arc of the circle.**

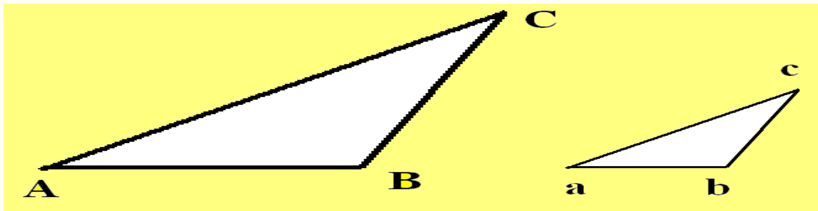


The line segment $\overline{\text{AB}}$ is called the chord associated with the arc length s. A **sector** (orange area above) of a circle is the area bounded by the two radii and the intercepted arc s. A **segment** (blue area) of circle is represented by the area bounded by the arc s and the chord $\overline{\text{AB}}$.

Any angle $\angle ACB = \alpha$ having its vertex on the major arc of the circle and sides terminating at the points A and B defining the minor arc of the circle is called **an angle associated with the arc** $\overset{\frown}{\text{AB}}$**.** In the next chapter we will show that the angles θ and α associated with the same arc $\overset{\frown}{\text{AB}}$ are always related to one another. In order to understand this relationship you must first become familiar with the terminology used to study angles and arc lengths associated with a circle.

Similarity

Two objects are said to be **similar** if the objects have the **same shape** but are **scaled to have different sizes.** Similarity is denoted using the mathematical symbol $\sim$



If triangle $\triangle ABC$ is similar to triangle $\triangle abc$, then this is written as $\triangle ABC \sim \triangle abc$. Think of triangle $\triangle ABC$ as a scaled version of triangle $\triangle abc$. If all the sides of triangle $\triangle abc$ have been scaled upward to produce triangle $\triangle ABC$, then this implies there exits a scale factor s where the sides of triangle $\triangle abc$ can be multiplied by s to produce

$$s\,\overline{ac} = \overline{AC}$$

$$s\,\overline{ab} = \overline{AB}$$

$$s\,\overline{bc} = \overline{BC}$$

where s is the positive nonzero scale factor. This in turn implies that **the ratio of the sides of two similar figures are proportional to one another.** This can be expressed by writing

$$\frac{\overline{AC}}{\overline{ac}} = \frac{\overline{AB}}{\overline{ab}} = \frac{\overline{BC}}{\overline{bc}} = s \tag{2.8}$$

Note the difference between congruence and similarity. Congruence requires the **same size and shape**, while similarity requires only the **shape to be the same.**

In Cartesian coordinates **a translation** is the process of moving an object a certain distance without changing the object. Every point on the object is moved the same distance and in the same direction. The figure 2-22 illustrates the translation of the letter A. All points (x_0, y_0) within the figure A get translated to the new position $(x_0 + h, y_0 + k)$ where h and k are given constants.

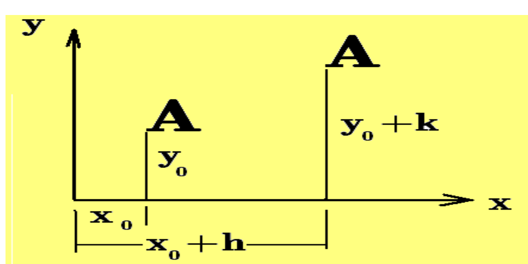


Figure 2-22. Translation of the letter A.

A **rotation** of an object occurs whenever a center of rotation is selected and then every point on the object is rotated through the same angle about the center point of rotation.

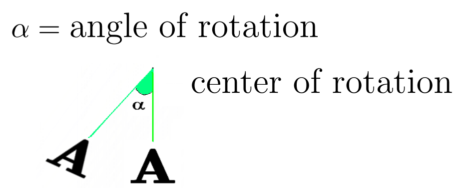


Figure 2-23. Rotation of object.

A **reflection** of an object occurs whenever a line is used to create a mirror image of the object. The line is called **a line of symmetry**. A square has four lines of symmetry.

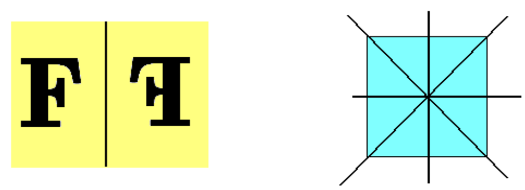
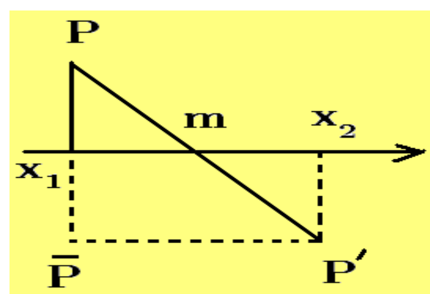


Figure 2-24. Lines of symmetry of an object.

Glide reflection

A glide reflection $(P \to P')$ is composed of a reflection $(P \to \bar{P})$ in a line x followed by a translation $(\bar{P} \to P')$ in the direction the line. A glide reflection is illustrated in the figure on the right. Note that in a glide reflection the point $m = \frac{x_1+x_2}{2}$ will always be the midpoint of the line segment $\overline{x_1x_2}$.



Similar figure

In general, two figures are said to be similar if

(a) They both have the same shape.

(b) One shape is a translation, rotation, reflection, glide reflection contraction or expansion of the other shape.

(c) There is a uniform scaling of one shape to the other. This scaling is an enlargement or shrinking which can be treated as a transformation of the first shape to the second shape.

Imagine two objects which appear to have the same shape, say a small object and a large object. If you scale the smaller object by a multiplication factor s and you can find a value of s which makes the smaller object **congruent** to the larger object, then you can say the two objects are **similar.**

Two triangles are similar if two angles of one triangle are congruent to two angles of the other triangle. This is sometimes called angle-angle similarity and implies all the angles of the triangle are known because there has to be $180°$ in every triangle. The only difference between the triangles is due to the scale factor s either expanding ($s > 1$) or contracting ($0 < s < 1$) the second triangle.

Two right triangles are similar if an acute angle of one equals an acute angle of the other. Any line drawn through the triangle and parallel to a triangle side will create a smaller triangle similar to the original triangle. Note that similar triangles have corresponding sides which are parallel to one another.

One can have **side-side-side (SSS) similarity if the corresponding sides of two triangles are proportional**. One can also use **side-angle-side (SAS) similarity if the angle of one triangle is congruent to an angle of another triangle and the lengths of the sides that include the angle are proportional**.

Two polygons are similar if their interior angles are equal and their corresponding sides are proportional. All circles are similar because the diameter (or radius) of one can be scaled to that of the other. All squares are similar because their interior angles are all equal.

Note that similar polygons can be divided into similar triangles with the ratio of the corresponding sides squared equal to the ratio of the areas as illustrated in the figure 2-25. (Euclid, book 6, proposition 20.)

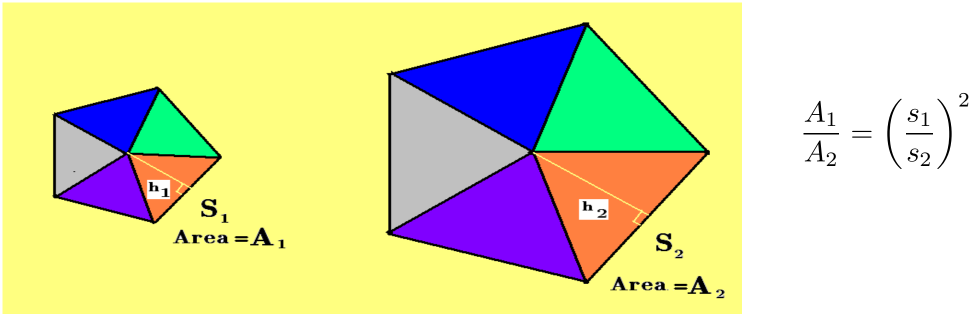


Figure 2-25. Ratio of areas for polygons

Example 2-4. Similarity and area

Given the two similar triangles illustrated in the figure 2-26. One can say the sides a, b, c and altitudes h are proportional so that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{h_1}{h_2} \tag{2.9}$$

These ratios are called **similarity ratios** associated with the two triangles $\triangle AB_1C_1$ and $\triangle AB_2C_2$. Using the theory of proportions one can show that **the perimeters of these triangles are also proportional** in that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{a_1 + b_1 + c_1}{a_2 + b_2 + c_2}$$

(See Chapter 1, Example 1-5.)

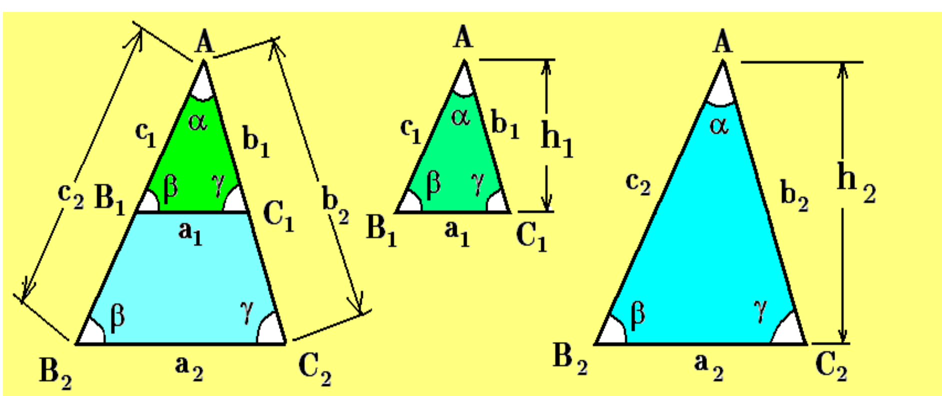


Figure 2-26. Similar triangles $\triangle AB_1C_1 \sim \triangle AB_2C_2$

Make note of the fact that the sides of the similar triangles have units of length and the areas of the similar triangles have units of length squared. This suggests that when **dealing with the ratio of areas associated with similar triangles**, one would expect to be dealing with quantities having units of length squared. Let

$$[AB_1C_1] = \text{Area triangle } \triangle AB_1C_1 = \frac{1}{2}a_1h_1$$
$$[AB_2C_2] = \text{Area triangle } \triangle AB_2C_2 = \frac{1}{2}a_2h_2$$

The similarity ratios from the equations (2.9) imply $h_2 = h_1\frac{a_2}{a_1}$ so that one can write the ratio of the areas associated with similar triangles as

$$\frac{[AB_1C_1]}{[AB_2C_2]} = \frac{\frac{1}{2}a_1h_1}{\frac{1}{2}a_2h_2} = \frac{a_1}{a_2}\frac{h_1}{h_1\frac{a_2}{a_1}} = \left(\frac{a_1}{a_2}\right)^2$$

This shows that when **dealing with similar triangles the ratio of the triangle areas is equal to the square of the similarity ratio** or

$$\frac{[AB_1C_1]}{[AB_2C_2]} = \frac{a_1^2}{a_2^2} = \frac{b_1^2}{b_2^2} = \frac{c_1^2}{c_2^2} = \frac{h_1^2}{h_2^2}$$

Note the above result will not hold if the triangles are not similar.

■

Examine the figure 2-27 and make the observation that not all rectangles are similar. One can say also that not all isosceles triangles are similar. Remember that for triangles to be similar it is necessary that two angles from one triangle be equal two angles from the other triangle.

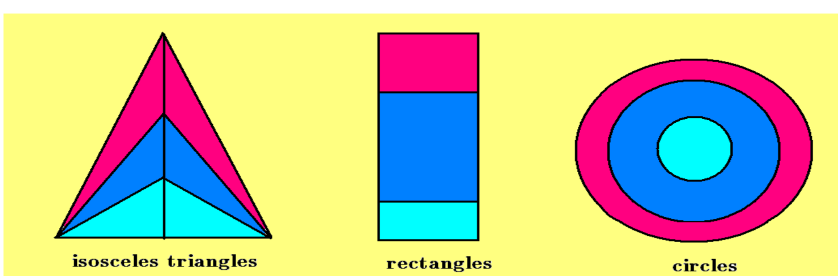


Figure 2-27. Which objects are similar?

Certain special triangles are recognized as being similar.

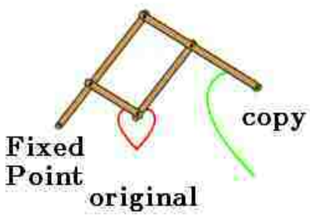
(i) A line drawn through a triangle and parallel to a base creates two similar triangles.

(ii) In a right triangle, if an altitude is constructed from the right angle to the hypotenuse, then three similar right triangles are created. The three triangles are the ones left and right of the altitude being similar to one another as well as being similar to the whole right triangle.

(iii) If two triangles have a common equal angle with the sides of this angle being proportional, then the triangles are similar.

(iv) Two equilateral triangles are similar with proportional sides.

Example 2-5.

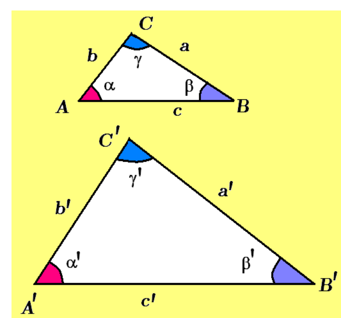


Similar figures can be created using a pantograph which is a mechanical device used for tracing and scaling such things as maps, designs, curves, blue prints, photos, etc. The length of the rods are adjustable to obtain the desired scaling. In the figure on the left the red curve is the original and the green curve is the copy. The figure is courtesy of Wikepedia.

■

Summary

Similar triangles have corresponding sides in the same proportion. For example,



$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Similar triangles have two corresponding angles equal. (Note that this implies all three corresponding angles are congruent since there must be π radians in every triangle.) For example,

$$\alpha = \alpha', \ \beta = \beta', \ \gamma = \gamma'$$

Similar triangles will result if two corresponding sides are in the same proportion and the included angles are equal. For example,

$$\frac{a}{a'} = \frac{c}{c'} \text{ and } \beta = \beta'$$

Dilatation

A dilatation expands or contracts a figure. The expanded or contracted figure has the exact same shape as the original figure and only differs in size. Associated with every dilatation is a scale factor and center of dilatation. Given the triangle $\triangle ABC$, select a point 0 outside the triangle as the center of dilatation and then construct the straight lines $\overline{0\text{A}}, \overline{0\text{B}}, \overline{0\text{C}}$ and extend these lines as illustrated in the figure 2-28.

Select the points A', B', C' such that

$$\overline{0C'} = s\,\overline{0\text{C}}, \quad \overline{0B'} = s\,\overline{0\text{B}}, \quad \overline{0A'} = s\,\overline{0\text{A}}$$

where s is a scale factor. If $0 < s < 1$, then a contraction occurs. If $s > 1$, then an expansion occurs. If $s = 1$, then an identity results.

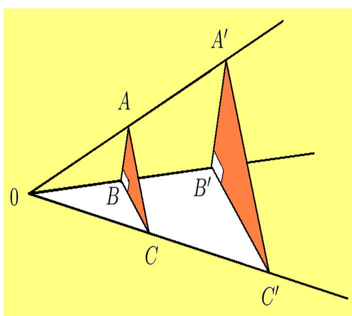


Figure 2-28. **Dilatation of triangle.**

One can slide the triangle $\triangle ABC$, keeping it parallel to triangle $\triangle A'B'C'$ to see the effect of scaling or seeing how one triangle is transformed into another similar triangle. One can then say that triangle $\triangle ABC$ is similar to triangle $\triangle A'B'C'$ written $(\triangle ABC \sim \triangle A'B'C')$. Note that any of the ratios

$$\frac{\overline{0C'}}{\overline{0\text{C}}} = \frac{\overline{0B'}}{\overline{0\text{B}}} = \frac{\overline{0A'}}{\overline{0\text{A}}}$$

will give you the scale factor s.

This type of scaling can be performed with any shaped plane figure.

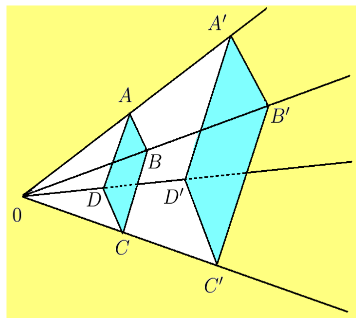


Figure 2-29. **Dilatation of quadrilateral.**

■

Example 2-6. **Slope values for perpendicular lines**

The following is an examination of two lines in Cartesian coordinates which intersect so that opposite angles are all equal to $\frac{\pi}{2}$ radians. Whenever this occurs the lines are said to be perpendicular to one another. In such cases one can use **similarity** to find out certain properties that the perpendicular lines must satisfy.

Examine the sketch in figure 2-30 where lines ℓ_1 and ℓ_2 have a perpendicular intersection at point C. Extend the lines until they intersect with the x-axis at points A and B respectively. These extended lines intersect the x-axis with angles α and β as measured counterclockwise from the x-axis to the line. The intersections with the x-axis forms the triangle $\triangle ABC$. Construct a line segment through the vertex C which is perpendicular to the x-axis at the point D. Let the constructed line segment have length h and note that it divides the line segment $\overline{AB}$ into two parts. Label these two parts $x_1 = \overline{AD}$ and $x_2 = \overline{DB}$. Next construct the similar right triangles $\triangle CDA$ and $\triangle BDC$ as illustrated in the figure 2-30.

Observe that the angles γ and α are complementary angles so that $\alpha + \gamma = \frac{\pi}{2}$. The right triangles constructed are similar triangles with their sides proportional and so one can write the ratios

$$\frac{h}{x_2} = \frac{x_1}{h} \qquad \Rightarrow \qquad h^2 = x_1 x_2 \tag{2.10}$$

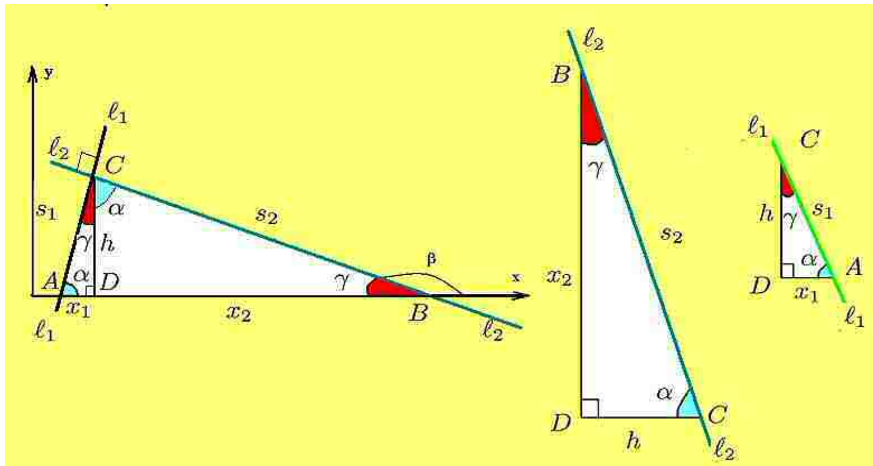


Figure 2-30. Perpendicular intersection of lines ℓ_1 and ℓ_2

Examine $\triangle ABC$ and show the slope of line ℓ_1 is given by the positive slope

$$m_1 = \frac{\text{change in y-values}}{\text{change in x-values}} = \frac{h}{x_1} \tag{2.11}$$

and the slope of line ℓ_2 is given by the negative slope

$$m_2 = \frac{\text{change in y-values}}{\text{change in x-values}} = \frac{-h}{x_2} \tag{2.12}$$

One can use the results from equation (2.10) and write

$$m_2 = \frac{-h}{x_2} = \frac{-h\,x_1}{x_2\,x_1} = \frac{-h\,x_1}{h^2} = \frac{-x_1}{h} = \frac{-1}{m_1} \tag{2.13}$$

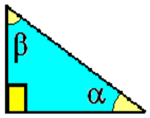
The equation (2.13) tells us that if **two lines ℓ_1 and ℓ_2, with slopes $m_1 \neq 0$ and $m_2 \neq 0$ intersect perpendicularly, then the product of the slopes must equal minus one or $m_1 m_2 = -1$.** Alternatively, one can say that **if two lines are perpendicular, then one slope must be the negative reciprocal of the other slope, provide neither slope is zero.**

Note that if one of the lines is perpendicular to the x-axis, then its equation is $x = x_0$ a constant and then the other perpendicular line has the equation $y = y_0$ a constant.

■

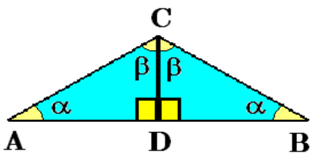
Exercises

▶ **2-1.**



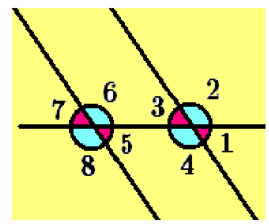
In the right triangle illustrated prove the angles α and β are complimentary angles.

▶



If $\triangle ADC$ is a reflection of $\triangle BDC$, then prove that
$\triangle ADC \cong \triangle BDC$

▶ **2-3.**



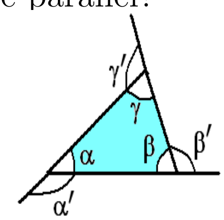
The figure illustrates two parallel lines intersected by a transversal. If the measure of angle 1 ($m\angle 1$) is $30^\circ(\frac{\pi}{6}rad)$, then

(i) find the measure of all the other angles.

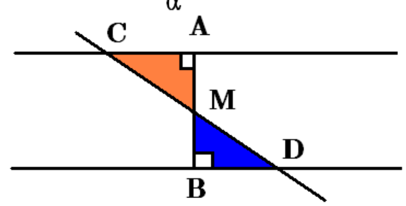
(ii) Determine which angles are supplementary.

(iii) What conditions must the angles satisfy for the lines to be parallel?

▶ **2-4.** Find the exterior angles α', β', γ', given α, β, γ.
What are the exterior angles if the triangle sides are extended opposite to the direction illustrated?



▶ **2-5.** Line segment $\overline{AB}$ is perpendicular to the parallel lines illustrated. The point M is the midpoint of line segment $\overline{AB}$ and line segment $\overline{CD}$ passes through the point M. Show $\overline{AC} = \overline{BD}$ and $\overline{CM} = \overline{DM}$.



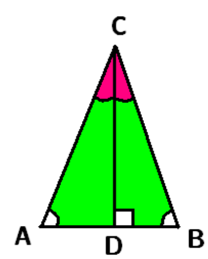
▶ **2-6.** Prove the diagonals of a parallelogram bisect one another and the opposite vertex angles are equal using a two-column (statement | reason) proof.

▶ **2-7.** If $\triangle ABC$ is an isosceles triangle, show

(i) The angle C bisector is perpendicular to the base $\overline{AB}$.

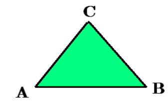
(ii) The angles opposite the equal sides are equal.

(iii) The bisector of angle C divides the base in half.

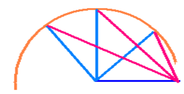


▶ **2-8.** For the isosceles triangle illustrated in the previous problem, extend a side through the vertex C and then bisect the exterior angle and show the bisector line is parallel to the base of the triangle.

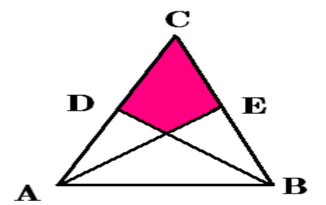
▶ **2-9.** Prove the angles of an equilateral triangle are equal.



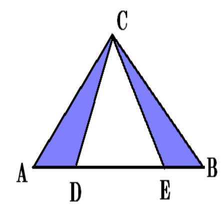
▶ **2-10.** Given two legs of a triangle show that the triangle with maximum area that can be constructed is a right triangle.



▶ **2-11.** Given the isosceles triangle $\triangle ABC$ with line segments $\overline{AE}$ and $\overline{BD}$. If $\overline{CD} = \overline{CE}$ show that $\overline{BD} = \overline{AE}$



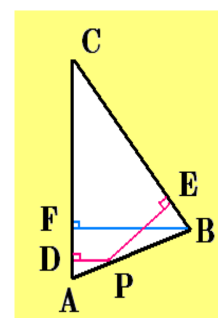
▶ **2-12.** Given the isosceles triangle $\triangle ABC$. If $\overline{AD} = \overline{EB}$ show that $\overline{CD} = \overline{CE} \Rightarrow \triangle CDE$ is isosceles.



▶ **2-13.** Given the isosceles triangle $\triangle ABC$ with $\overline{AC} = \overline{CB}$. Let P denote any point on the side $\overline{AB}$ and construct the lines $\overline{PD}$ and $\overline{PE}$ perpendicular to the isosceles triangle sides followed by the altitude $\overline{BF}$.

(a) Find three similar triangles.

(b) Use the law of proportions to show $\overline{DP} + \overline{PE} = \overline{BF}$



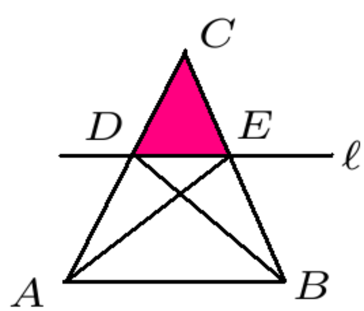
▶ **2-14.** If $\triangle ABC$ is an isosceles triangle with line ℓ parallel to the base $\overline{AB}$, prove that

(i) $\triangle CDB \cong \triangle CEA$

(ii) $\triangle ADB \cong \triangle BEA$

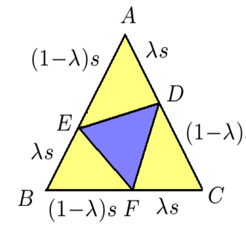
(iii) $\triangle CDE \sim \triangle CAB$

(iv) $\dfrac{\overline{CD}}{\overline{CA}} = \dfrac{\overline{CE}}{\overline{CB}}$

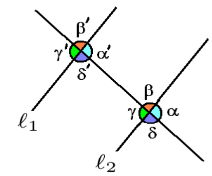


▶ **2-15.** Find the sum of the exterior angles of a triangle.

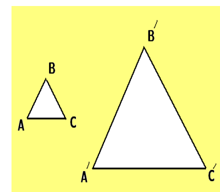
▶ **2-16.** Triangle $\triangle ABC$ is an equilateral triangle with sides of length s and λ is a parameter $0 \leq \lambda \leq 1$. If the distances $\overline{\text{BE}} = \overline{\text{AD}} = \overline{\text{CF}}$ prove that $\triangle DEF$ is an equilateral triangle for all values of λ.



▶ **2-17.** What conditions must be placed upon the angles $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ in order that lines ℓ_1 and ℓ_2 be parallel?



▶ **2-18.** Given the triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar triangles with $\overline{\text{AB}} \parallel \overline{A'B'}$ and $\overline{\text{BC}} \parallel \overline{B'C'}$. Make constructions if necessary to show $\overline{\text{AC}} \parallel \overline{A'C'}$.



▶ **2-19.**

Given the right triangle $\triangle ABC$. From the right angle B construct the median, angle bisector and altitude. Define the angles $\alpha_0 = \angle EBF$ and $\alpha_1 = \angle DBE$.

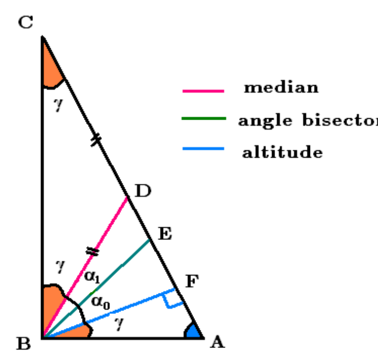
(a) Show $\overline{\text{BD}} = \frac{1}{2}\overline{\text{AC}} = \overline{\text{CD}} \Rightarrow \triangle BDC$ is isosceles.

(b) Show in $\triangle ABF$, $\angle A + \angle ABF = \frac{\pi}{2} = \angle A + \gamma$

(c) Show $\angle BCE = \angle CBD = \angle ABF = \frac{\pi}{2} - \angle A = \gamma$

(d) Show $\gamma + \alpha_1 = \frac{\pi}{4} = \gamma + \alpha_0$

(e) Show $\alpha_0 = \alpha_1$

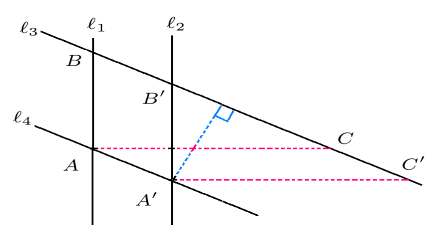


▶ **2-20.** Find the area of the triangle having the vertices with coordinates

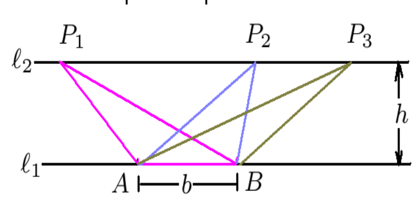
(i) (1,0), (4,5), (5,7) (ii) (3,4), (7,1), (8,5) (iii) (-3,-5), (3,5), (-10,8)

▶ **2-21.** Given two sets of parallel lines, $\ell_1 \parallel \ell_2$ and $\ell_3 \parallel \ell_4$

Show $\triangle ABC \cong \triangle A'B'C'$



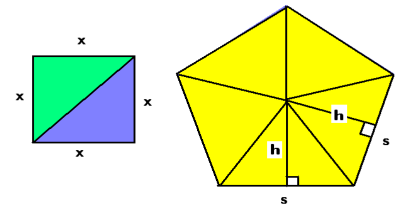
▶ **2-22.** Given two parallel lines ℓ_1, ℓ_2 with points A and B fixed on line ℓ_1 and point P is allowed to move along line ℓ_2. Find the area [APB] no matter where P is on line ℓ_2



▶ **2-23.** Use the fact that the whole equals the sum of its parts to calculate the following areas in terms of triangle areas.

(a) The area of a square.

(b) The area of a regular pentagon if h is the apothem and s is the length of each side.



▶ **2-24.**

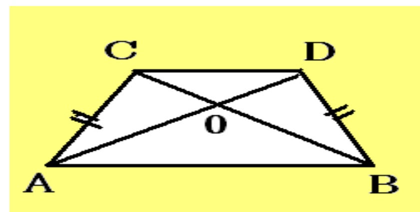
Given the isosceles trapezoid with diagonals $\overline{\text{AD}}$ and $\overline{\text{CB}}$ intersecting at point 0.

(a) Show $\triangle ACB \cong \triangle BDA$

(b) Show $\overline{\text{A0}} = \overline{\text{0B}}$ and $\overline{\text{0C}} = \overline{\text{0D}}$

(c) Show $\overline{\text{ODC}} \sim \overline{\text{0AB}}$

(d) Find a relationship between $\overline{\text{CD}}$ and $\overline{\text{AB}}$



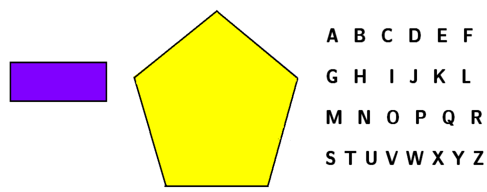
▶ **2-25.**

Find all lines of symmetry associated with the following figures.

(a) A rectangle.

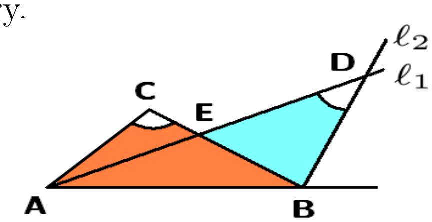
(b) A regular hexagon and regular pentagon.

(c) Examine each letter of the alphabet for symmetry.

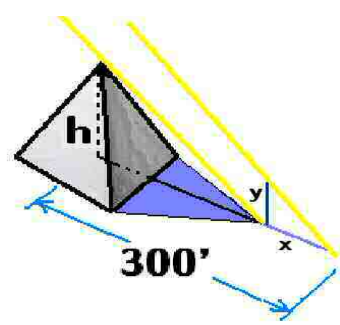


(a) rectangle (b) pentagon (c) letters of alphabet

▶ **2-26.** In triangle $\triangle ABC$ the line ℓ_1 bisects angle A and line ℓ_2 bisects the exterior angle at vertex B. Show that $\frac{1}{2}\angle ACB = \angle EDB$



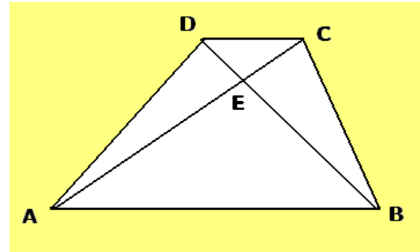
▶ **2-27.** To measure the height h of a pyramid a six foot pole y is placed in the ground at the tip of the pyramid shadow and the shadow x from pole is measured to be $4'$. If the tip of the shadow from the pole is $300'$ from the center of the pyramid, then find the height of the pyramid.



► **2-28.** Given the trapezoid ABCD illustrated with diagonals $\overline{\text{AC}}$ and $\overline{\text{DB}}$.

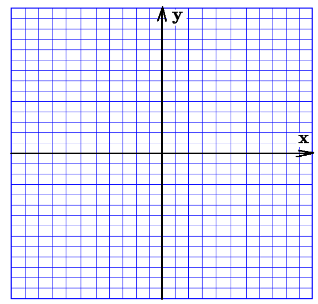
(i) What triangles are similar?

(ii) If $\overline{\text{ED}} = 3$, $\overline{\text{AE}} = 10$, $\overline{\text{EB}} = 8$ and $\overline{\text{DC}} = 4$, then find $\overline{\text{EC}}$ and $\overline{\text{AB}}$.



► **2-29.** Given triangle $\triangle ABC$ with vertices $A: \ (-3,1)$, $B: \ (8,5)$, $C: \ (12,-6)$ Find the slopes of $\overline{\text{AC}}$, $\overline{\text{BC}}$, $\overline{\text{AC}}$. Is the triangle a right triangle?

► **2-30.** (i) Graph the line $\ell_1: \ \frac{x}{5} + \frac{y}{6} = 1$
(ii) Find the line ℓ_2 which is perpendicular to line ℓ_1 which passes through the point $(1,1)$.
(iii) Graph the line ℓ_2.



► **2-31.** Given line ℓ_1 with slope m_1 and line ℓ_2 with slope m_2.

(i) What condition is necessary for the lines to be parallel?

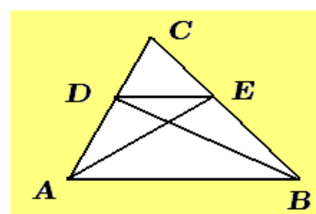
(ii) What condition is necessary for the lines to be perpendicular?

► **2-32.**

Given triangle $\triangle ABC$ with parallel lines $\overline{\text{DE}}$ and $\overline{\text{AB}}$ followed by the construction of the lines $\overline{\text{DB}}$ and $\overline{\text{AE}}$. If $\overline{\text{CD}} = \overline{\text{CE}}$ and $\overline{\text{DA}} = \overline{\text{EB}}$, then

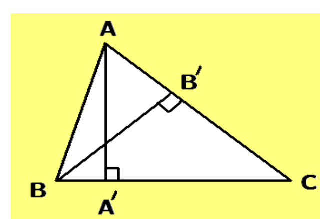
(a) Show that $\triangle DAE \cong \triangle EBD$

(b) Show that $\triangle DAB \cong \triangle EBA$



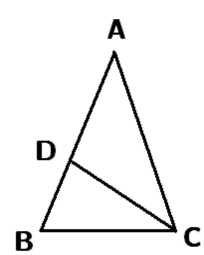
► **2-33.**

Let $\overline{AA'}$ and $\overline{BB'}$ denote altitudes associated with triangle $\triangle ABC$. Show that $(\overline{AA'}) \cdot (\overline{\text{BC}}) = (\overline{BB'}) \cdot (\overline{\text{AC}})$



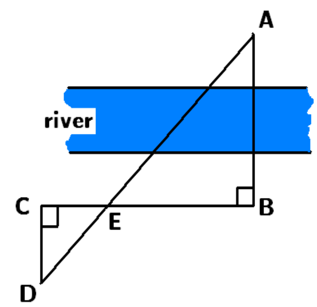
► **2-34.**

Given isosceles triangle $\triangle ABC$ with $\overline{\text{AC}} = \overline{\text{AB}}$ and $\overline{\text{BC}} = \overline{\text{CD}}$. Prove that $\triangle ABC \sim \triangle CBD$.



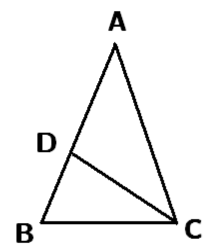
▶ **2-35.**

A surveyor wants to calculate the distance between two points A and B on the opposite sides of a river. From point B he measures a distance $\overline{\mathrm{BC}} = 500'$, where $\overline{\mathrm{BC}}$ is perpendicular to line $\overline{\mathrm{AB}}$. He then measures a distance $\overline{\mathrm{CD}} = 100'$, where $\overline{\mathrm{CD}}$ is perpendicular to line $\overline{\mathrm{BC}}$. The line-of-sight from point D to A intersects line $\overline{\mathrm{BC}}$ at point E. The distance $\overline{\mathrm{EC}}$ is measured and found equal to $50'$. Find the distance $\overline{\mathrm{AB}}$.



▶ **2-36.**

Given isosceles triangle $\triangle ABC$ with $\overline{\mathrm{AC}} = \overline{\mathrm{AB}}$ and $\overline{\mathrm{BC}} = \overline{\mathrm{CD}}$. Prove that $\triangle ABC \sim \triangle CBD$.

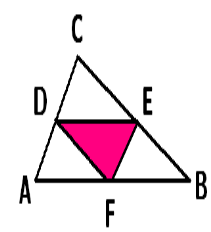


▶ **2-37.** In triangle $\triangle ABC$ let D, E denote the midpoints of sides $\overline{\mathrm{AC}}$ and $\overline{\mathrm{CB}}$ and then join these points. Use similar triangles to show $\overline{\mathrm{DE}} = \frac{1}{2}\overline{\mathrm{AB}}$. Write out in words what this result is telling you. It will be used in many geometric proofs.

▶ **2-38.** Given a triangle inscribed within a circle. Explain what must be done to circumscribe a triangle about the circle such that the circumscribe triangle is similar to the inscribed triangle.

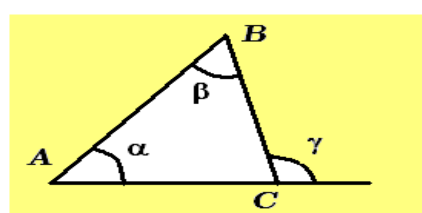
▶ **2-39.**

Given triangle $\triangle ABC$ with points D, E, F lying on the mid-points of the sides as illustrated. Prove that the triangles $\triangle ABC$ and $\triangle DEF$ are similar.



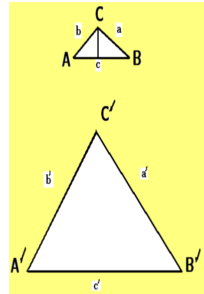
▶ **2-40.**

Given the triangle $\triangle ABC$ with exterior angle $\gamma = 110°$. If $\beta - \alpha = 20°$, then find all the interior angles of triangle $\triangle ABC$.

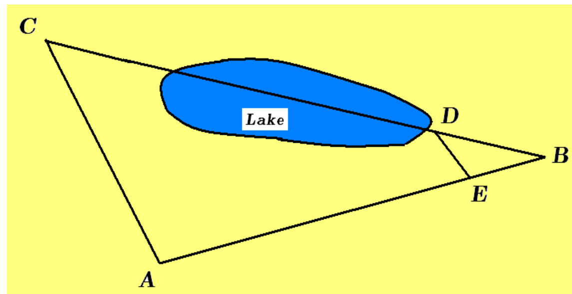


▶ **2-41.** Given that triangle $\triangle ABC$ is similar to triangle $\triangle A'B'C'$ ($\triangle ABC \sim \triangle A'B'C'$). Let $[ABC]$ denote the area of triangle $\triangle ABC$ and $[A'B'C']$ denote the area of triangle $\triangle A'B'C'$. Show that

$$\frac{[ABC]}{[A'B'C']} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{c^2}{c'^2}$$

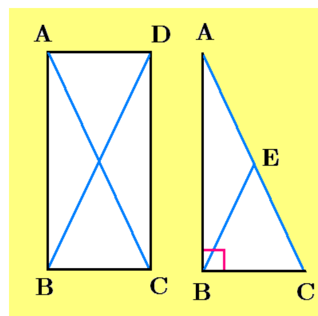


▶ **2-42.** To find the distance $\overline{\text{CD}}$ and $\overline{\text{CB}}$ across a lake, a surveyor measures the distances $\overline{\text{AC}}$, $\overline{\text{DB}}$, $\overline{\text{BE}}$, $\overline{\text{DE}}$ where $\overline{\text{DE}} \parallel \overline{\text{AC}}$. From the known information explain how the distances $\overline{\text{CD}}$ and $\overline{\text{AE}}$ can be calculated.



▶ **2-43.**

(a) Prove the diagonals of a rectangle are equal and bisect one another.
(b) If $\overline{\text{BE}}$ is the median of a right triangle, then show $\overline{\text{BE}} = \frac{1}{2}\overline{\text{AC}}$
(c) Show the median $\overline{\text{BE}}$ of a right triangle divides the triangle into two isosceles triangles.



▶ **2-44.** In a Cartesian coordinate system what happens to a general point (x_0, y_0) during
(a) a reflection about the line $x = 0$?
(b) a reflection about the line $y = 0$?
(c) a reflection about the line $y = x$?

▶ **2-45.** Show that in any triangle $\triangle ABC$ with sides a, b, c one can write

$$a + b \geq \sqrt{ab}, \quad a + c \geq \sqrt{ac}, \quad b + c \geq \sqrt{bc}$$

Hint: Expand $(\sqrt{X} - \sqrt{Y})^2 \geq 0$.

▶ **2-46.** Prove that a triangle with two equal medians must be isosceles.

Geometry

Chapter 3

Shapes and properties

Sum of interior angles of simple polygons

We know the sum of the interior angles of a triangle is 180° or π radians. Let us investigate the sum of the interior angles of polygons with 4,5,6,... sides. Let us begin by examining the interior angles of a quadrilateral, pentagon, hexagon and heptagon as illustrated in the figure 3-1.

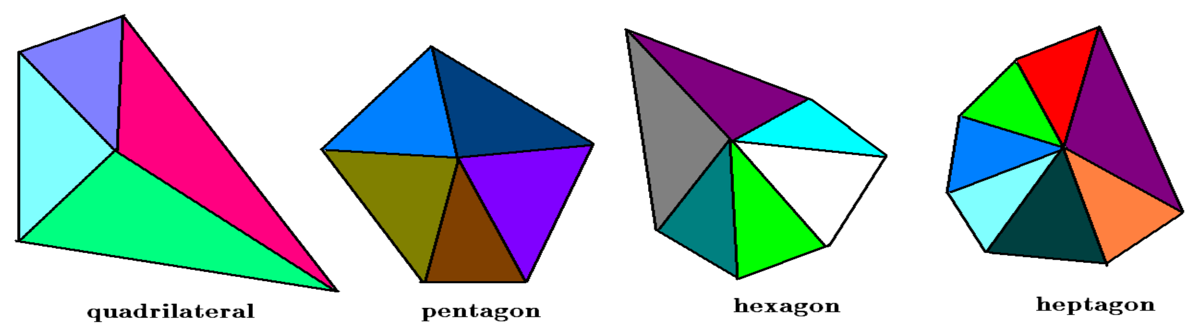


Figure 3-1.
Triangles inside selected convex polygons.

Select a point inside the quadrilateral and then draw lines from the selected point to each vertex of the quadrilateral. This creates four triangles inside the quadrilateral. The interior angles of each triangle is 180° or π radians. The sum of the angles from each triangle is $4(180^\circ)$ or 4π radians. The central angles around the point selected inside the triangle sum to $360^\circ = 2(180^\circ)$ or 2π radians. Therefore, the sum of the interior angles of a quadrilateral is

$$\text{Sum of angles of triangles} - \text{Sum of angles around selected central point} \tag{3.1}$$

This gives

$$4(180^\circ) - 2(180^\circ) = 2(180^\circ) \quad \text{or} \quad 4\pi - 2\pi = 2\pi \text{ radians}$$

as the sum of the interior angles of a quadrilateral.

In a similar fashion select a point inside a pentagon and construct lines from this point to each of the five vertices. This creates five triangles. The sum of the interior angles of a pentagon can be determined by the use of equation (3.1) which produces the relation

$$5(180°) - 2(180°) = 3(180)° \quad \text{or} \quad 5\pi - 2\pi = 3\pi \text{ radians}$$

as representing the sum of the interior angles of a pentagon.

Note that a pattern is developing. By selecting an interior point inside a hexagon, then one can create six triangles so that the equation (3.1) can be used to determine the sum of the interior angles of a hexagon. This gives

$$6(180°) - 2(180°) = 4(180°) \quad \text{or} \quad 6\pi - 2\pi = 4\pi \text{ radians} \tag{3.2}$$

This pattern can be extended to determine the sum of the interior angles of an **n-gon** as

$$n(180°) - 2(180°) = (n-2)(180°) \quad \text{or} \quad n\pi - 2\pi = (n-2)\pi \text{ radians}$$

In summary, one can state that **the summation of the interior angles of an n-gon will be $(n-2)\pi$ radians.**

Sum of exterior angles of simple polygons

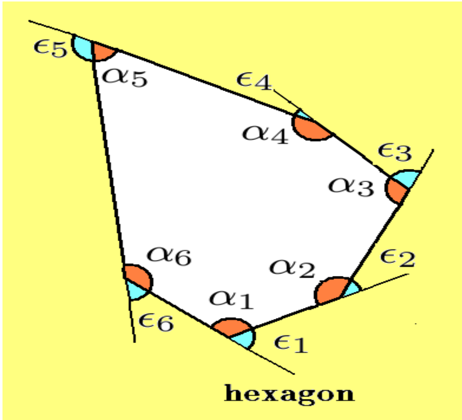
One can set up a pattern for calculating **the summation of the exterior angles of a n-gon.** Examine the hexagon illustrated with the exterior angles ϵ_i, $i = 1,2,3,4,5,6$ and interior angles α_i, $i = 1,2,3,4,5,6$. Note the summation of an exterior angle with its corresponding interior angle will always produce the result π because these angles are supplementary.

The Greek letter sigma $\sum$ is used to denote a summation of terms. The notation

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + \cdots + f(n)$$

can be used as a shorthand to express a summation of the terms $f(1) + f(2) + f(3) + \cdots + f(n)$. The index i below the Greek letter sigma is called the starting summation index and the n on top of the sigma is called the ending index for i. One then substitutes for i all the integers ranging from the starting index to the ending index to obtain the sum.

For the hexagon illustrated one can construct interior angles α and exterior angles ϵ as illustrated. These angles are supplementary angles and so one can write



$$\begin{aligned} \alpha_1 + \epsilon_1 &= \pi \\ \alpha_2 + \epsilon_2 &= \pi \\ \alpha_3 + \epsilon_3 &= \pi \\ \alpha_4 + \epsilon_4 &= \pi \\ \alpha_5 + \epsilon_5 &= \pi \\ \alpha_6 + \epsilon_6 &= \pi \end{aligned} \tag{3.3}$$

The summation notation can be used to express the summation of the above equations. The summation of all of the above equations can be expressed in the form

$$\sum_{i=1}^{6} \alpha_i + \sum_{i=1}^{6} \epsilon_i = 6\pi \tag{3.4}$$

because equals added to equals the results are equal.

Here $\sum_{i=1}^{6} \alpha_i$ represents the summation of the interior angles of the hexagon and $\sum_{i=1}^{6} \epsilon_i$ represents the summation of the exterior angles of the hexagon. We know $\sum_{i=1}^{6} \alpha_i = (6-2)\pi$ from equation (3.2). Therefore, equation (3.4) becomes

$$\sum_{i=1}^{6} \epsilon_i = 6\pi - (6-2)\pi = 2\pi \tag{3.5}$$

The equation (3.5) can be generalized in the case of an n-gon instead of a hexagon. Just replace 6 everywhere it occurs in equation (3.5) by n to obtain

$$\sum_{i=1}^{n} \epsilon_i = n\pi - (n-2)\pi = 2\pi \tag{3.6}$$

which tells us that **the summation of the exterior angles of an n-gon must equal 2π.**

In summary, we have discovered that **the summation of the exterior angles of an n-gon will always equal 2π and the summation of the interior angles of an n-gon is $(n-2)\pi$.**

Area of regular polygons

Consider the regular polygons illustrated in the figure 3-2. These polygons have equal sides and equal interior angles. The symmetry associated with regular polygons allows one to take the summation of the interior angles of an n-gon to obtain $(n-2)\pi$. Dividing this value by the number of sides n produces the angular value at a single vertex. The **center point** of a regular polygon is the point inside the boundaries which is equidistant from each vertex.

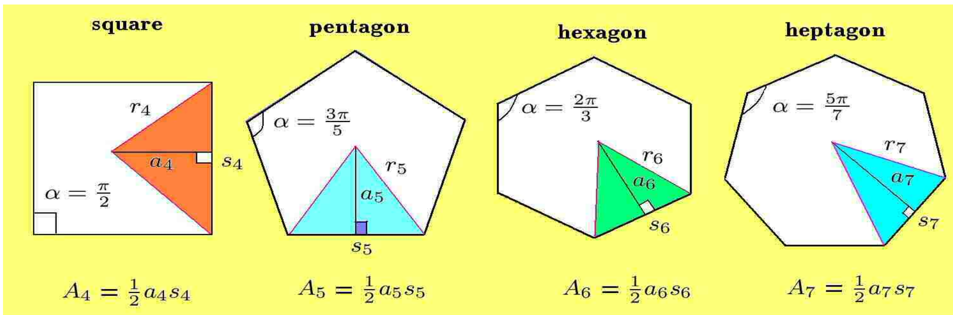


Figure 3-2.
Selected regular polygons

The **apothem** of a regular polygon is the distant from the center point to the midpoint of a side. The center point and apothem for selected regular polygons are illustrated in the figures 3-2 and 3-4. Note that the distance from the center of a regular polygon to a vertex gives the radius R of a circumscribed circle and the distance from the center to the midpoint of a side gives the apothem a which represents the radius of the inscribed circle. Each regular polygon has an interior isosceles triangle consisting of a side s_i, $i = 4, 5, 5, 7, \ldots$ and two radii r_i, $i = 4, 5, 6, 7, \ldots$. This terminology is illustrated in the figure 3-3 for the special case of a pentagon.

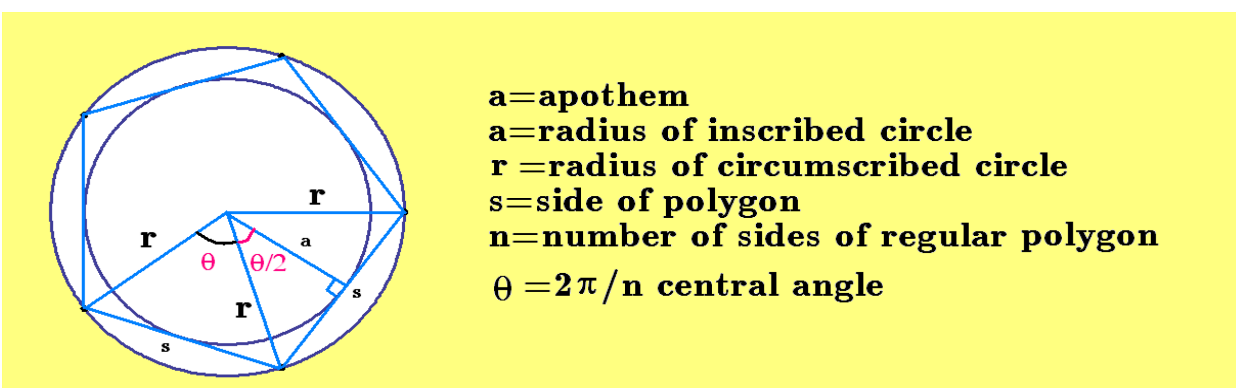


Figure 3-3.
Inscribed and circumscribed circles for pentagon.

Each regular polygon with n-sides $(n \geq 4)$ has an apothem a_n, $n = 4, 5, 6, 7 \ldots$ which represents the height of a single isosceles triangle which is part of the regular polygon. The situation is illustrated in the figure 3-4.

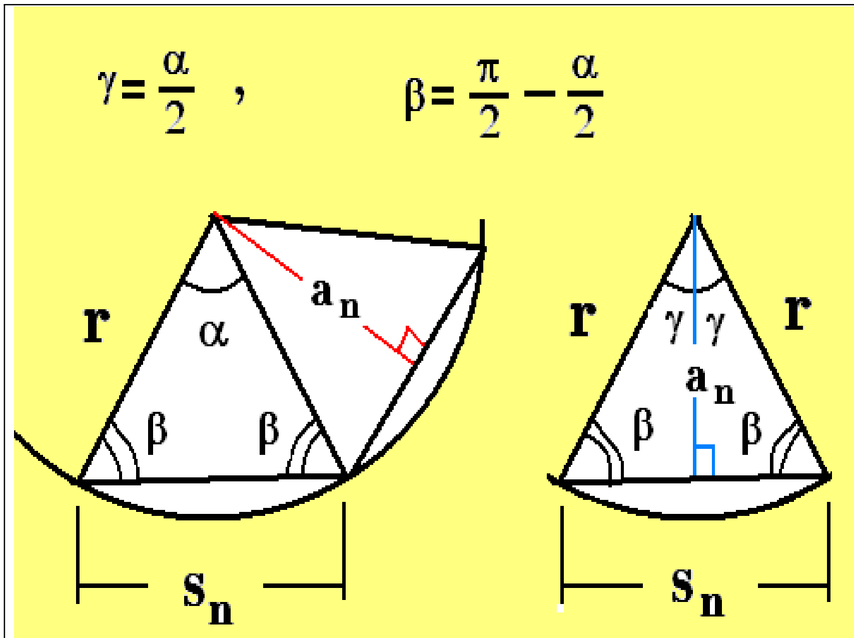


Figure 3-4.

Section of an n-gon

Let A_n, $n = 4, 5, 6, 7, \ldots$ denote the area of a single triangle associated with a regular polygon of n-sides and let A_{tn} denote the total area associated with a regular polygon of n-sides. Also let P_n denote the perimeter of a regular polygon having n equal sides. Using a summation of triangle areas one can verify

	Area of single triangle	Total area of polygon
square $n=4$	$A_4 = \frac{1}{2} a_4 s_4$	$A_{t4} = 4A_4 = \frac{1}{2} a_4 (4 s_4) = \frac{1}{2} a_4 P_4 = \frac{1}{2}\left(\frac{s_4}{2}\right)(4 s_4) = s_4^2$
pentagon $n=5$	$A_5 = \frac{1}{2} a_5 s_5$	$A_{t5} = 5A_5 = \frac{1}{2} a_5 (5 s_5) = \frac{1}{2} a_5 P_5$
hexagon $n=6$	$A_6 = \frac{1}{2} a_6 s_6$	$A_{t6} = 6A_6 = \frac{1}{2} a_6 (6 s_6) = \frac{1}{2} a_6 P_6$
heptagon $n=7$	$A_7 = \frac{1}{2} a_7 s_7$	$A_{t7} = 7A_7 = \frac{1}{2} a_7 (7 s_7) = \frac{1}{2} a_7 P_7$
	$\vdots$	$\vdots$
n-gon	$A_n = \frac{1}{2} a_n s_n$	$A_{tn} = nA_n = \frac{1}{2} a_n (n s_n) = \frac{1}{2} a_n P_n$

Observe that in the special case of a square the apothem is given by $a_4 = \frac{s_4}{2}$ so that the total area for the square is the side squared.

Generalizing the calculation of area for other regular polygons one finds

$$\text{Area of regular polygon} = \frac{1}{2}(\text{apothem})(\text{perimeter})$$

which states the general formula for the area of a regular polygon is half the product of the apothem times the perimeter of the regular polygon.

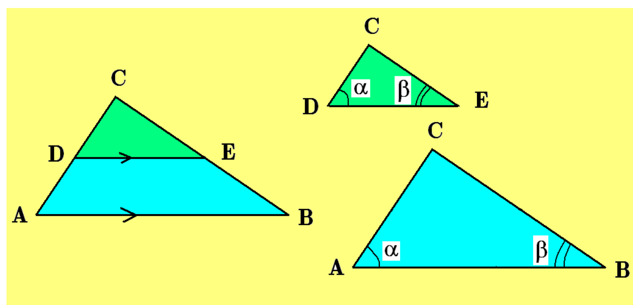
Thales theorem of proportion

In the triangle $\triangle ABC$ illustrated, assume the line segment $\overline{DE}$ is parallel to the base $\overline{AB}$. Show the given figure has the following proportional line segments

$$\frac{\overline{CD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EB}}$$

Solution

One can demonstrate that the triangles $\triangle DEC$ and $\triangle ABC$ are similar because corresponding angles are equal to one another. It is given that the line segment $\overline{DE}$ is parallel to the base line segment $\overline{AB}$. Therefore, one can pick up the base line segment and lay it on the line segment $\overline{DE}$ to show



$$\alpha = \angle CAB = \angle CDE \quad \text{and} \quad \beta = \angle CED = \angle CBA$$

because this is an example where corresponding angles are equal. Therefore, two angles of one triangle are congruent to two angles of the other triangle which demonstrates the triangles are similar. Similar triangles have sides that are proportional so that one can write the ratio equality $\frac{\overline{CA}}{\overline{CD}} = \frac{\overline{CB}}{\overline{CE}}$ which can be written as

$$\frac{\overline{CD}+\overline{DA}}{\overline{CD}} = \frac{\overline{CE}+\overline{EB}}{\overline{CE}} \quad \Rightarrow \quad 1+\frac{\overline{DA}}{\overline{CD}} = 1+\frac{\overline{EB}}{\overline{CE}} \quad \Rightarrow \quad \frac{\overline{DA}}{\overline{CD}} = \frac{\overline{EB}}{\overline{CE}}$$

One can now use the cross product property of proportions to rewrite this last proportion into the form desired. This result is sometimes referred to as the side splitter theorem.

Converse of Thales theorem

If there exists a triangle $\triangle ABC$ where sides $\overline{\text{AB}}$ and $\overline{\text{AC}}$ are intersected by a line which produces a line segment $\overline{\text{DE}}$ such that

$$\frac{\overline{\text{AD}}}{\overline{\text{DB}}} = \frac{\overline{\text{AE}}}{\overline{\text{EC}}}, \tag{3.7}$$

then one can say that $\overline{\text{DE}}$ is parallel to the triangle base $\overline{\text{BC}}$ written ($\overline{\text{DE}} \parallel \overline{\text{BC}}$).

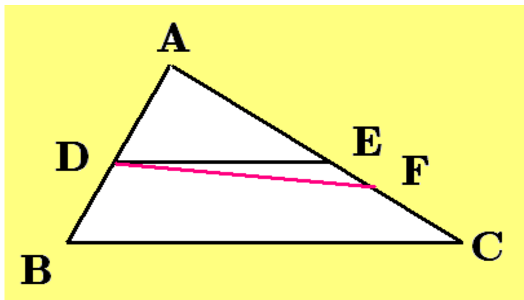


Figure 3-5. Triangle $\triangle ABC$ intersected by line segment $\overline{\text{DE}}$

Proof

The proof is by contradiction. Assume the line segment $\overline{\text{DE}}$ is not parallel to $\overline{\text{BC}}$. We know that one can construct a line through point D which is parallel to the base $\overline{\text{BC}}$ so we construct this parallel line in red and label its intersection with side $\overline{\text{AC}}$ as point F. We now have $\overline{\text{DF}} \parallel \overline{\text{BC}}$ by construction. By Thales proportional theorem, if $\overline{\text{DF}} \parallel \overline{\text{BC}}$, then the following must hold true

$$\frac{\overline{\text{AD}}}{\overline{\text{DB}}} = \frac{\overline{\text{AF}}}{\overline{\text{FC}}} \tag{3.8}$$

By hypothesis the equation (3.7) must hold as well as equation (3.8). We can employ the axiom, things equal to the same thing are equal to one another to show

$$\frac{\overline{\text{AF}}}{\overline{\text{FC}}} = \frac{\overline{\text{AE}}}{\overline{\text{EC}}} \tag{3.10}$$

Add 1 to both sides of equation (3.10) to show

$$\begin{aligned} \frac{\overline{\text{AF}}}{\overline{\text{FC}}} + 1 =& \frac{\overline{\text{AE}}}{\overline{\text{EC}}} + 1 \\ \frac{(\overline{\text{AF}} + \overline{\text{FC}})}{\overline{\text{FC}}} =& \frac{(\overline{\text{AE}} + \overline{\text{EC}})}{\overline{\text{EC}}} \\ \frac{\overline{\text{AC}}}{\overline{\text{FC}}} =& \frac{\overline{\text{AC}}}{\overline{\text{EC}}} \end{aligned} \tag{3.10}$$

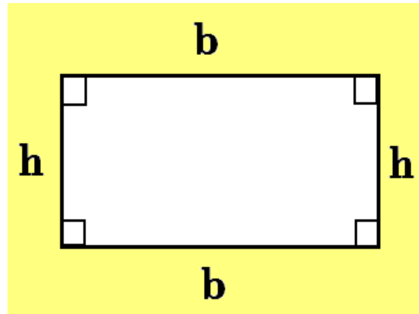
The equation (3.10) implies that $\overline{\text{FC}} = \overline{\text{EC}}$ which requires that the line segments $\overline{\text{DE}}$ and $\overline{\text{DF}}$ be the same and so $\overline{\text{DE}}$ is parallel to $\overline{\text{BC}}$ which contradicts our original assumption.

Area of special quadrilaterals

Let us find ways to find the area and perimeter associated with the special quadrilaterals known as, the rectangle, the parallelogram, the rhombus, the trapezoid and the kite.

The rectangle

A rectangle is a quadrilateral with two pair of parallel sides. The length of the sides are unequal and each of the interior angles is a right angle. We have previously demonstrated that the area of a rectangle is given as the base times the height.



$$\text{Area } A = bh = (\text{base})(\text{height}) \qquad \text{Perimeter } P = 2(h+b)$$

The parallelogram

A parallelogram with base b and height h has two sets of parallel sides. If one constructs a diagonal as illustrated, then the area of the parallelogram is the sum of the areas of two triangles giving

$$\text{Area of parallelogram} = bh = (\text{base})(\text{height})$$

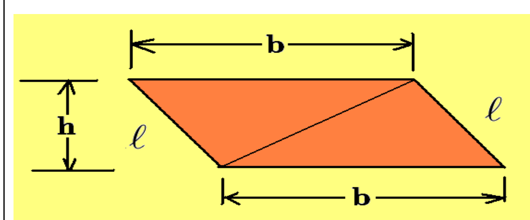


Figure 3-6. Parallelogram

The perimeter of the parallelogram is

$$\text{Perimeter} = P = 2b + 2\ell$$

where ℓ is the slant height associated with the parallelogram side.

The rhombus

A rhombus is a parallelogram having four congruent sides. It is a special case of a parallelogram where the base length equals the side length. The diagonals of a rhombus intersect each other and divides the rhombus into four congruent right triangles. The area of the rhombus is the sum of the areas of these four right triangles. Let A_1, A_2, A_3, A_4 denote the areas of these four right triangles. One finds

$$A_1 = \frac{1}{2}b_1h_1, \qquad A_2 = \frac{1}{2}b_1h_2, \quad A_3 = \frac{1}{2}b_2h_2, \qquad A_4 = \frac{1}{2}b_2h_1$$

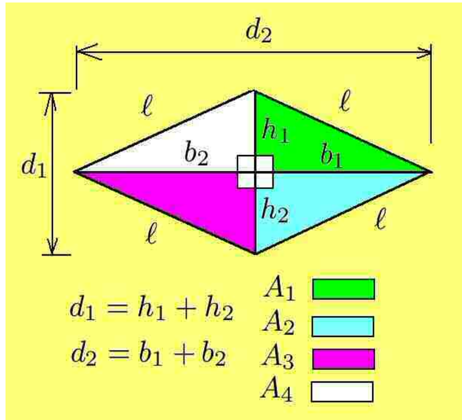


Figure 3-7. Rhombus with diagonals d_1 and d_2

Therefore

$$A_1 + A_2 = \frac{1}{2}(h_1 + h_2) = \frac{1}{2}d_1 b_1$$

$$A_3 + A_4 = \frac{1}{2}(h_1 + h_2)b_2 = \frac{1}{2}d_1 b_2$$

and consequently

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{2}d_1(b_1 + b_2) = \frac{1}{2}d_1 d_2$$

This shows the area of a rhombus equals half the product of the diagonals.

In summary,

$$\text{Area of rhombus } A = \frac{1}{2}d_1 d_2 \qquad \text{Perimeter of rhombus } P = 4\ell$$

The trapezoid

A trapezoid is a quadrilateral where two opposite sides are parallel as in figure 3-8. The parallel sides are said to represent the bases b_1 and b_2 of the trapezoid. The nonparallel sides are called the legs of the trapezoid. If a trapezoid has congruent legs, then the trapezoid is said to be isosceles.

The angles α_1 and α_2 are called the base angles associated with base b_1 and the angles β_1 and β_2 are known as the base angles associated with base b_2 of the trapezoid. An isosceles trapezoid is a trapezoid with a line of symmetry passing perpendicular to the parallel sides. In an isosceles trapezoid the base angles are congruent because of the line of symmetry. Note also that the diagonals of an isosceles trapezoid are equal and the opposite angles are supplementary.

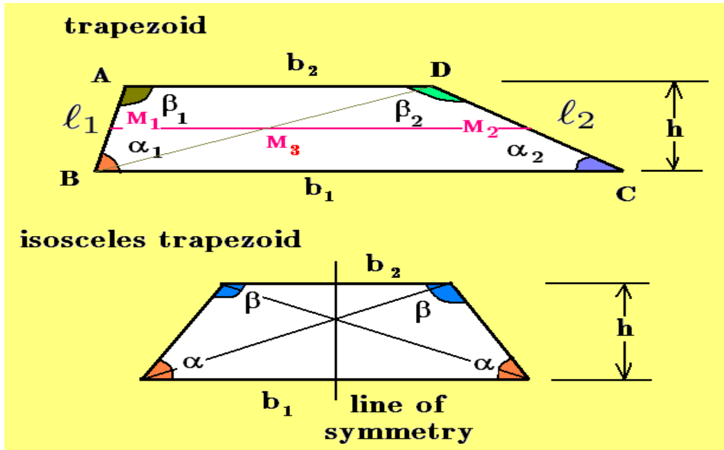


Figure 3-8. Trapezoid and isosceles trapezoid

The midpoint line of a trapezoid is defined as the line segment connecting the midpoints M_1 and M_2 of the trapezoids legs. Examine the figure 3-8 and observe that by using Thales theorem of proportion $\overline{\text{B}M_3} = \overline{M_3\text{D}}$ and by the midpoint theorem associated with triangle $\triangle ABD$

$$\overline{M_1M_3} = \frac{1}{2}\overline{\text{AD}} = \frac{1}{2}b_2$$

Similarly, in triangle $\triangle BDC$

$$\overline{M_3M_2} = \frac{1}{2}\overline{\text{BC}} = \frac{1}{2}b_1$$

follows from the midpoint theorem and by addition

$$\overline{M_1M_3} + \overline{M_3M_2} = \overline{M_1M_2} = \frac{1}{2}\left(\overline{\text{AD}} + \overline{\text{BD}}\right) = \frac{1}{2}(b_1 + b_2) \tag{3.11}$$

Therefore, the midpoint line segment $\overline{M_1M_2}$ has length which represents the average value of the bases. The perimeter of the trapezoid is obtained by adding the lengths of the trapezoid edges. One finds

$$\text{Perimeter } P = \ell_1 + \ell_2 + b_1 + b_2$$

where ℓ_1 and ℓ_2 are the legs of the trapezoid and b_1, b_2 are the base lengths. By constructing the diagonal line segment $\overline{\text{BD}}$, as illustrated in the figure 3-8, one forms two triangles where the area of each triangle is $\frac{1}{2}$ the base times the height or

$$\text{Area } \triangle BDC = \frac{1}{2}b_1h \quad \text{Area } \triangle ABD = \frac{1}{2}b_2h \quad \Rightarrow \quad \text{Area trapezoid} = \frac{1}{2}(b_1 + b_2)h$$

since the total area is the sum of its partial areas. Hence, the area of a trapezoid is the average base length times the perpendicular distance between the parallel sides.

The kite

The kite is illustrated in the figure 3-9 and is defined as a quadrilateral with adjacent sides which are equal. This produces a figure with a line of symmetry. The diagonals of a kite intersect one another to form four right angles. By symmetry the diagonal d_1 is divided by the line segment $\overline{\text{AC}}$ and the line segment $\overline{\text{BD}}$ divides the other diagonal into two parts h_1 and h_2 where $d_2 = h_1 + h_2$. The area of a kite can be obtained by adding together the area of triangles $\triangle ABD$ and $\triangle BDC$. This produces the result

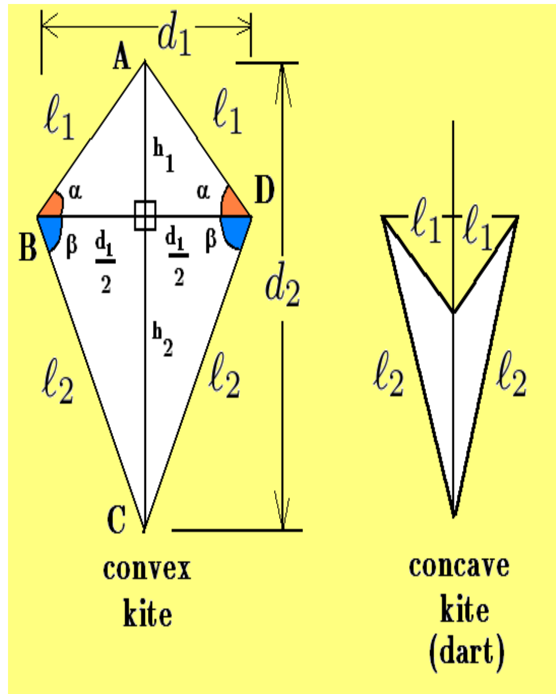


Figure 3-9.
Convex and concave kite

$$\text{Area } A = \frac{1}{2}d_1h_1 + \frac{1}{2}d_1h_2 = \frac{1}{2}d_1(h_1+h_2) = \frac{1}{2}d_1d_2$$

Hence, the area associated with a kite is half the product of the diagonals.

The perimeter of the kite is $P = 2(\ell_1 + \ell_2)$.

Example 3-1. Show the diagonals of a kite intersect perpendicularly.

Solution

A kite is illustrated in the figure 3-10. The problem is to show $\overline{AC} \perp \overline{BD}$

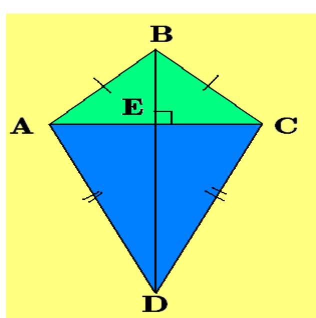


Figure 3-10. Kite with diagonals $\overline{DB}$ and $\overline{AC}$

	Statements	Reasons
1.	$\overline{AB} = \overline{BC}$	Definition of kite
2.	$\overline{AD} = \overline{CD}$	Definition of kite
3.	$\overline{BD} = \overline{BD}$	Common side to 2 triangles
4.	$\triangle ABD \cong \triangle CBD$	Side-Side-Side
5.	$\angle ABD \cong \angle CBD$	From congruent triangles
6.	$\overline{AB} = \overline{BC}$, $\angle ABE = \angle CBE$, $\overline{BE} = \overline{BE}$	Side-Angle-Side observation
7.	$\triangle ABE \cong \triangle CBE$	Side-Angle-Side congruence
8.	$\angle AEB \cong \angle BEC$	From congruent triangles
9.	$\angle AEB + \angle BEC = 180°$	Supplementary angles
10.	$2\angle AEB = 180°$	$\angle BEC = \angle AEB$
11.	$\angle AEB = 90° = \angle BEC$	Diagonals intersect $\perp$

■

Area of irregular polygons

The figure 3-11 illustrates selected irregular shaped polygons. The problem is to find the area and perimeter associated with these plane figures. It will be assumed that the Cartesian coordinates of the polygon vertices are known. To find the area of irregular shaped polygons such as the ones illustrated in the figure 3-11 one can proceed using one of the following methods.

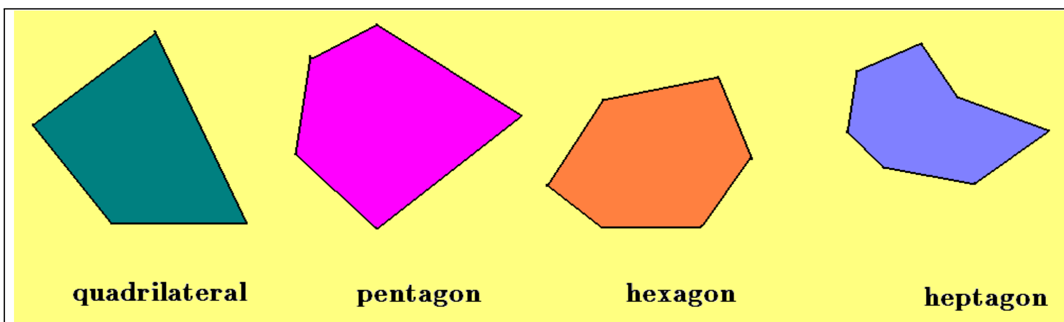


Figure 3-11. Irregular shaped polygons

Construct triangles

Start at a vertex of a given irregular polygon and construct triangles by drawing lines to the other vertices as illustrated in the figure 3-12(a). The problem is then reduced to finding the area of each triangle. The total area of the polygon is then a summation of these areas.

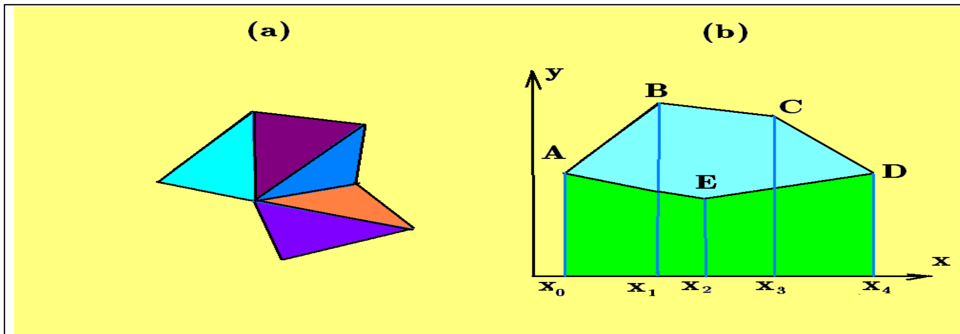


Figure 3-12. Area associated with irregular shaped polygons

Example 3-2. Leonhard Euler (1707-1783) a famous Swiss mathematician asked the question: How many ways W_n can a convex polygon of n sides be broken up into triangles by constructing diagonals which do not intersect one another? This turn out to be a difficult problem but Euler solved it and obtained the answer

$$W_n = \frac{2 \cdot 6 \cdot 10 \cdots (4n-10)}{(n-1)!}$$

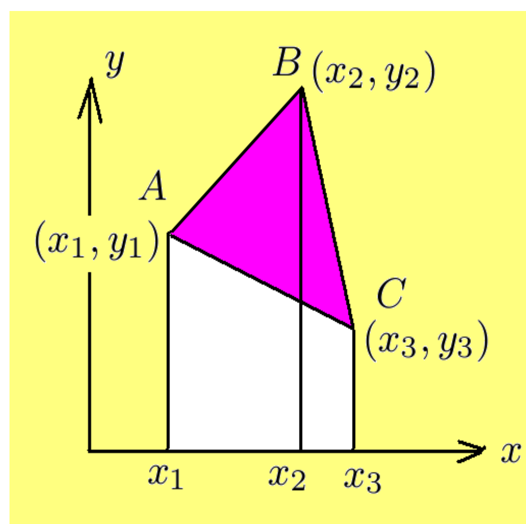
where $n! = n(n-1)(n-2)\cdots(3)(2)(1)$

This problem and many others like it is one example of how questions in geometry led to a new branch of mathematics called combinatorics.

■

Area of triangle

To find the area of a single triangle with known coordinates, one can drop perpendicular lines from each vertex to the x-axis as illustrated in the accompanying figure. The area of the triangle $\triangle ABC$ is then given by



$$\begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{AB}x_2x_1 \end{bmatrix} + \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{BC}x_3x_2 \end{bmatrix} - \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{AC}x_3x_1 \end{bmatrix}$$

The area of the trapezoids is the average height times the base so that one can write

$$\text{Area}\,\triangle ABC \;= \frac{1}{2}(y_2+y_1)(x_2-x_1) + \frac{1}{2}(y_2+y_3)(x_3-x_2) - \frac{1}{2}(y_1+y_3)(x_3-x_1)$$

Because the position of the vertices can vary, the sign associated with the area can change. To keep the area a positive quantity the absolute value of this answer is used The above answer for the area is often written in the form of an absolute value

$$\text{Area}\,\triangle ABC \;=\left|\frac{1}{2}\left[(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + (y_3x_1 - y_3x_1)\right]\right| \tag{3.12}$$

Area of irregular polygon

To find the area of the irregular polygon in figure 3-12(b) one can drop perpendicular lines from each vertex to the x-axis as illustrated. The area of the polygon is then obtained as an addition followed by a subtraction of areas associated with trapezoids. For example, in the figure 3-12(b) the area of the polygon ABCDE is calculated using

$$\begin{bmatrix}\text{Area of}\\ \text{trapezoid}\\ \text{AB}x_1x_0\end{bmatrix}+\begin{bmatrix}\text{Area of}\\ \text{trapezoid}\\ \text{BC}x_3x_1\end{bmatrix}+\begin{bmatrix}\text{Area of}\\ \text{trapezoid}\\ \text{CD}x_4x_3\end{bmatrix}-\begin{bmatrix}\text{Area of}\\ \text{trapezoid}\\ \text{AE}x_2x_0\end{bmatrix}-\begin{bmatrix}\text{Area of}\\ \text{trapezoid}\\ \text{ED}x_4x_3\end{bmatrix}$$

Generalization

Another way to calculate the area of a polygon with n-sides having the vertices

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3),\ \cdots,\ (x_{n-1}, y_{n-1}),\ (x_n, y_n)$$

is obtained from the multiplication of the coordinates in the following pattern.

$$\text{Area n-gon} = \left|\frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n & x_1\\ y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n & y_1\end{bmatrix}\right|$$

Construct lines with positive and negative slopes as illustrated below.

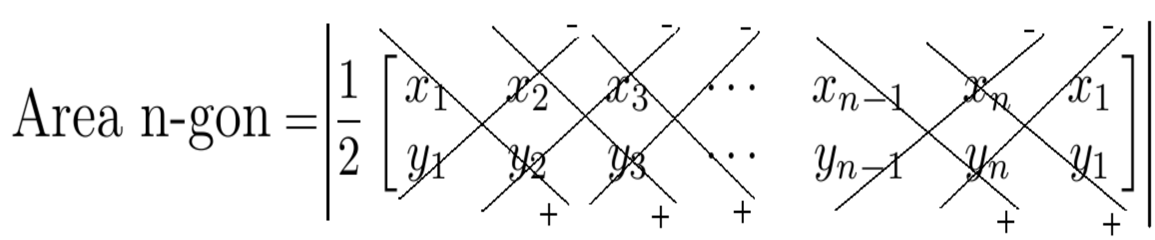


Figure 3-13. Coordinate pattern with lines

The coordinates touching a line are multiplied together with a + sign for the lines with negative slope and a - sign for lines with a positive slope. This gives the area formula

$$\text{Area n-gon} = \left| \frac{1}{2} \left[(x_1y_2 - y_1x_2) + (x_3y_1 - y_3x_2) + \cdots + (x_ny_{n-1} - x_{n-1}y_n) + (x_1y_n - x_ny_1) \right] \right|$$

where again the absolute value sign has to be used to keep the area positive. Note this formula can be used to calculate the area formula in equation (3.12).

Midpoint theorem

In triangle $\triangle ABC$ of figure 3-14 show the line segment $\overline{M_1M_2}$ connecting the midpoints of sides $\overline{\text{AB}}$ and $\overline{\text{AC}}$ satisfies the following properties.

(a) The segment $\overline{M_1M_2}$ is parallel to the third side $\overline{\text{BC}}$

(b) The segment $\overline{M_1M_2}$ is half the length of side $\overline{\text{BC}}$

Solution

In order to prove the above assertions a parallelogram M_1BCM_3 will be constructed where the segment $\overline{M_1M_2}$ is parallel to the side $\overline{\text{BC}}$ and from the parallelogram the above properties can be established. Begin by extending the line segment $\overline{M_1M_2}$ to point M_3 such that $\overline{M_1M_2} = \overline{M_2M_3}$ and then construct the line segment $\overline{M_3\text{C}}$.

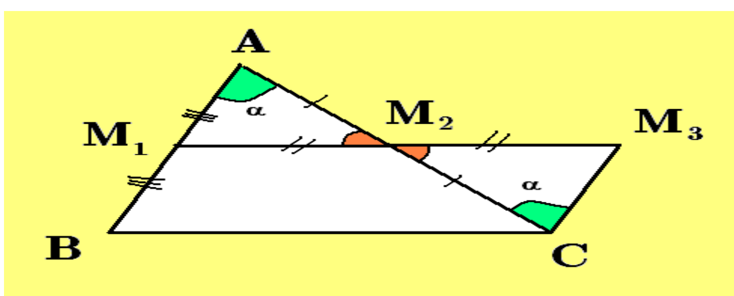


Figure 3-14. **Line segment $\overline{M_1M_2}$ connecting midpoints.**

Observe that

(i) $\overline{\text{A}M_1} = \overline{M_1\text{B}}$ by definition of a midpoint.

(ii) $\overline{\text{A}M_2} = \overline{M_2\text{C}}$ by definition of a midpoint.

(iii) $\overline{M_1M_2} = \overline{M_2M_3}$ by construction.

(iv) Observe the angles $\angle M_1M_2A = \angle CM_2M_3$ because opposite angle are equal.

(v) The triangle $\triangle AM_1M_2$ is congruent to triangle $\triangle CM_2M_3$ because of SAS

(vi) We know corresponding sides of congruent triangles are equal so that sides

$\overline{M_3\text{C}} = \overline{M_1\text{A}} = \overline{M_1\text{B}}$

(vii) Also corresponding angles from congruent triangles are equal so that $\angle M_1AM_2 = \angle M_2CM_3 = \alpha$ which implies $\overline{\text{AB}} \parallel \overline{\text{C}M_3}$

(viii) Using the converse of the Thales proportion theorem $\dfrac{\overline{\text{A}M_1}}{\overline{M_1\text{B}}} = 1 = \dfrac{\overline{\text{A}M_2}}{\overline{M_2\text{C}}}$ which implies $\overline{M_1M_2} \parallel \overline{\text{BC}}$

This demonstrates that M_1BCM_3 is a parallelogram with opposite sides parallel and equal. Therefore, one can write

$$\overline{\text{BC}} = \overline{M_1M_3} = 2\overline{M_1M_2} \quad \Rightarrow \quad \overline{M_1M_2} = \frac{1}{2}\overline{\text{BC}}$$

Parallel intercept theorem

Show that if parallel lines produce equal intercepts on one transversal, then they will produce equal intercepts on any other transversal. The situation is illustrated in the figure 3-15.

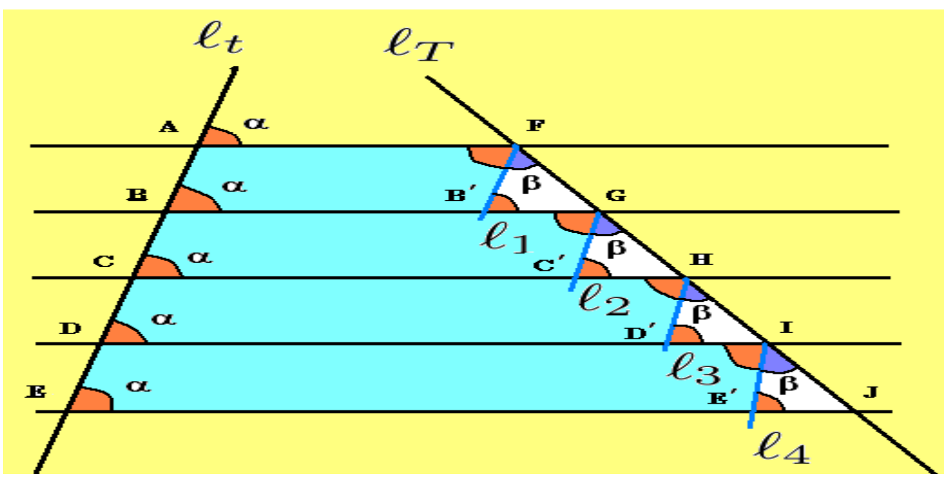


Figure 3-15. **Parallel lines cut by transversal ℓ_t produces equal intercepts.**

Solution:

To prove this theorem we will produce a series of parallelograms and congruent triangles to demonstrate the conclusion. By hypothesis the transversal line ℓ_t cuts the parallel lines to produce the equal line segments

$$\overline{\text{AB}} = \overline{\text{BC}} = \overline{\text{CD}} = \overline{\text{DE}} \tag{3.13}$$

Let ℓ_T denote any other transversal line which intersects the parallel lines at the points F, G, H, I, J as illustrated in the figure 3-15. One can now construct the lines $\ell_1, \ell_2, \ell_3, \ell_4$ through the points F, G, H, I respectively which are all parallel to the transversal line ℓ_t and producing the parallelograms

$$ABB'F, \quad BCC'G, \quad CDD'H, \quad DEE'I$$

illustrated in the above figure. The opposite sides of the parallelogram are parallel and equal to that

$$\overline{\text{AB}} = \overline{\text{F}B'}, \quad \overline{\text{BC}} = \overline{\text{G}C'}, \quad \overline{\text{CD}} = \overline{\text{H}D'}, \quad \overline{\text{DE}} = \overline{\text{I}E'} \tag{3.14}$$

and because of the hypothesis things equal to the same things are equal to one another, one finds

$$\overline{\text{F}B'} = \overline{\text{G}C'} = \overline{\text{H}D'} = \overline{\text{I}E'} \tag{3.15}$$

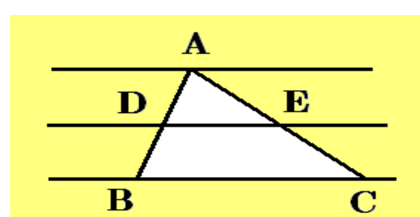
The angles α are all equal because they are corresponding angles in one situation and opposite vertex angles of a parallelogram in another situation and alternate interior angles in still another situation. The angles $(\alpha+\beta)$ are corresponding angles in one situation and the angles β are corresponding angles in another situation. Therefore, the triangles

$$\triangle FB'G, \quad \triangle GC'H, \quad \triangle HD'I, \quad \triangle IE'J$$

are all congruent (ASA) to one another, which implies the corresponding sides are equal or $\overline{\text{FG}} = \overline{\text{GH}} = \overline{\text{HI}} = \overline{\text{IJ}}$ which was to be demonstrated.

Corollary

A line which bisects one side of a triangle (point D) which is at the midpoint of $\overline{\text{AB}}$ and is parallel to the triangle base $\overline{\text{BC}}$, will bisect the third side (point E) at the mid-point of $\overline{\text{AC}}$ and the line segment $\overline{\text{DE}}$ will have a length one half the third side. This is sometimes referred to as the mid-segment theorem.

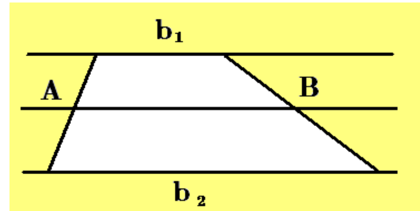


$$\overline{\text{DE}} = \frac{1}{2}\overline{\text{BC}}, \;\; \overline{\text{AD}} = \overline{\text{DB}}, \;\; \overline{\text{AE}} = \overline{\text{EC}}$$

Corollary

A line through the midpoints of a trapezoids legs will be parallel to the trapezoid bases and have a length equal to the average value of the bases.

$$\overline{\text{AB}} = \frac{1}{2}(b_1 + b_2)$$



Tools for geometric constructions

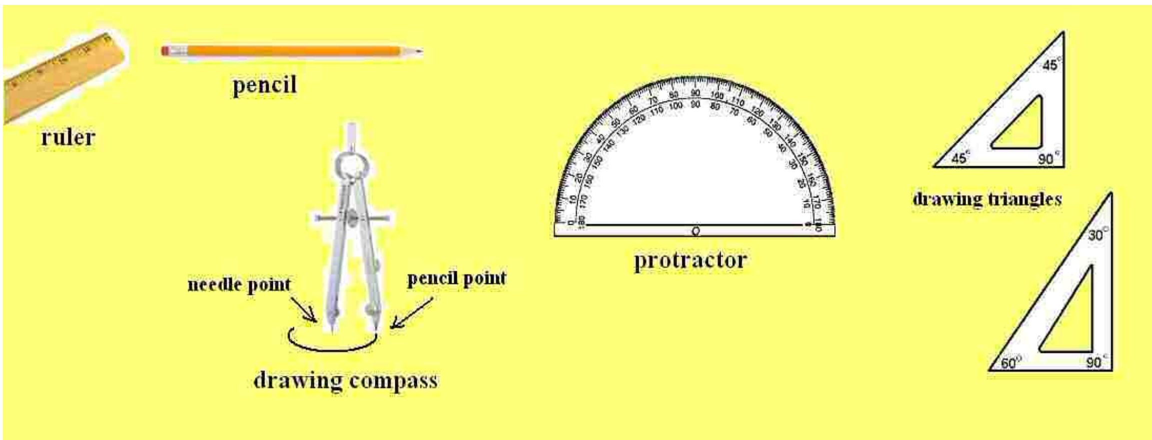


Figure 3-16. Tools for geometry

The tools for geometric constructions are illustrated in the figure 3-16. In addition to the pencil and ruler there is the drawing compass[1] which is used for drawing circles and measuring distances on maps used for navigation. It is usually a device with two arms which can be spread apart because of a hinge or screw adjustment device. The bottom of one arm of the drawing compass is a needle point and at the bottom of the other arm is a pencil point or drawing lead. Also illustrated is a protractor for measuring angles. Sometimes the protractor has two scales. One scales goes right to left across the top measuring angles from 0 to 180 degrees in the positive direction. The second, lower scale goes left to right measuring angles from 0 to 180 degrees in the negative direction. You can purchase a special protractor which measures in units of radians or grads if you desire. The drawing triangles can be used as a straight edge and have angles of 30, 45 and 60 degrees because these angles occur quite frequently.

[1] Make note of the difference between a drawing compass and a directional compass which points to magnetic North. They are different types of compasses.

Note in the following paragraphs various constructions are performed using only a pencil, straight edge (ruler), and a drawing compass. These were the only instruments available to the ancient Greeks who studied geometry. Constructions using a straight edge and drawing compass are an important part of learning geometry.

Find midpoint of a line segment using drawing compass and ruler

(Euclid, Book 1, Propostion 10)

1) Given line segment $\overline{AB}$

2) Widen the compass so the distance between the needle point and the lead point is greater than $\frac{1}{2}\overline{AB}$. Maintain this distance on the compass throughout the construction.

3) Place the needle point at A and make marks above and below the line segment $\overline{AB}$ as illustrated in the figure 3-17

4) Place the needle point at B and make marks above and below the line segment $\overline{AB}$ which intersect the previous marks. Where these marks cross are labeled as points C and D.

5) Use a straight edge to draw a line connecting the points C and D.

6) The intersection of the line segments $\overline{CD}$ and $\overline{AB}$ are perpendicular and $\overline{CD}$ bisects the line segment $\overline{AB}$.

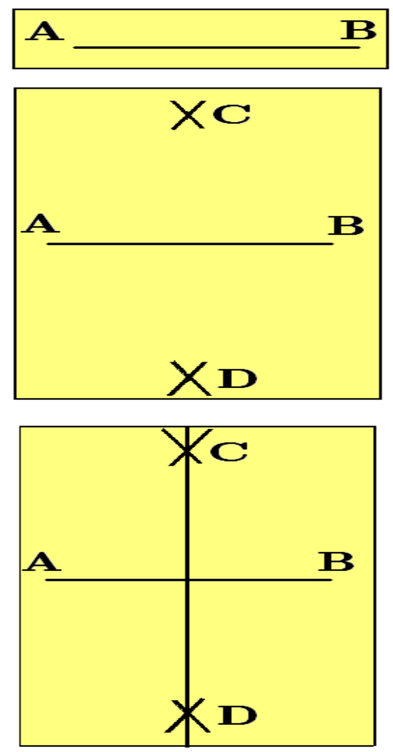


Figure 3-17.
Finding the midpoint

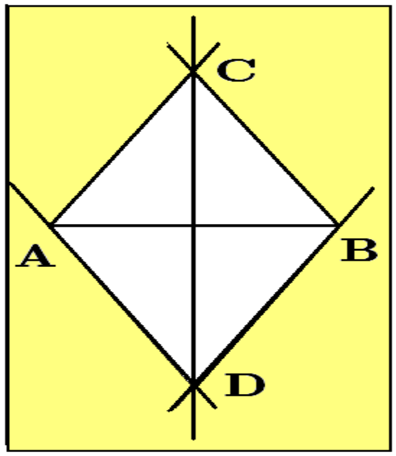
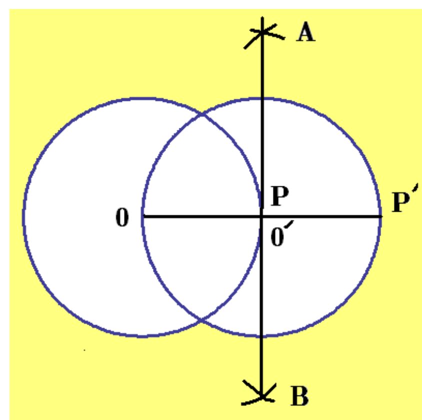


Figure 3-18.
Addditional constructions

The reasons why the above constructions bisect the line segment $\overline{AB}$ can be seen by constructing the line segments as illustrated in the figure 3-18 to form a rhombus. We know the diagonals of a rhombus bisect each other and the diagonals of the rhombus are perpendicular to one another. Note that finding the midpoint of a line segment is used quite frequently in geometry. Recall that if you know the Cartesian coordinates (x_1, y_1) and (x_2, y_2) of the line segment endpoints, then the midpoint is given by $(x_m, y_m) = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$.

Construct a tangent line to a circle

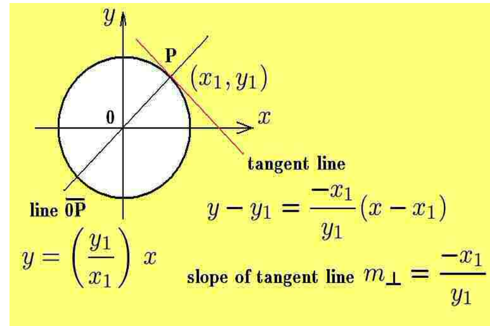
Using the drawing compass construct a circle with center labeled 0. Select a point P on the circumference of the circle and without changing the compass setting place the needle point of the compass at point P and draw another circle identical in size to the first circle. Next draw the line $\overline{0P}$ and extend the line to intersect the second circle at the point P'. This construction makes point P the midpoint of the line segment $\overline{0P'}$. Now you can go through the procedure for finding the midpoint of the line segment $\overline{0P'}$.



Widen the compass and place the needle point at 0 and make two arcs above and below the line segment $\overline{0P'}$, then move the needle point of the compass to the point P' and do the same thing. Call the intersection points of these arcs A and B. Use a straight edge and draw the line segment $\overline{\text{AB}}$ which is now a tangent line to the circle at point P.

Note that this construction is the same used for finding the midpoint of a line segment. Observe that the tangent line constructed is perpendicular to the circle radius and diameter. (Euclid, book 3, Proposition 18)

In Cartesian coordinates, if (x_1, y_1) are the coordinates of point P on the given circle, then the slope of the line passing through the points 0 and P is $m = \frac{\text{change in y}}{\text{change in x}} = \frac{y_1}{x_1}$ and the point-slope formula for the equation of the line $\overline{0P}$ is $y = \frac{y_1}{x_1}x$. Recall that the slope on any line which is perpendicular to a line with slope m has a slope $\frac{-1}{m}$ which is the negative reciprocal of slope m.



Therefore, the slope of any line perpendicular to the line $\overline{0P}$ must have the slope $m_\perp = \frac{-1}{m} = -\frac{x_1}{y_1}$. The tangent line to the circle is perpendicular to the line $\overline{0P}$ and passes through the point (x_1, y_1). Using the point-slope formula for the equation of the line with slope $m_\perp$ which passes through the point (x_1, y_1) is therefore

$$y - y_1 = \left(\frac{-x_1}{y_1}\right)(x - x_1) \tag{3.16}$$

This is the equation of the tangent line to the circle with point of tangency (x_1, y_1).

Construction of line parallel to a given line
(Euclid, Book 1, Proposition 31)

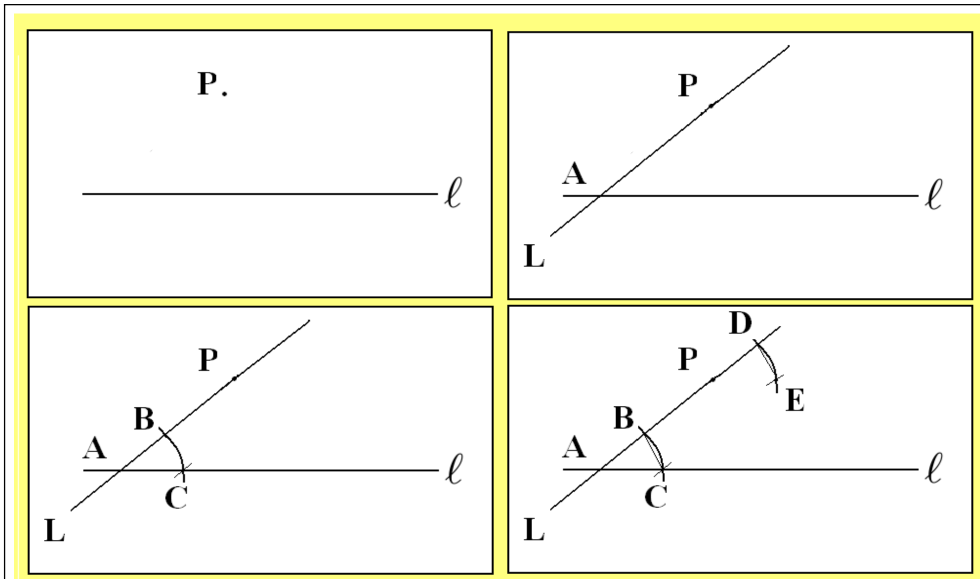


Figure 3-19.
Steps for constructing parallel line.

Given a line ℓ and a point P not on the given line. The problem is to construct another line through point P which is parallel to the given line. The situation is illustrated in the figures 3-19 and 3-20.
(i) Construct a line L through point P which intersect the given line ℓ at an angle at point A.
(ii) Place the needle of the drawing compass at point A and draw an arc which intersects the lines ℓ and L at the points B and C as illustrated above.
(iii) Using the same compass distance $\overline{\text{AB}}$ place the needle of the drawing compass at point P and make the same kind of arc as in step (ii) above. This arc intersects line L at point D.
(iv) Adjust the drawing compass distance to measure the distance $\overline{\text{BC}}$, then place the compass needle at the point D and make an arc intersecting the previous arc at point E. Note the distances $\overline{\text{DE}} = \overline{\text{BC}}$.
(v) Construct the straight line through the points P and E and call this line ℓ'.

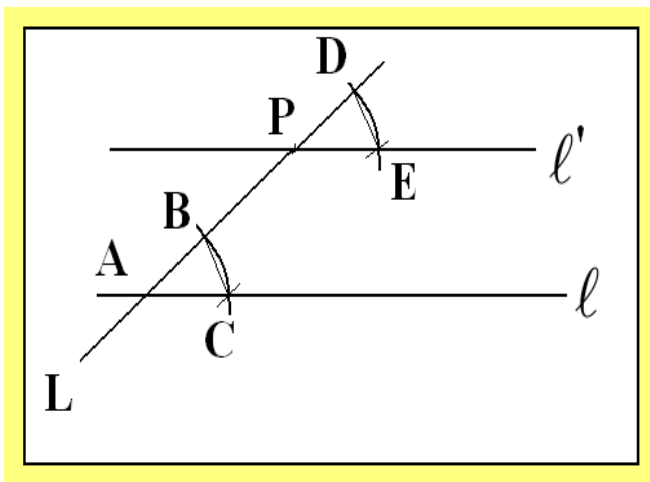


Figure 3-20.
Constructing parallel line to given line ℓ through given point P.

(vi) The line ℓ and ℓ' are parallel. This is because the angles $\angle DPE = \angle BAC$ because corresponding angles are equal.

Line through given point P perpendicular to given line ℓ (Euclid, Book 1, Proposition 12)
Case 1: Given a line ℓ and point P **not on line** ℓ.
The problem is to construct a line through P which is perpendicular to the line ℓ.

1) Widen the compass so it intersects the given line ℓ when the needle is place

at the point P and then make two arcs which intersect line ℓ at points A and B as illustrated in the figure 3-21.

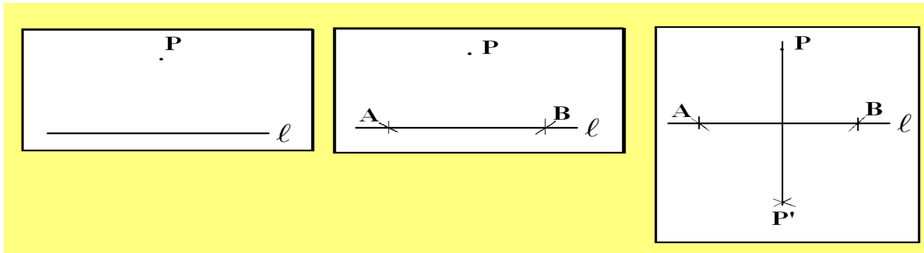


Figure 3-21.
Constructions for perpendicular line PP'

2) Widen compass again and place needle of compass at point A and make arc below line, then place needle of compass at point B and make another arc below line ℓ which intersects the first arc constructed. Call the point of intersection of these arcs point p'.

3) Use a straight edge and construct the line segment $\overline{PP'}$ which is now perpendicular to the given line ℓ.

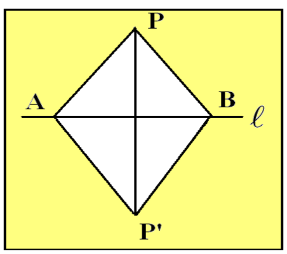


Figure 3-22.
Kite formed by $APBP'$

The reason why this construction works is because the constructions form a kite and we know the diagonals of a kite intersect perpendicular to one another.

Case 2:(Euclid, Book 1, Proposition 11)
Given a line ℓ and point P **on the line** ℓ.

Construct line perpendicular to given line and through point P on the given line.

1) Put needle of compass at point P and make two arcs intersecting the given line ℓ at points A and B as in figure 3-23.

2) Widen compass and put the needle of the compass on point A and make a mark at point C. Put needle of compass at point B and make intersection with mark at C.

3) Construct line with straight edge through the two points P and C. The constructed line is perpendicular to the given line and passes through the given point P.

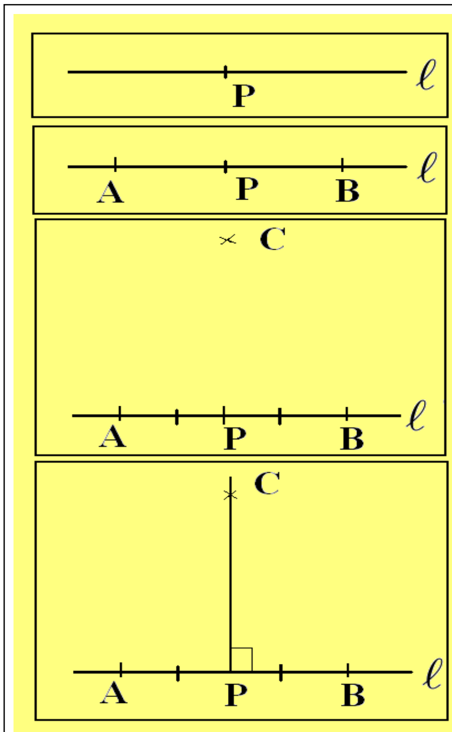


Figure 3-23.
Line $\overline{CP}$ perpendicular to line ℓ

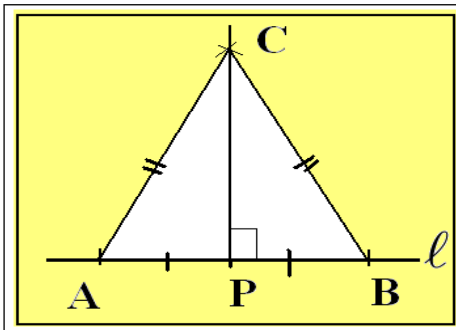


Figure 3-24.
Line $\overline{CP}$ perpendicular to line ℓ

The reason why this construction works is because triangle $\triangle ACP$ of figure 3-24 is congruent to triangle $\triangle BCP$ ($\triangle ACP \cong \triangle BCP$) due to the fact the sides $\overline{AC} = \overline{CB}$, $\overline{AP} = \overline{BP}$ and $\overline{CP} = \overline{CP}$ being a common side. Since the triangles are congruent, their angles are equal so we have $\angle APC \cong \angle BPC$ and $\angle APC + \angle BPC = 180^\circ$, which implies that $\angle APC = \angle BPC = 90^\circ$ and line segment $\overline{CP}$ is perpendicular to line ℓ.

Example 3-3. **(Theorem) If a point P lies on a perpendicular bisector of a line segment $\overline{AB}$, then it is equidistant from the endpoints of the line segment.**

Proof

If $\overline{CM}$ in figure 3-25 is a perpendicular bisector of the line segment $\overline{AB}$, then $\overline{AM} = \overline{MB}$.

Let the point P denote an arbitrary point on the perpendicular bisector $\overline{CM}$ and construct the line segments $\overline{PA}$ and $\overline{PB}$. This creates two right triangles with common side $\overline{PM}$. The triangles $\triangle APM$ and $\triangle PMB$ are congruent so that $(\triangle APM \cong \triangle PMB)$ because of side-angle-side (SAS). Therefore $\overline{AP} = \overline{PB}$ since corresponding sides of congruent triangles are equal.

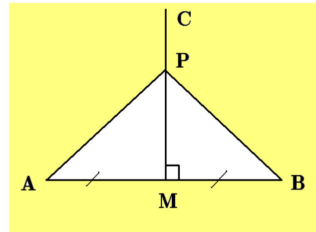


Figure 3-25.
Line segment $\overline{PD}$ is perpendicular bisector of $\overline{AB}$

The converse of this theorem is also true. One can say that **if the point P is equidistant from the endpoints A and B, then the point P must lie on the perpendicular bisector of the line segment $\overline{AB}$.**

Here our assumption is that $\overline{PA} = \overline{PB}$ and we must show the point P lies on the perpendicular bisector. One can construct the line through point P which bisects the angle $\angle APB$ and intersects the line segment $\overline{\text{AB}}$ at the point M. Then the triangles $\triangle APM$ and $\triangle PMB$ are congruent $(\triangle APM \cong \triangle PMB)$ because of SAS and consequently $\overline{AM} = \overline{MB}$ since corresponding sides of congruent triangles are equal. In these triangles angles $\angle AMP = \angle BMP = \frac{\pi}{2}$ so that $\overline{\text{PM}}$ is perpendicular to $\overline{\text{AB}}$. This demonstrates the point P must lie on the perpendicular bisector of the line segment $\overline{AB}$.

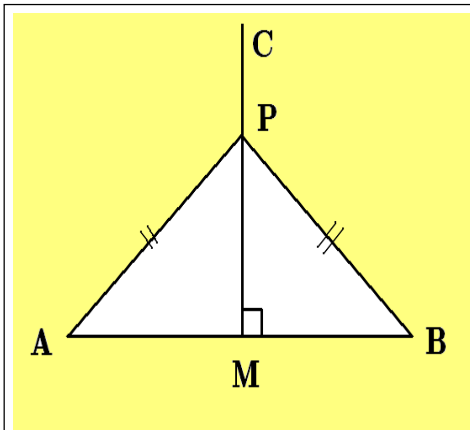


Figure 3-26.
Line segments $\overline{PB} = \overline{PA}$.

■

Example 3-4. **(Theorem)** If one constructs the perpendicular bisector to each side of a general triangle ABC, then these perpendicular bisectors meet at a point of concurrency called **the circumcenter**. The distances from the circumcenter, labeled 0, to each vertex of the triangle ABC are all equal and one can write $\overline{0A} = \overline{0B} = \overline{0C}$. A circle with center point at the concurrent circumcenter and radius $\overline{0A}$ produces a circumscribed circle enclosing the triangle.

Another name for the circumscribe circle is the circumcircle. This is a circle which passes through each vertex of a given triangle. The radius of the circumcircle is called the circumradius. (Euclid, Book 4, Proposition 5)

Observe that the circumcenter lies

(*i*) inside the triangle when it is acute

(*ii*) on the hypotenuse when a right triangle

(*iii*) outside the triangle when it is obtuse

To calculate the coordinates of the circumcenter for triangle $\triangle ABC$ when A has the coordinates (x_1, y_1), B has the coordinates (x_2, y_2) and C has the coordinates (x_3, y_3) one can proceed as follows.

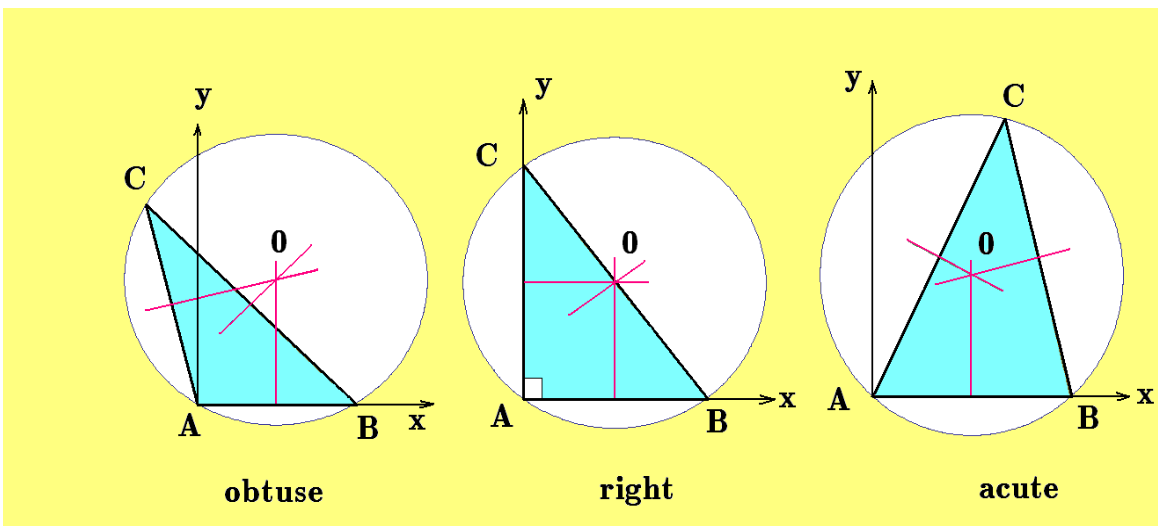


Figure 3-27. Construction of circumcenter for acute, right and obtuse triangles.

(*i*) Calculate the slope associated with each side of the triangle $\triangle ABC$

$$m_1 = \text{slope } \overline{AC} = \frac{(y_3 - y_1)}{(x_3 - x_1)}, \quad m_2 = \text{slope } \overline{AB} = \frac{(y_2 - y_1)}{(x_2 - x_1)}, \quad m_3 = \text{slope } \overline{BC} = \frac{(y_2 - y_3)}{(x_2 - x_3)}$$

because of the definition that the slope of a line segment is the change in y-values divided by the change in x-values.

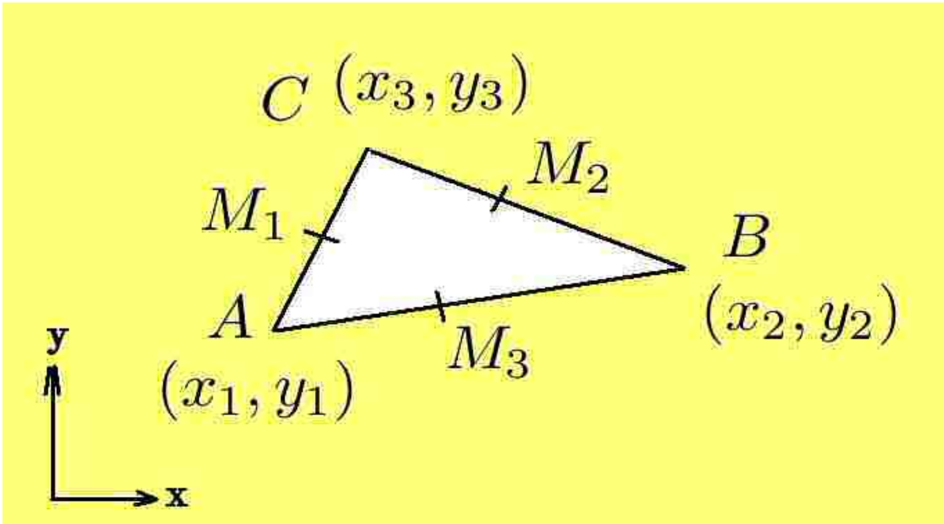


Figure 3-28. General triangle $\triangle ABC$

(*ii*) Calculate the midpoints associated with each side of the triangle

$$\text{midpoint } \overline{AC} = M_1 = \left(\frac{1}{2}(x_1+x_3), \frac{1}{2}(y_1+y_3)\right)$$
$$\text{midpoint } \overline{AB} = M_3 = \left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)\right)$$
$$\text{midpoint } \overline{BC} = M_2 = \left(\frac{1}{2}(x_2+x_3), \frac{1}{2}(y_2+y_3)\right)$$

because the midpoint is determined by taking the average of the x-values and y-values associated with the endpoints of a line segment.

(*iii*) Calculate the slopes of the perpendicular bisectors through the points M_1, M_2 and M_3

$$\text{slope through } M_1 = m_{\perp AC} = -1/m_1$$
$$\text{slope through } M_3 = m_{\perp AB} = -1/m_2$$
$$\text{slope through } M_2 = m_{\perp BC} = -1/m_3$$

because if two lines are perpendicular the product of their slopes must equal -1.

(*iv*) Calculate the equations of the lines representing the perpendicular bisectors

$$\text{line} \perp \overline{AC} \text{ through } M_1 \text{ is } y - \frac{1}{2}(y_1+y_3) = m_{\perp AC}\left(x - \frac{1}{2}(x_1+x_3)\right)$$
$$\text{line} \perp \overline{AB} \text{ through } M_3 \text{ is } y - \frac{1}{2}(y_1+y_2) = m_{\perp AB}\left(x - \frac{1}{2}(x_1+x_2)\right)$$
$$\text{line} \perp \overline{BC} \text{ through } M_2 \text{ is } y - \frac{1}{2}(y_2+y_3) = m_{\perp BC}\left(x - \frac{1}{2}(x_2+x_3)\right)$$

These lines result using the point-slope formula for the equation of a line.

The circumcenter is calculated by solving for the values of x and y which satisfy **any two of the above equations simultaneously.** One finds the circumcenter has the coordinates

$$
\begin{aligned}
x_c =& \frac{(x_1^2+y_1^2)(y_3-y_2)+(x_2^2+y_2^2)(y_1-y_3)+(x_3^2+y_3^2)(y_2-y_1)}{2\,(x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1))}\\
y_c =& \frac{(x_1^2+y_1^2)(x_2-x_3)+(x_2^2+y_2^2)(x_3-x_1)+(x_3^2+y_3^2)(x_1-x_2)}{2\,(x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1))}
\end{aligned}
\tag{3.17}
$$

In the special cases where a slope is **zero or undefined**, then **modifications to the above equations are necessary.** These modifications will usually simplify one or more of the equations for a line to the form of x=a constant or y=some constant. The resulting algebra in solving for the circumcenter is also simplified.

Example 3-5. (Special case)

In the special case of a right triangle, assign the Cartesian coordinates

$$A(0, y_2), \quad B(0,0), \quad C(x_1, 0)$$

to the vertices of the triangle as illustrated in the figure 3-29.

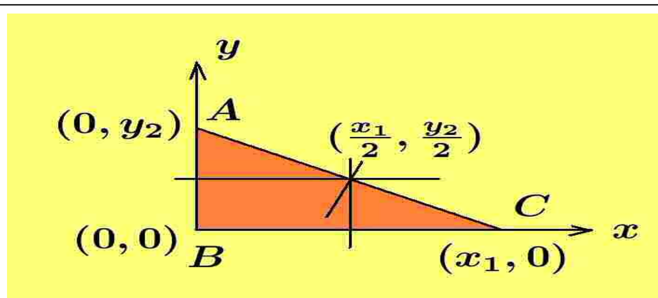


Figure 3-29.
Right triangle with three perpendicular bisectors.

In this special case the midpoints are

$$\text{for } \overline{AB},\ (0, \frac{y_2}{2}), \qquad \text{for } \overline{BC}\ (\frac{x_1}{2}, 0), \qquad \text{for } \overline{AC}\ (\frac{x_1}{2}, \frac{y_2}{2})$$

The line representing the perpendicular bisector of segment $\overline{AB}$ is $y = \frac{y_2}{2} = a\ constant$. The line representing the perpendicular bisector of line segment $\overline{BC}$ is $x = \frac{x_1}{2}$. To find the equation for the perpendicular bisector of line segment $\overline{AC}$, first note that the slope of $\overline{AC}$ is

$$m_{AC} = -\frac{y_2}{x_1} = \frac{\text{change in y}}{\text{change in x}}$$

with its negative reciprocal $m_{\perp AC} = \frac{x_1}{y_2}$ being the slope of the line perpendicular to $\overline{AC}$. The perpendicular bisector line must pass through the midpoint of $\overline{AC}$ and so one can use the point-slope formula to obtain the equation

$$y - \frac{y_2}{2} = \frac{x_1}{y_2}(x - \frac{x_1}{2})$$

The three perpendicular bisector lines meet at the midpoint of the line segment $\overline{AC}$ and has the coordinates $(\frac{x_1}{2}, \frac{y_1}{2})$ and represents the circumcenter for the circumscribed circle.

■

Altitudes

An **altitude of a triangle** is a line segment constructed through the vertex of a triangle which is perpendicular to the side opposite the vertex.

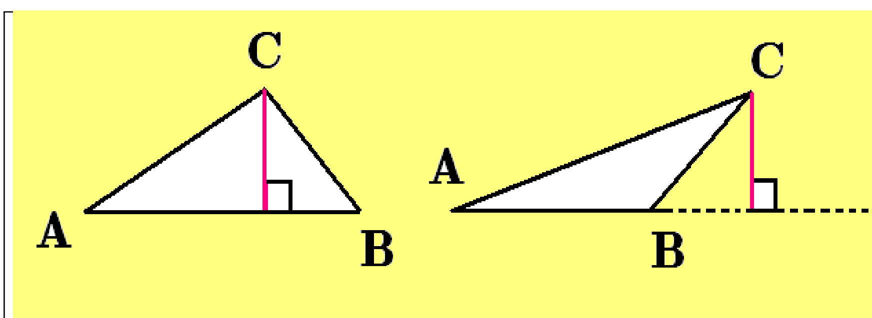


Figure 3-30.
Altitude from vertex C

Sometimes the side opposite the vertex must be extended in order to meet the perpendicular line. Every triangle has three altitudes which meet at a common point of concurrency called the **orthocenter**.

Example 3-6. Calculate the orthocenter associated with the general triangle illustrated in the figure 3-31.

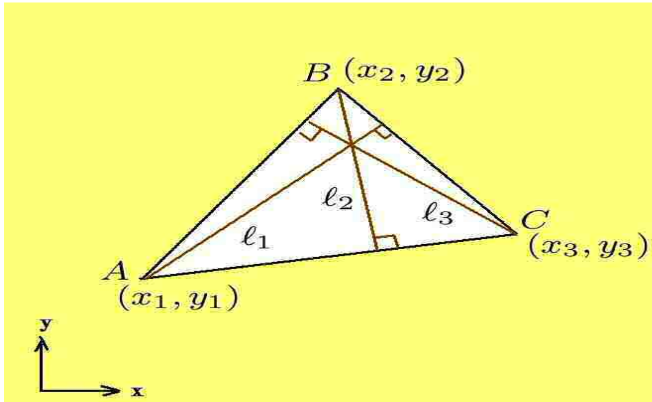


Figure 3-31.
Triangle with three altitudes.

Solution

The slopes associated with the triangle sides are given by

$$\begin{aligned} \text{Slope } \overline{\text{AB}} = m_{AB} =& \frac{y_2 - y_1}{x_2 - x_1} \\ \text{Slope } \overline{\text{BC}} = m_{BC} =& \frac{y_3 - y_2}{x_3 - x_2} \\ \text{Slope } \overline{\text{AC}} = m_{AC} =& \frac{y_3 - y_1}{x_3 - x_1} \end{aligned} \tag{3.18}$$

The slopes of the lines perpendicular to the triangle sides have values which are the negative reciprocals of the slopes given in equation (3.18). The equations of the lines representing the altitudes of triangle $\triangle ABC$ are given by

$$\begin{aligned} &\text{line } \ell_1 = \text{Altitude through vertex A} \qquad && y - y_1 = -\frac{x_3 - x_2}{y_3 - y_2}(x - x_1) \\ &\text{line } \ell_2 = \text{Altitude through vertex B} \qquad && y - y_2 = -\frac{x_3 - x_1}{y_3 - y_1}(x - x_2) \\ &\text{line } \ell_3 = \text{Altitude through vertex C} \qquad && y - y_3 = -\frac{x_2 - x_1}{y_2 - y_1}(x - x_3) \end{aligned} \tag{3.19}$$

Select any two of the lines from the equations (3.19) and solve these simultaneous equations to obtain the coordinates (x_H, y_H) of the orthocenter

$$\begin{aligned} x_H =& \frac{\lambda(y_1 - y_3) + \mu(y_3 - y_2) + \nu(y_2 - y_1)}{y_2(x_1 - x_3) + y_1(x_3 - x_2) + y_3(x_2 - x_1)} \\ y_H =& -\frac{\lambda(x_1 - x_3) + \mu(x_3 - x_2) + \nu(x_2 - x_1)}{y_2(x_1 - x_3) + y_1(x_3 - x_2) + y_3(x_2 - x_1)} \end{aligned} \quad \text{where} \quad \begin{aligned} \lambda =& x_3 x_1 + y_3 y_1 \\ \mu =& x_2 x_3 + y_2 y_3 \\ \nu =& x_1 x_2 + y_1 y_2 \end{aligned} \tag{3.20}$$

■

Example 3-7. **(Theorem) The larger side of a triangle subtends the greater angle.**
(Euclid, Book 1, Proposition 18)
Solution

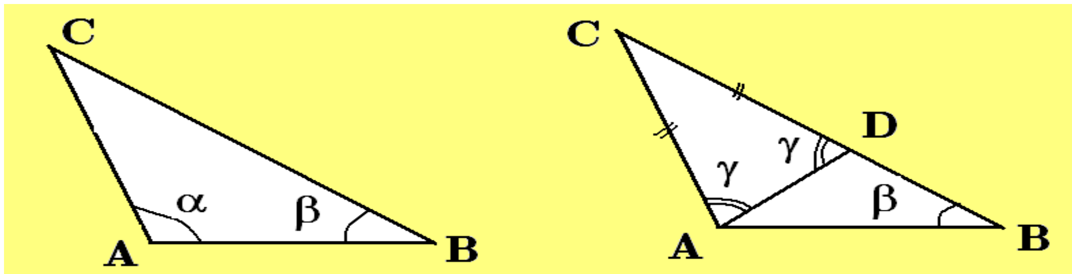


Figure 3-32.
Triangle ABC

Given $\triangle ABC$ with $\overline{BC} > \overline{AC}$ as illustrated in the figure 3-32. Let angle A equal α ($\angle A = \alpha$) and let angle B equal β ($\angle B = \beta$). We wish to demonstrate that $\alpha > \beta$. Using a drawing compass place the needle point at the vertex C and pencil point at the vertex A and make an arc which intersect the line segment $\overline{BC}$ at the point D as illustrated. Use a straight edge and construct the line $\overline{AD}$. By construction the triangle $\triangle ACD$ is an isosceles triangle with $\overline{AC} = \overline{CD}$ and so the angles opposite these sides are also equal. Let $\angle CAB = \angle ABC = \gamma$ as illustrated in the figure 3-32. One can then make the following arguments.

$$(i)\ \alpha > \gamma \quad \text{The whole is greater than any of its parts}$$
$$(ii)\ \gamma \text{ is the exterior angle of } \triangle ABD \text{ and therefore}$$
$$\gamma > \beta$$
$$\text{Therefore} \quad \alpha > \gamma > \beta$$

The converse of the above (Euclid, Book 1, Proposition 19)

In any triangle the greater angle is subtended by the greater side.

Here we assume that $\alpha > \beta$ and wish to show this implies $a > b$.

The proof is by contradiction. In the given triangle one of the conditions

$$a = b, \qquad a < b, \quad \text{or} \quad a > b$$

must hold.

If $a = b$, then triangle ABC would become an isosceles triangle with $\alpha = \beta$. But we know $\alpha > \beta$. Therefore, the original assumption that $a = b$ must be false.

Assume the $a < b$ and note that we have just shown the larger angle is always opposite the larger side, therefore our assumption implies $\alpha < \beta$, which is again a contradiction to the facts given and so one can conclude our original assumption is false.

The only case left is the case $a > b$ which must hold.

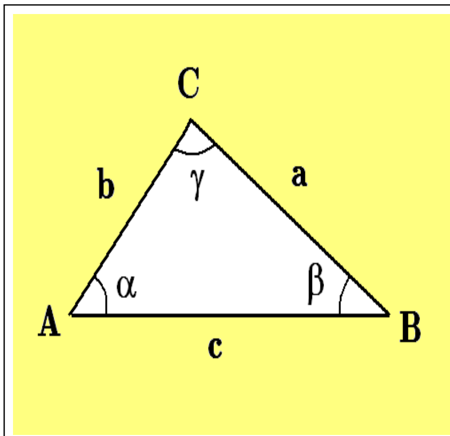


Figure 3-33.
Triangle ABC

■

Example 3-8. (Theorem) The triangle inequality

(Euclid, Book 1, Proposition 20)

The sum of two sides of any triangle will always be greater than the third side

Given the $\triangle ABC$ illustrated in the figure 3-34 with sides a, b, c. Show that one can write

$$b + c > a \quad \text{or} \quad a + c > b \quad \text{or} \quad a + b > c$$

This is known as the triangle inequality.

Extend the line segment $\overline{BC}$ in figure 3-34, then use a drawing compass with the needle set at vertex C and the pencil point set at vertex A. Construct the arc of a circle which intersects the extended line $\overline{BC}$ at the point D. Add the line segment $\overline{AD}$ and observe that this construction is such that $\overline{DC} = \overline{AC} = b$ and results in the figure 3-34. Observe that the constructed triangle $\triangle ADC$ is an isosceles triangle with base angles

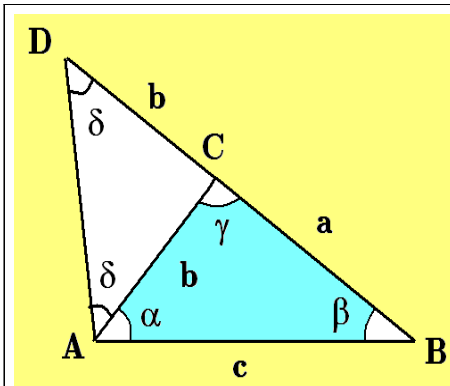


Figure 3-34.
Triangle inequality

$$\angle ADC = \angle DAC = \delta$$

Associated with the triangle $\triangle ABD$ is the inequality $\angle DAB = \alpha + \delta > \delta$ since the whole is greater than any of its parts.

Using the fact that the larger angle is always opposite the larger side, one finds the larger side opposite angle $\angle DAB$ is $\overline{DC}+\overline{CB}=b+a$ and the smaller side opposite the angle $\delta=\angle ADC$ is $\overline{AB}=c$. Consequently, in triangle $\triangle ABD$ one can say

$$\text{if} \quad \alpha+\delta=\angle DAB>\delta \quad \text{then} \quad a+b>c$$

which is called the triangle inequality.

It is left as an exercise to derive the inequalities $a+c>b$ and $b+c>a$. These derivations are identical to what has been done above, but with all the symbols changed around.

■

Example 3-9. Triangle inequality for real numbers

The absolute value of a real number x is defined

$$\mid x \mid = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \tag{3.21}$$

For z and r real numbers, the inequality

$$\mid z \mid \leq r \quad \text{is equivalent to} \quad -r \leq z \leq r$$

In general, if x and y are real numbers, then one can write

$$-\mid x \mid \leq x \leq \mid x \mid \qquad \text{and} \qquad -\mid y \mid \leq y \leq \mid y \mid$$

Adding these inequalities one finds

$$-[\mid x \mid + \mid y \mid] \leq x+y \leq \mid x \mid + \mid y \mid$$

which by definition can be expressed

$$\mid x+y \mid \leq \mid x \mid + \mid y \mid \tag{3.22}$$

which is the triangle inequality for real numbers.

■

Example 3-10. **Show the shortest distance from a point P to a line ℓ is always the perpendicular distance.**

Solution

Given a line ℓ and a point P not on the line ℓ. Construct the perpendicular line to ℓ which passes through point P and intersects line ℓ at point A. One can now extend the line $\overline{PA}$ downward to the point B, where B is selected such that $\overline{PA} = \overline{AB}$. Let Q denote any point on line ℓ which is different from point A and then construct the line segments $\overline{QP}$ and $\overline{QB}$.

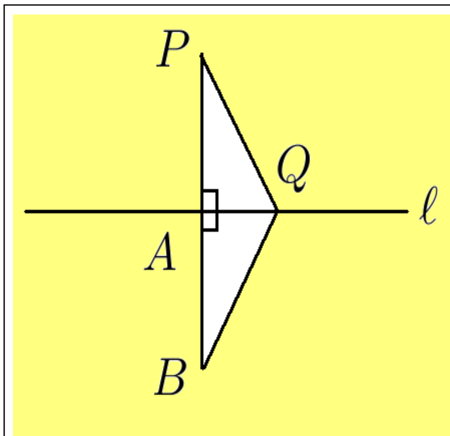


Figure 3-35.
Shortest distance to line ℓ

The triangles $\triangle PAQ$ and $\triangle QAB$ are congruent right triangles ($\triangle PAQ \cong \triangle QAB$). They are congruent using the SAS (side-angle-side) postulate with the line segment $\overline{AQ}$ the common side. Therefore, one can write $\overline{PQ} = \overline{QB}$. Using the triangle inequality associate with triangle $\triangle PQB$ one finds

$$\overline{PQ} + \overline{QB} > \overline{PB} \tag{3.23}$$

but $\overline{PB} = \overline{PA} + \overline{AB} = 2\overline{PA}$ and $\overline{PQ} + \overline{QB} = 2\overline{PQ}$ so that equation (3.23) reduces to

$$2\overline{PQ} > 2\overline{PA} \qquad \text{or} \qquad \overline{PQ} > \overline{PA} \tag{3.24}$$

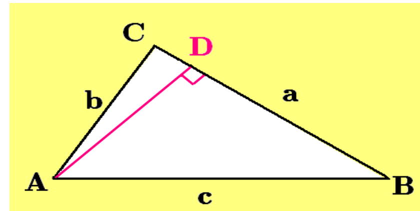
This demonstrates that any line through point P to the line ℓ, which differs from the perpendicular line through P to the given line ℓ, will always have a longer length than the perpendicular distance $\overline{PA}$.

■

Example 3-11. **Another look at the triangle inequality**

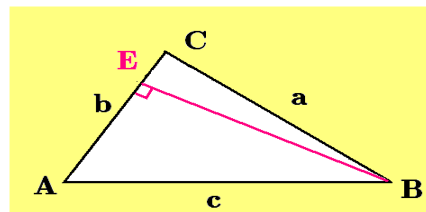
Construct the altitudes from each vertex to the opposite sides of a general triangle $\triangle ABC$, with sides a,b,c as illustrated in the figures below.

In the triangle on the right with altitude from vertex A, note that the line segments $\overline{CD}$ and $\overline{DB}$ are perpendicular to the line segment $\overline{AD}$. Using the results from the previous example one finds the inequalities



$$\overline{AB} > \overline{BD}$$
$$\overline{CA} > \overline{DC}$$

Add these inequalities



$$\overline{AB} + \overline{CA} > \overline{BD} + \overline{DC}$$
$$\text{or} \qquad c + b > a \tag{3.25}$$

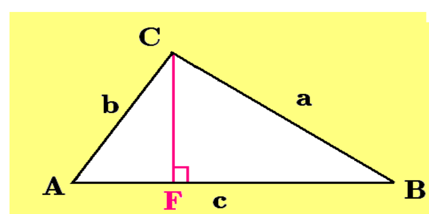
In the triangle with altitude from vertex B, note that the line segments $\overline{AE}$ and $\overline{EC}$ are perpendicular the line segment $\overline{EB}$. This is the same situation as in the previous example and so one can produced the inequalities

$$\overline{CB} > \overline{EC}$$
$$\overline{AB} > \overline{AE}$$

Add these inequalities

$$\overline{CB} + \overline{AB} > \overline{AE} + \overline{EC}$$
$$\text{or} \qquad a + c > b \tag{3.26}$$

In the triangle with the altitude from vertex C, note that the line segments $\overline{AF}$ and $\overline{FB}$ are perpendicular to the line segment $\overline{CF}$.



This is the same situation as in the previous example which results in the inequalities

$$\overline{CB} > \overline{FB}$$
$$\overline{CA} > \overline{AF}$$
$$\text{Add these inequalities} \qquad \overline{CB} + \overline{CA} > \overline{AF} + \overline{FB} \tag{3.27}$$
$$\text{or} \qquad a + b > c$$

The equations (3.25), (3.26), (3.27) show that **the sum of two sides of any triangle will always be greater than the third side** which is the triangle inequality. ■

Radius perpendicular to chord

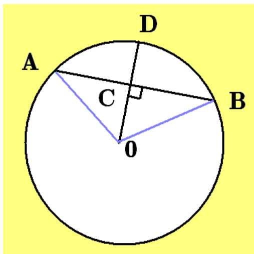


Figure 3-36.

(Euclid, book3, proposition3)
If $\overline{AB}$ is the chord of a circle and a radius $\overline{0D}$ is constructed which is perpendicular to the chord, then this radius will bisect the chord. The situation is illustrated in the figure 3-36. Label the intersection of radius $\overline{0D}$ with the chord $\overline{AB}$ as the point C. Construct also the line segments $\overline{0A}$ and $\overline{0B}$ to form the two right triangles $\triangle A0C$ and $\triangle B0C$.

Observe that these right triangles are congruent ($\triangle A0C \cong \triangle B0C$) because of the RHS (Right triangle-Hypotenuse-Side) postulate where $\overline{0C}$ is a common side to both right triangles and the radii $\overline{0A}$ and $\overline{0B}$ represent the hypotenuses of these right triangles. Therefore, $\overline{AC} = \overline{CB}$ because corresponding parts of congruent triangles are equal to one another. Consequently, point C is the midpoint of the chord $\overline{AB}$. This demonstrates that any radius $\overline{0D}$ which is perpendicular to a chord $\overline{AB}$ will bisect the chord.

Example 3-12. Show two chords are congruent if and only if they are equidistant from the center of the circle. (Euclid, book 3, proposition 14)

Proof

Associated with two circles having the same radius r assume that the chords $\overline{AB}$ and $\overline{CD}$ are given with $\overline{AB}$ congruent to $\overline{CD}$ ($\overline{AB} \cong \overline{CD}$). Construct the line $\overline{0E}$ which is perpendicular to $\overline{AB}$ and the construct the line $\overline{0F}$ which is perpendicular to the chord $\overline{CD}$. Next construct the radial distances $\overline{0C}$ and $\overline{0A}$.

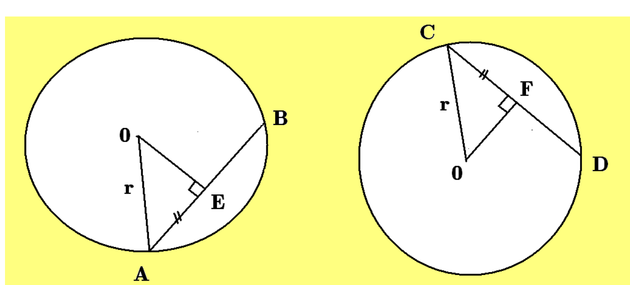


Figure 3-37. Circles with congruent chords

We know that if $\overline{0F}$ is perpendicular to $\overline{CD}$, then $\overline{CF} = \overline{FD}$ which implies $2\overline{CF} = \overline{CD}$. In a similar fashion, if $\overline{0E}$ is perpendicular to $\overline{AB}$, then $\overline{AE} = \overline{EB}$ and one can write $2\overline{AE} = \overline{AB}$. These results hold because we know a line from the center of the circle which is perpendicular to a chord must bisect the chord. By hypothesis $\overline{AB} = \overline{CD}$, therefore one can write

$$2\overline{AE} = 2\overline{CF} \quad \Rightarrow \quad \overline{AE} = \overline{CF} \tag{3.28}$$

which implies triangle $\triangle A0E$ is congruent to triangle $\triangle C0F$ $(\triangle A0E \cong \triangle C0F)$ because of RHS (Right triangle-Hypotenuse-Side) using the hypotenuse r and sides $\overline{AE}$ and $\overline{CF}$. Therefore, $\overline{0E} = \overline{0F}$ because corresponding parts of congruent triangles are equal to one another.

Conversely, assume that $\overline{0F} = \overline{0E}$ with $\overline{0E} \perp \overline{AB}$ and $\overline{0F} \perp \overline{CD}$. In this case the two right triangles $\triangle C0F$ and $\triangle A0E$ are congruent by RHS (Right triangle-Hypotenuse-Side) with hypotenuse r and sides $\overline{0F} = \overline{0E}$. Therefore, $\overline{AE} = \overline{CF}$ and we know $\overline{0E}$ bisects $\overline{AB}$ and $\overline{0F}$ bisects $\overline{CD}$ so that $\overline{AB} = 2\overline{AE}$ and $\overline{CD} = 2\overline{CF}$ Consequently

$$\overline{AB} = 2\overline{AE} = 2\overline{CF} = \overline{CD}$$

and the two chords are equal.

■

The inscribed angle

For the circle illustrated in the figure 3-38 assume that there is a sector formed with the central angle β which intersects the circumference of the circle at the points A and B. One can select **any point** P on the major arc formed by the points A and B and then construct the chords $\overline{PA}$ and $\overline{PB}$. The **inscribed angle** $\alpha = \angle APB$ is defined as the angle formed by the two chords $\overline{PA}$ and $\overline{PB}$ which have the common endpoint P on the major arc associate with the minor arc $\widehat{AB}$. The point P is also known as the vertex of the inscribed angle $\angle APB$. The other endpoints A and B define the minor intercepted arc $s = \widehat{AB}$ associated with both the central angle β and inscribed angle α.

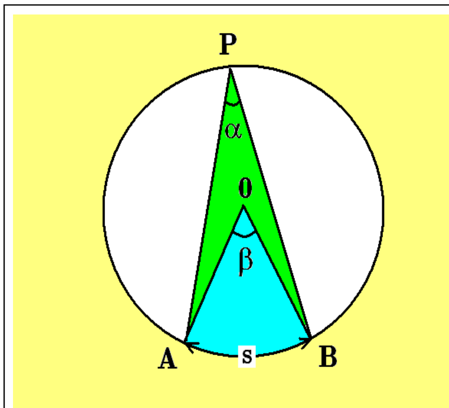
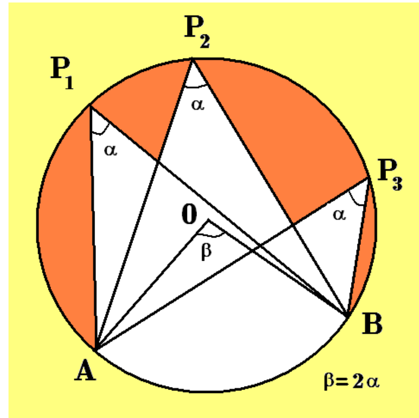


Figure 3-38. Inscribed angle

The inscribed angle theorem

(Euclid, book 3, propositions 20,21)
The inscribed angle theorem states that the inscribed angle α, associated with the central angle β, will not change as the vertex point P moves along the major arc formed by the central angle. Further, the inscribed angle α will always have the value of half of the central angle ($\alpha = \frac{1}{2}\beta$). Another way of saying this is that the central angle β will always be twice the inscribed angle α.



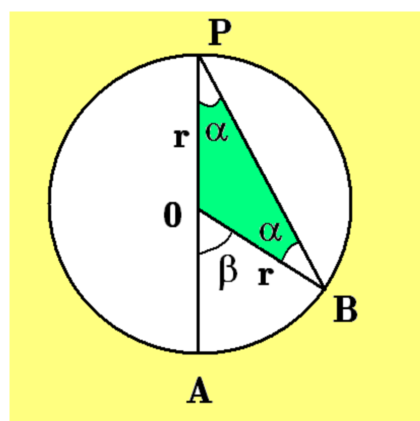
The proof of this theorem is broken up into three cases.

Case 1: Consider the special case where one chord $\overline{\text{PA}}$ is the diameter of the circle.

Case 2: Investigate the special case where the center of the circle in the interior of triangle $\triangle PAB$.

Case 3: Investigate the special case where the center of the circle is exterior to the triangle $\triangle PAB$.

Case 1: Assume the line segment $\overline{PA}$ passes through the center of the circle (0) and then construct the radius $\overline{0B} = r$. In this special case the triangle $\triangle P0B$ is an isosceles triangle with equal angles $\angle 0PB = \angle 0BP = \alpha$ and the angle $\angle P0B = \pi - \beta$ because the central angle β and angle $\angle P0B$ are supplementary angles. The sum of the angles in any triangle must add to π radians. Therefore in triangle $\triangle P0B$



$$\alpha + \alpha + (\pi - \beta) = \pi \qquad \text{or} \qquad \beta = 2\alpha \tag{3.29}$$

This demonstrates the central angle is twice the inscribed angle or $\alpha = \frac{1}{2}\beta$ as stated in the theorem.

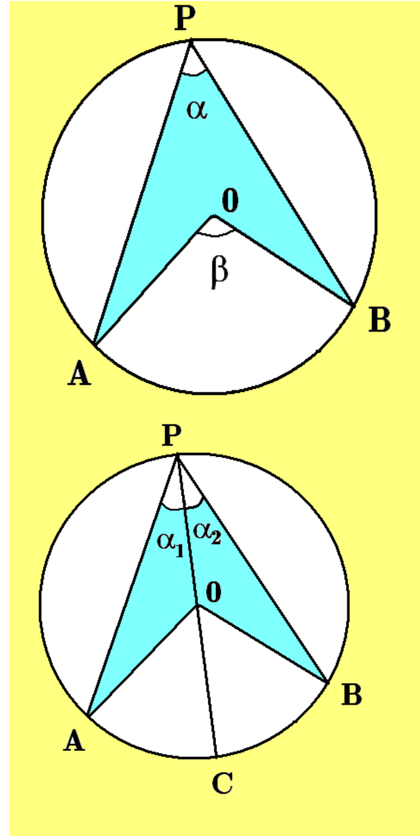
Case 2: Construct the line $\overline{P0}$ and then extend the line to intersect the circle at the point C as illustrated in the figures on the right. The constructed line divides the inscribed angle α into two smaller angles labeled α_1 and α_2 such that $\alpha = \alpha_1 + \alpha_2$. Use the results from case 1 above to verify that

$$\alpha_1 = \frac{1}{2}\angle A0C$$
$$\alpha_2 = \frac{1}{2}\angle C0B$$

Adding these equations gives

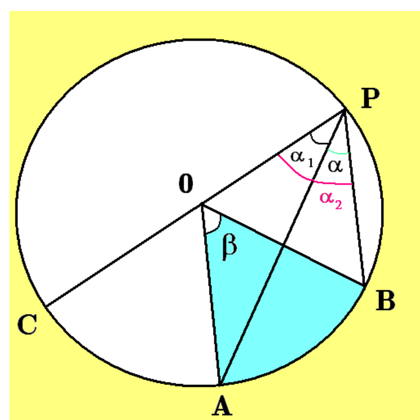
$$\alpha_1 + \alpha_2 = \frac{1}{2}\left[\angle A0C + \angle C0B\right]$$

Note that $\alpha = \alpha_1 + \alpha_2$ and $\beta = \angle A0C + \angle C0B$ so that $\alpha = \frac{1}{2}\beta$ which was to be proved.



Case 3: Construct the line $\overline{P0}$ and then extend the line to intersect the circle at point C. The constructed line forms the inscribed angles α_1 and α_2 as illustrated. Note that in this case we have $\alpha = \alpha_2 - \alpha_1$. Use case 1 above and verify that

$$\alpha_1 = \frac{1}{2}\angle C0A$$
$$\alpha_2 = \frac{1}{2}\angle B0C$$



Subtracting these equations produces the result

$$\alpha_2 - \alpha_1 = \frac{1}{2}\left[\angle B0C - \angle C0A\right]$$

But $\alpha = \alpha_2 - \alpha_1$ and $\beta = \angle A0B = \angle B0C - \angle C0A$ so that again we have $\alpha = \frac{1}{2}\beta$

(Theorem)

Let ABCD denote a quadrilateral inside a circle, then the opposite angles sum to two right angles. (Euclid, book3, proposition 22)

Proof:

Using the inscribed angle theorem from the previous discussion, one can verify the equal angles associated with the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$. Using the fact that for every triangle the sum of the interior angles must equal two right angles or π radians, one can verify the following equations.

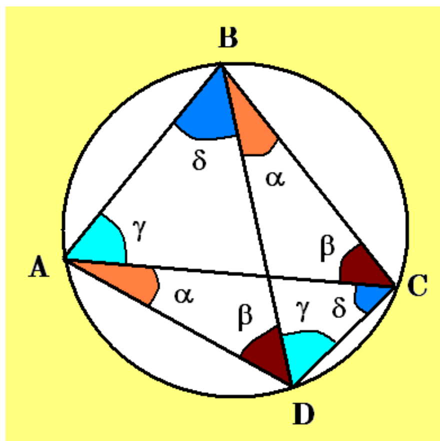


Figure 3-39. Inscribed angles associated with quadrilateral

In triangle $\triangle ABC$ the sum of the interior angles produces

$$(\alpha + \delta) + \beta + \gamma = \pi \tag{3.30}$$

The equation (3.30) shows that the opposite angles associated with a quadrilateral inscribed inside a circle must sum to two right angles. Quadrilaterals with vertices on the circumference of a circle are called **cyclic quadrilaterals**.

How to construct a regular dodecagon using a protractor

First construct a circle of radius r using a drawing compass. We know that the dodecagon has 12 sides and so divide the $360°$ of the circle into 12 parts to obtain $\frac{360°}{12} = 30°$. Use a protractor and make marks $(\cdot)$ on the circumference of the circle every $30°$ as illustrated in the figure 3-40. Draw lines from the points marked to the center of the circle and also draw line segments connecting the marked points on the circumference of the circle to form a loop around the inside of the circle creating the dodecagon.

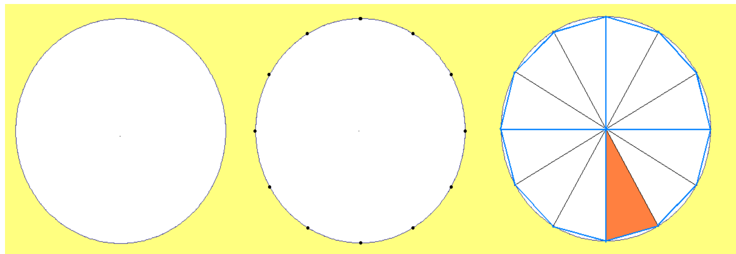


Figure 3-40.

Area of a circle

Examine the dodecagon figure constructed in the figure 3-40 and observe that the lines through the center of the circle create 12 isosceles triangles around the inside of the circle. Each side of the isosceles triangles have length equal to the radius r of the circle.

Examine the figure 3-41 and make the following observations. As the number of sides of the inscribed polygon increases, then the side s gets smaller and the apothem h approaches the radius r of the circle. One can **approximate the area of the circle** by adding up the areas of the 12 isosceles triangles created inside the dodecagon of figure 3-41. Let h, the apothem, denote the height of the isosceles triangle and let s denote the base of the triangle. We know the area of one triangle is given by $\frac{1}{2}$ the base times the height or $A_t = \frac{1}{2}sh$.

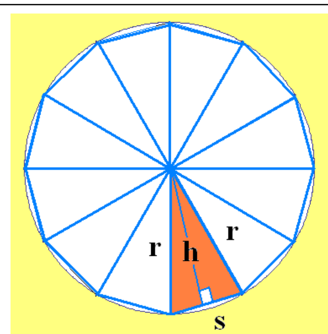


Figure 3-41.
Dodecagon with inside isosceles triangles with base s and height h

Therefore the approximate area for the circle is

$$A_{circle} \approx (12)A_t = (12)\frac{1}{2}sh = \frac{1}{2}h(12s) \tag{3.31}$$

where the symbol $\approx$ is used to represent 'approximation' and the quantity $(12s)$ represents the **perimeter** of the dodecagon.

Instead of 12 isosceles triangles, suppose there were n such triangles, where n is a large number. One would expect the area approximation would get better as n increased. Using the notation $\lim_{n\to\infty}$ to denote the limit as n increases without bound, the area of the circle approximation becomes

$$A_{circle} = \lim_{n\to\infty} \frac{1}{2}h(ns) \tag{3.32}$$

where n has replaced the 12 in equation (3.31). The quantity ns in equation (3.32) now represents the **perimeter** of the constructed n-sided polygon created by n triangles, all having base s, inside the circle. The value of ns approaches the circumference of the circle which we know is $2\pi r$. As n gets larger, the base s gets smaller and the value of h approaches the value of **the circle radius** r so that equation (3.32) becomes

$$A_{circle} = \lim_{n\to\infty} \frac{1}{2}h(ns) = \frac{1}{2}r(2\pi r) = \pi r^2 \tag{3.33}$$

This gives the result that **the area of a circle is given by π times the radius squared.**

In summary a circle with radius r has the following properties.

$$\begin{aligned}
&(i) && \textbf{The diameter } \boldsymbol{d} \textbf{ is twice the radius or } \boldsymbol{d = 2r}\\
&(ii) && \textbf{The circumference } \boldsymbol{c} \textbf{ is given by } \boldsymbol{c = 2\pi r = \pi d}\\
&(iii) && \textbf{The area } \boldsymbol{A} \textbf{ is given by } \boldsymbol{A = \pi r^2}\\
&(iv) && \textbf{The circle is a figure with constant width.}
\end{aligned} \tag{3.34}$$

Exercises

▶ **3-1.** Find the sum of the interior angles of

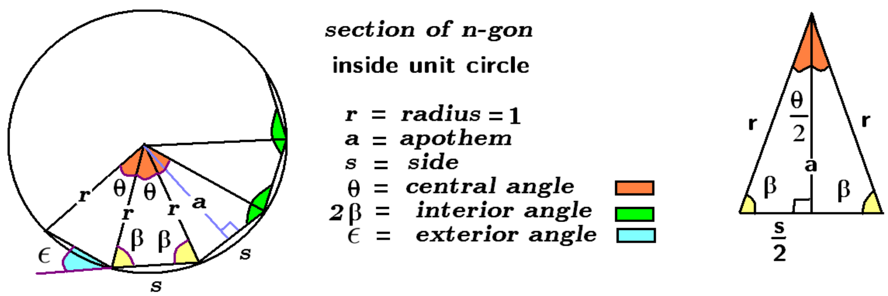
(a) A triangle

(b) A quadrilateral

(c) A pentagon

(d) A hexagon

▶ **3-2.** Consider a regular polygon with n sides inscribed within a unit circle as illustrated.



Fill in the following table.

Regular polygons with			
sides n	central angle θ $(radians)$	isosceles base angle β $(radians)$	exterior angle ϵ $(radians)$
3			
4			
5			
6			
7			
8			
⋮	⋮	⋮	⋮
n			

▶ **3-3.** Perform the following summations.

$$(a)\ \sum_{i=1}^{5} i^2 \qquad (b)\ \sum_{j=0}^{5}(2j+1) \qquad (c)\ \sum_{k=1}^{5} k^3 \qquad (d)\ \sum_{i=0}^{5} 2^i$$

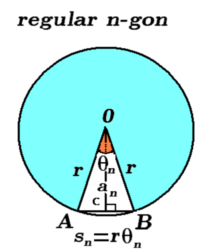
▶ **3-4.** Given a regular pentagon with radius$=1$, apothem$=\frac{1}{4}(1+\sqrt{5})$, and one side$=2\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}$

(a) Find the perimeter and area of the pentagon.

(b) Find the radius of the circumscribed circle.

(c) Find the radius of the inscribed circle.

▶ **3-5.**



An n-sided regular polygon is inscribed inside a circle having radius r.
(i) The apothem a_n satisfies $0 < a_n < r$ because the perpendicular distance $\overline{0C} = a_n$ is the shortest distance to line $\overline{AB}$ and also by the larger side opposite larger angle theorem.
(ii)The chord satisfies $\overline{AB} < \widehat{AB} = s_n = r\theta_n$, because shortest distance from point A to B is a straight line.

(a) For an n-sided inscribed polygon, find the central angle θ_n.

(b) For an n-sided inscribed polygon, find its perimeter p_n.

(c) Show the area of the inscribed n-sided regular polygon is given by $A_n = \frac{1}{2}(apothem)(perimeter) = \frac{1}{2}(a_n)(p_n)$

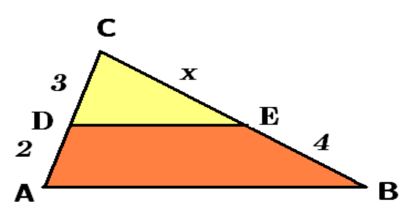
(d) Show in the limit $\lim_{n\to\infty} a_n = r, \quad \lim_{n\to\infty} p_n = 2\pi r$

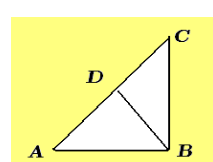
(e) Show the area of the circle is $\lim_{n\to\infty} A_n = \frac{1}{2}(radius)(circumference\ of\ circle)$ because of step (d). That is, as n increases the inscribed polygon becomes a circle.

▶ **3-6.** Find the sum of the exterior angles of

(a) A triangle, (b) A quadrilateral, (c) A pentagon, (d) A hexagon

▶ **3-7.** For the triangle given $\overline{DE} \parallel \overline{AB}$.
Find x and the ratio $\frac{\overline{DE}}{\overline{AB}}$

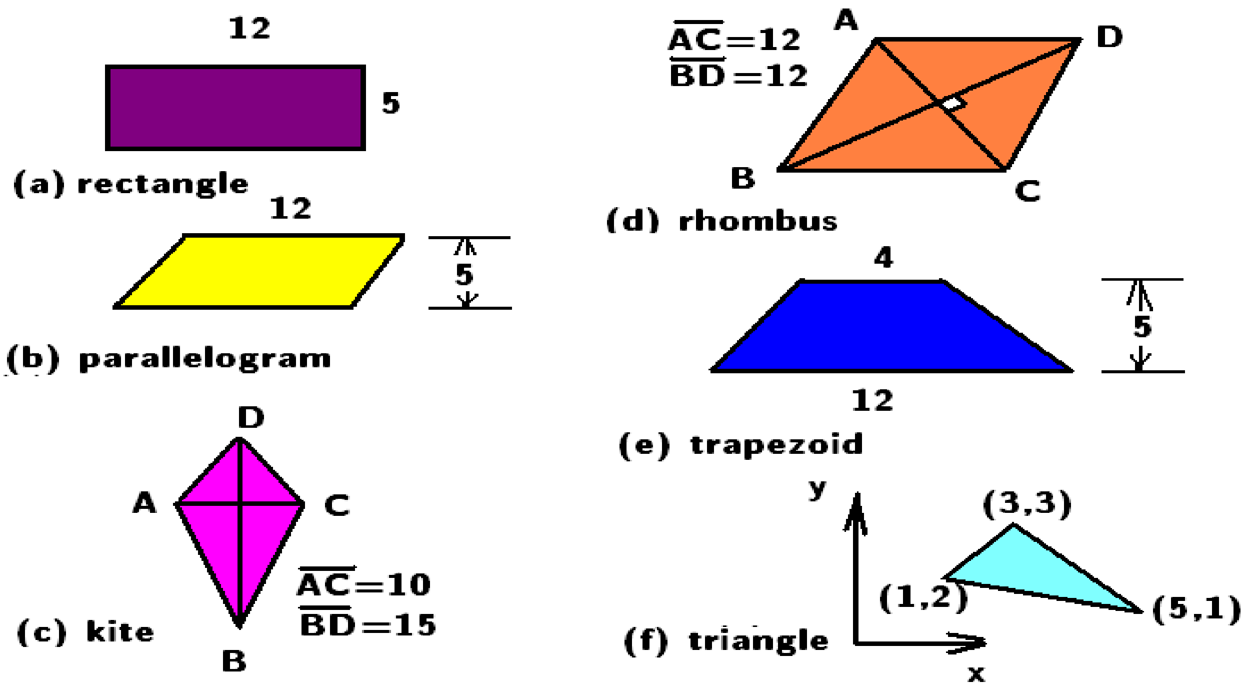




▶ **3-8.**

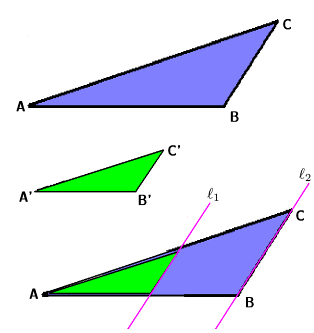
Given triangle $\triangle ABC$ with line $\overline{BD}$ such that $\overline{AD} = \overline{BD}$ and $\overline{AD} = \overline{DC}$. Prove that $\overline{CB} \perp \overline{AB}$.

▶ **3-9.** Find the area of the following figures. Assume all lengths are in units of meters.

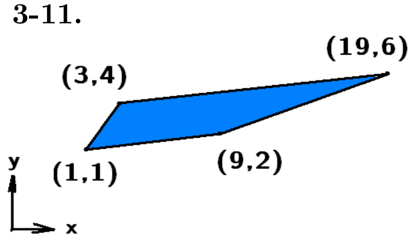


▶ **3-10.**

Given two similar triangles $\triangle ABC$ and $\triangle A'B'C'$. Place the smaller triangle inside the larger triangle and construct **appropriate parallel lines** $\ell_1 \parallel \ell_2 \parallel \ell_3$ to show that the ratio of the sides associated with similar triangles being proportional can be proven using Thales parallel intercept theorem.

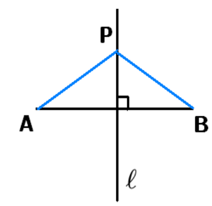


▶ **3-11.**



Find the equation of the line ℓ through the midpoints of the sides of the given trapezoid. Is this line parallel to the trapezoid bases?

▶ **3-12.**
If line ℓ is a perpendicular bisector of line segment $\overline{AB}$, then prove that any point P on line ℓ is equidistant from the endpoints.



▶ **3-13.** Find the circumcenter for the triangle $\triangle ABC$ with vertices

$(a)\quad A:\ (0,0),\ B:\ (14,0),\ C:\ (7,49) \qquad (b)\quad A:\ (0,0),\ B:\ (14,0),\ C:\ (5,15)$

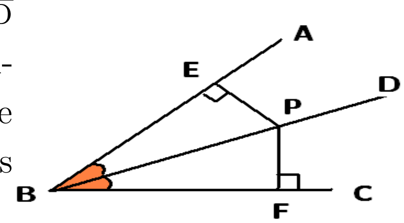
▶ **3-14.** Prove the diagonals of a parallelogram bisect one another.

▶ **3-15.** Prove the opposite sides of a parallelogram are equal.

▶ **3-16.** Prove a diagonal of a parallelogram creates two equal triangles.

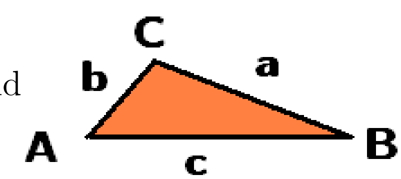
▶ **3-17.** Prove that parallel lines are everywhere the same distance apart.

▶ **3-18.** Given an angle $\angle ABC$ with angle bisector $\overline{\text{BD}}$ and arbitrary point P on the angle bisector line. Construct the perpendicular lines $\overline{\text{PF}}$ and $\overline{\text{PE}}$. Prove the theorem that every point P on the angle bisector is equidistant from the angle sides.



▶ **3-19.**

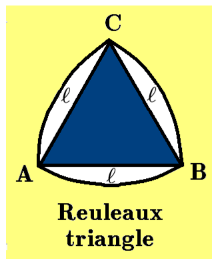
Given triangle $\triangle ABC$. Move the vertex C around to show $|a-b| < c < a+b$



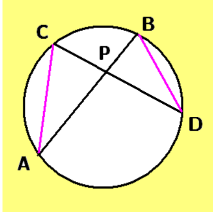
▶ **3-20.** Generalize the triangle inequality to polygons. Prove the side of any polygon will always be less than the sum of the other sides.

▶ **3-21.**

A Reuleaux triangle is constructed as follows. (i) Construct an equilateral triangle with side of length ℓ. (ii) Set drawing compass width equal to ℓ and place the needle of compass at each vertex to draw the arcs $\widehat{\text{AC}}, \widehat{\text{AB}}, \widehat{\text{BC}}$. Prove figure has constant width.



▶ **3-22.**

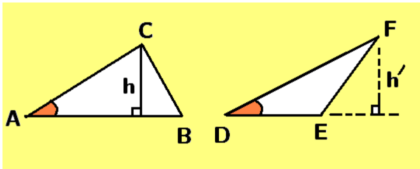


Select a point P inside a circle and construct two different chords $\overline{\text{AB}}$ and $\overline{\text{CD}}$ through the point P as illustrated. Use the inscribed angle theorem to show

(a) The triangles $\triangle APC$ and $\triangle DPB$ are similar.

(b) Show $\overline{\text{PA}} \cdot \overline{\text{PB}} = \overline{\text{PC}} \cdot \overline{\text{PD}}$

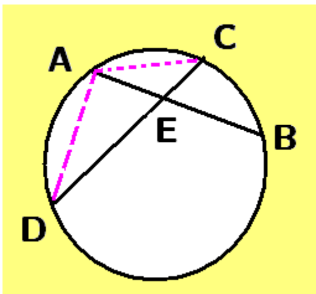
▶ **3-23.**



Given two triangles $\triangle ABC$ and $\triangle DEF$ where $\angle A = \angle D$. Prove that

$$\frac{[ABC]}{[DEF]} = \frac{\overline{\mathrm{AB}}\cdot\overline{\mathrm{AC}}}{\overline{\mathrm{DE}}\cdot\overline{\mathrm{DF}}}$$

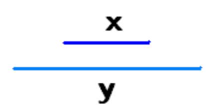
▶ **3-24.**



Let $\overline{\mathrm{AB}}$ equal a chord of a circle and let point C denote the midpoint of the arc $\widehat{\mathrm{AB}}$. Let $\overline{\mathrm{CD}}$ denote any other chord which intersects the chord $\overline{\mathrm{AB}}$ at point E. Prove that

$$\frac{\overline{\mathrm{CD}}}{\overline{\mathrm{CA}}} = \frac{\overline{\mathrm{CA}}}{\overline{\mathrm{CE}}}$$

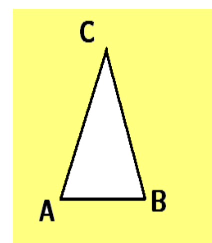
▶ **3-25.**



Given two unequal line segments x and y. Show that

$$\sqrt{xy} < \frac{(x+y)}{2} \quad \Rightarrow \quad \text{geometric mean} < \text{arithmetic mean}$$

▶ **3-26.**

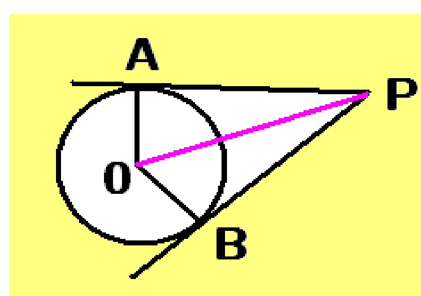


Given triangle $\triangle ABC$ is an isosceles triangle. Show that the line which bisects the exterior angle at vertex C is parallel to the base $\overline{\mathrm{AB}}$.

▶ **3-27.** Examine a rectangle with sides $(a+b)$ and $(c+d)$. Show graphically that

$$(a+b)(c+d) = ac + ad + bc + bd$$

▶ **3-28.**



If P is a point exterior to a circle and from point P one constructs two tangents to the circle which touch the circumference at points A and B, then

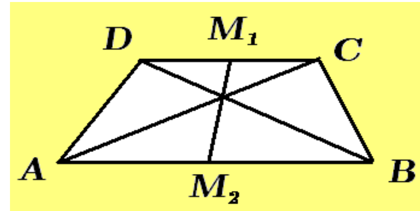
(i) Prove that $\overline{\mathrm{PA}} = \overline{\mathrm{PB}}$

(ii) Show line $\overline{\mathrm{0P}}$ bisects angle $\angle APB$

▶ **3-29.** Show that if two triangles are similar, then their altitudes are also similar.

▶ **3-30.**

Let M_1 and M_2 denote the midpoints of the bases associated with a trapezoid. Prove that the line segment $\overline{M_1M_2}$ will pass through the point of intersection of the diagonals.
in

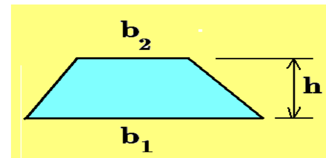


▶ **3-31.**

(a) Find the area of the trapezoid illustrated.

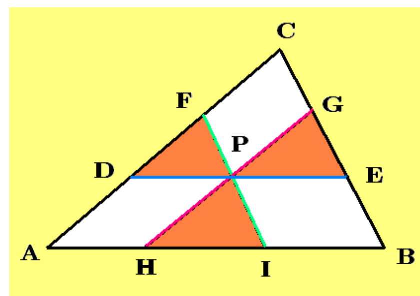
(b) What happens to the trapezoid as $b_2 \to 0$?

(c) Find the area of the trapezoid if $b_2 \to 0$.



▶ **3-32.**

Given the triangle $\triangle ABC$ select any point P within the triangle and then draw three different lines through the selected point where each line is parallel to a side of the triangle. Label the points of intersection with the sides to obtain the figure illustrated.



(a) Show triangles $\triangle ABC$, $\triangle PEG$, $\triangle DPF$, $\triangle HIP$ are similar triangles.

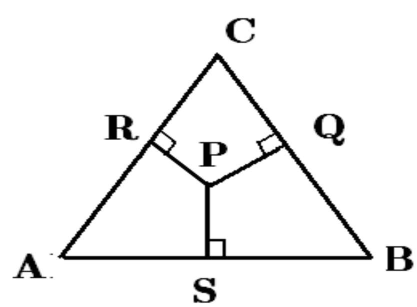
(b) Show that $\dfrac{[DPF]}{[ABC]} = \left(\dfrac{\overline{DP}}{\overline{AB}}\right)^2$, $\dfrac{[PEG]}{[ABC]} = \left(\dfrac{\overline{PE}}{\overline{AB}}\right)^2$, $\dfrac{[HIP]}{[ABC]} = \left(\dfrac{\overline{HI}}{\overline{AB}}\right)^2$

(c) Show that $\sqrt{\dfrac{[DPF]}{[ABC]}} + \sqrt{\dfrac{[HIP]}{[ABC]}} + \sqrt{\dfrac{[PEG]}{[ABC]}} = \dfrac{\overline{DP}}{\overline{AB}} + \dfrac{\overline{HI}}{\overline{AB}} + \dfrac{\overline{PE}}{\overline{AB}} = 1$

(d) Show that $[ABC] = \left(\sqrt{[DBF]} + \sqrt{[HIP]} + \sqrt{[PEG]}\right)^2$

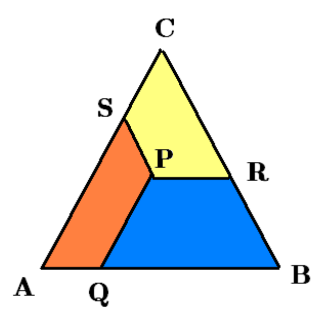
▶ **3-33.**

Select an arbitrary point inside an equilateral triangle $\triangle ABC$ with altitude equal to h. Construct the perpendicular lines $\overline{PQ}, \overline{PR}, \overline{PS}$ to the triangle sides and show that $\overline{PQ} + \overline{PR} + \overline{PS} = h$.



▶ **3-34.**

Select an arbitrary point P inside an equilateral triangle with sides of length ℓ. Construct the line segments $\overline{PQ}, \overline{PR}, \overline{PS}$ such that $\overline{PQ} \parallel \overline{AC}, \overline{PR} \parallel \overline{AB}, \overline{PS} \parallel \overline{BC}$.
Show that $\overline{PQ} + \overline{PR} + \overline{PS} = \ell$
Hint: Extend line $\overline{PQ}$, then form similar triangles and parallelogram.



Geometry

Chapter 4

The Pythagorean Theorem

The Pythagorean theorem states a property of right triangles that has been known for over three thousand years. Consider the right triangle illustrated in the figure 4-1.

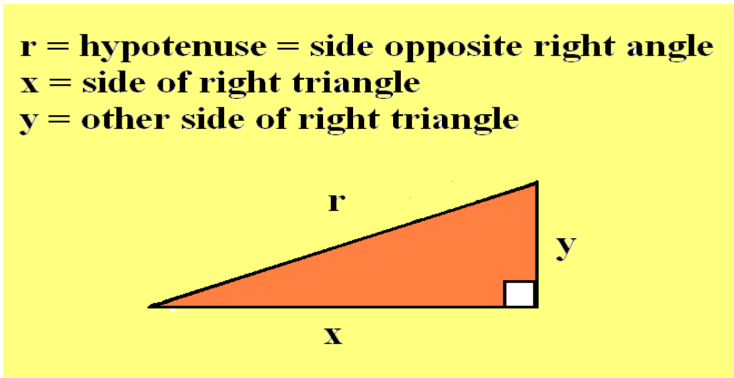


Figure 4-1. Right triangle with shorter legs meeting at a right angle.

The Pythagorean theorem states that **all right triangles** have the following property.

The Pythagorean Theorem

In every right triangle having sides x and y forming a right angle and having a hypotenuse r opposite the right angle, one can say that the square of the hypotenuse is equal to the sum of the squares of the other two sides.

or $\quad x^2 + y^2 = r^2$

In a right triangle the side opposite the right angle is called the hypotenuse. The sides of the right triangle which form the right angle are called the legs of the right triangle. The more formal term cathetus means either leg of a right triangle and the plural catheti refers to both sides making the right angle.

The Pythagorean theorem is named after Pythagoras[1] (570-495) BCE, who is supposedly the first person to have provided a written proof of the theorem. There

[1] Pythagoras of Samos (569-475) BCE a Greek philosopher, mathematician and religious leader.

is much evidence[2] indicating that the Hindus, Babylonians and Chinese knew about this theorem about 1000 years before Pythagoras.

Currently there are probably over 500 different proofs of the Pythagorean theorem. The book *Pythagorean Proposition* by Elisha Scott Loomis contains well over 350 proofs. Many alternative proofs of the Pythagorean theorem can be found on the internet. About 3000 years ago the Chinese were the first to come up with a geometric proof of this theorem. This geometric proof is illustrated in the figure 4-2, where the right triangle in 4-2(a) is reproduced four times and arranged as shown in figure 4-2(b).

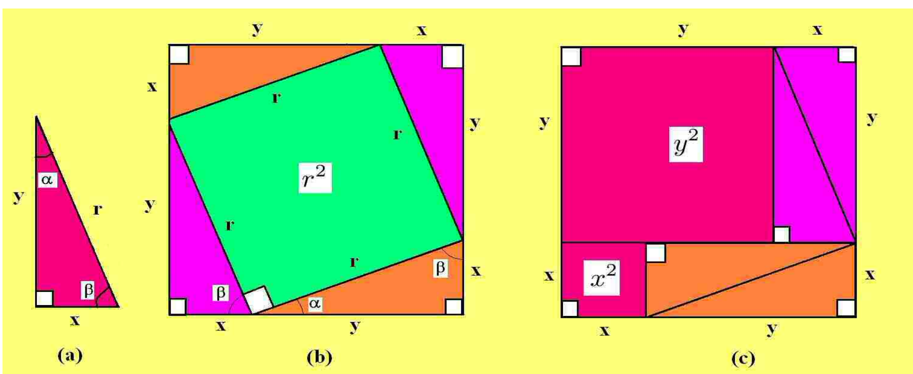


Figure 4-2. Geometric proof of Pythagorean theorem

This new arrangement, figure 4-2(b), shows a square with sides of length $(x+y)$ and inside this square is another square with sides of length r. We know there must be 180° in a triangle so it follows that the angles α and β must sum to 90°. This demonstrates that the square with sides of length r is indeed a square with 90° corners, because of the angular sum $\alpha + 90^\circ + \beta = 180^\circ$ along the straight line for any side $(x+y)$ of the larger square.

The area of the larger square can be found by calculating

$$Area = (x+y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 4\left(\frac{1}{2}xy\right) \tag{4.1}$$

which represents the sum of two squares plus the area of four triangles as pictured in figure 4-2(c). This area can also be expressed as the sum of the square with sides

[2] This theorem was developed around 800 BCE by Hindu mathematician Baudhayana in his book Baudhayana Sulba Sutra.

r plus 4-times the area of the triangle in figure 4-2(a). This alternative method of expressing the area of the square in figure 4-2(b) is written

$$Area = r^2 + 4\left(\frac{1}{2}xy\right) \tag{4.2}$$

Using the postulate that things equal to the same thing are equal to each other, one can write

$$Area = x^2 + y^2 + 4\left(\frac{1}{2}xy\right) = r^2 + 4\left(\frac{1}{2}xy\right)$$

which simplifies to

$$x^2 + y^2 = r^2 \tag{4.3}$$

which is the Pythagorean theorem.

This result can also be expressed geometrically by moving the triangles in figure 4-2(b) to their new positions in figure 4-2(c). By comparing the geometry of the figures 4-2(b) and 4-2(c), one can conclude that the result given by equation (4.3) is true.

In addition to the figure 4-2 there is another image to help you visualize the Pythagorean theorem. This image is given in figure 4-3 and represents the square of each side of a right triangle attached to the appropriate side.

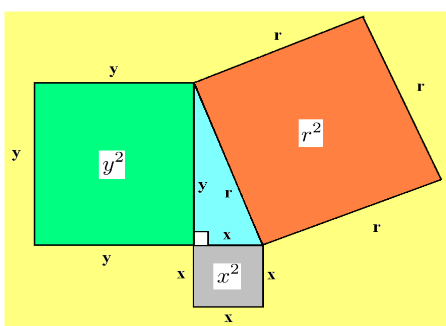


Figure 4-3. Image to help remember the Pythagorean theorem $\mathbf{x^2 + y^2 = r^2}$

Another proof of the Pythagorean theorem

Consider the right triangle $\triangle ABC$ illustrated in the figure 4-4 which has legs a,b and hypotenuse c. Construct a line perpendicular to the line $\overline{AB}$ which passes through the vertex C as illustrated. Label the point of intersection with side $\overline{AB}$ of length c as point D and note this perpendicular line divides the length c into two parts labeled c_1 and c_2 such that $c = c_1 + c_2$. Also define the distance $\overline{CD} = h$ as the height of the triangle and then construct the three similar triangles illustrated in the figure 4-5.

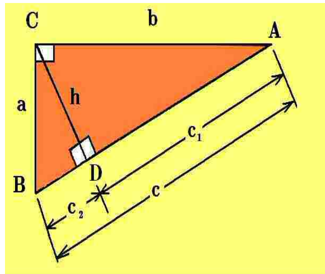


Figure 4-4. **Right triangle divided into three right triangles**

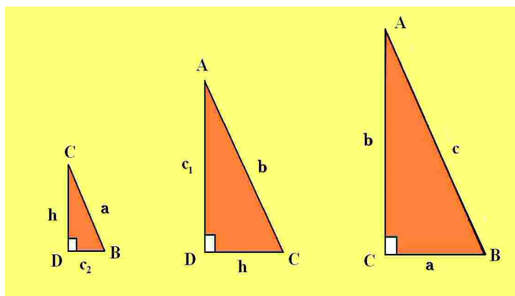


Figure 4-5. **Three similar right triangles from figure 4-4**

Comparing similar triangles $\triangle CDB$ and $\triangle ACB$ form the ratio

$$\frac{a}{c} = \frac{c_2}{a} \implies a^2 = c\,c_2 \tag{4.4}$$

Comparing the similar triangles $\triangle ADC$ with triangle $\triangle ACB$ one finds

$$\frac{b}{c} = \frac{c_1}{b} \implies b^2 = c\,c_1 \tag{4.5}$$

Now add the equations (4.4) and (4.5). By the postulate equals added to equals the results are equal, one obtains the equation

$$a^2 + b^2 = cc_1 + cc_2 = c\,(c_1 + c_2) = c\,c = c^2 \tag{4.6}$$

because $c_1 + c_2 = c$. This verifies that the sum of the squares of the legs of a right triangle must equal the hypotenuse of the right triangle squared.

Example 4-1.

Use the Pythagorean theorem to find the shortest distance from a point (x_0, y_0) to a given line ℓ.

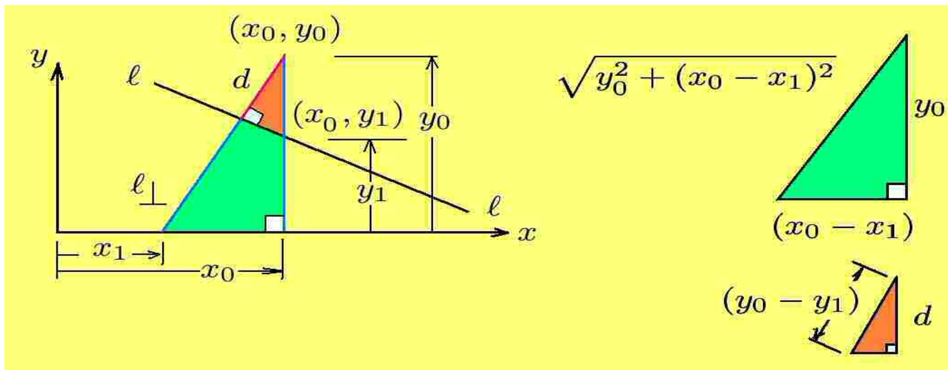


Figure 4-6. **Distance from point (x_0, y_0) to line ℓ**

Solution:

Let d denote the perpendicular distance from the point (x_0, y_0) to the line ℓ. The general equation for the line ℓ is $Ax + By + C = 0$ where A, B, C are given constants. The slope of the line ℓ is $m = \frac{-A}{B}$ and the slope of the line $\ell_\perp$ which is perpendicular to $Ax + By + C = 0$ is $m_\perp = \frac{B}{A}$ which is the negative reciprocal of the slope m. The equation of the perpendicular line which passes through the point (x_0, y_0) can be represented as $y - y_0 = \frac{B}{A}(x - x_0)$. This line intersects the x-axis at the point

$$x_1 = x_0 - y_0 \frac{A}{B} \tag{4.7}$$

where $y = 0$. Constructing the line $x = x_0$ one finds it intersects the line ℓ at the point (x_0, y_1) where

$$y_1 = \frac{-C}{B} - \frac{A}{B} x_0 \tag{4.8}$$

and there is created two similar right triangles as illustrated in the figure 4-6. Using the similar right triangles one finds the ratio

$$\frac{\sqrt{y_0^2 + (x - x_0)^2}}{y_0 - y_1} = \frac{y_0}{d} \tag{4.9}$$

where the Pythagorean theorem is used to calculate the hypotenuse of the larger right triangle. Using the equations (4.7) and (4.8) one finds

$$x_0 - x_1 = y_0\frac{A}{B}, \qquad y_0 - y_1 = \frac{Ax_0 + By_0 + C}{B}, \qquad \sqrt{y_0^2 + (x_0 - x_1)^2} = \frac{y_0}{B}\sqrt{A^2 + B^2} \tag{4.10}$$

The equation (4.10) simplifies the equation (4.9) to produce the relation

$$d = \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \tag{4.11}$$

for the perpendicular distance from (x_0, y_0) to the line ℓ. Observe that the square root has two signs and sometimes the distance d comes out negative. To make the distance d always positive one can select the sign of the square root to make the distance d positive.

In summary, from the general equation of a line $Ax + By + C = 0$ one can immediately write the distance formula

$$d = \frac{Ax + By + C}{\pm\sqrt{A^2 + B^2}} \tag{4.12}$$

where the sign on the square root is to be selected to make d a positive distance. The result given by equation (4.12) can be used in the following way.

(i) If one substitutes $x = x_0$ and $y = y_0$ into the equation (4.12) and gets a positive number d, then d represents the perpendicular distance of the point (x_0, y_0) from the line.

(ii) If one substitutes $x = x_0$ and $y = y_0$ into the equation (4.12) and gets zero, then the point (x_0, y_0) is on the given line.

■

Euclid's proof of Pythagorean theorem

The figure 4-7 is known as the Bride's Chair and was used by Euclid to prove the Pythagorean theorem. You can find this figure in Euclid's the Elements under the listing of proposition 47, book 1. In figure 4-7 there is the right triangle $\triangle ABC$ with sides a and b meeting to form a right angle, together with the side c opposite the right angle, known as the hypotenuse. Euclid constructed the squares a^2, b^2 and c^2 and placed these squares against the corresponding sides of triangle $\triangle ABC$ as illustrated. Euclid then made some straight line constructions to prove that $a^2 + b^2 = c^2$ for any right-angled triangle.

He did this by showing the area a^2, attached to side a, was equal to the rectangular area $BHJD$ within the square c^2 attached to the hypotenuse c, as illustrated in the figure 4-7. He then showed that the area b^2, attached to side b was equal to the rectangular area $HAEJ$ within the square c^2. This demonstrated that the sum of the areas $a^2 + b^2$ had to equal the area c^2 because the whole must equal the sum of its parts.

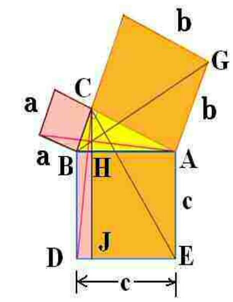


Figure 4-7. The Bride's Chair

In order to make Euclid's proof as simple as possible, consider the following basic concepts which will be employed multiple times during Euclid's proof of the Pythagorean theorem.

1) If you have two parallel lines a distance h apart and then construct a fixed line segment $\overline{B_1B_2} = b$ on the lower parallel line, one can then select a point P on the upper parallel line to construct the triangle $\triangle PB_1B_2$. This triangle will always have the area of half the base times the height h no matter where the point P is on the upper parallel. Note all the triangles have the same height and base. (Euclid, Book 1, Proposition 38)

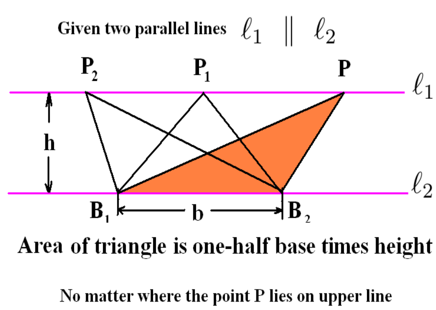


Figure 4-8. Area of triangle remains the same as point P moves on upper line.

2) Construct a rectangle ABCD between two parallel lines as illustrated in the figure 4-9 and the let point B move to a point B' on the upper parallel. Observe that the area of triangle $\triangle BDC$ is the same as the area of triangle $\triangle B'DC$ and this area

will always equal half the area of the rectangle ABCD because the two triangles have the same base. Alternatively one can say that the area of the rectangle is twice the area of triangle $\triangle B'DC$ because they have the same base. (Euclid, Book 1, Proposition 41)

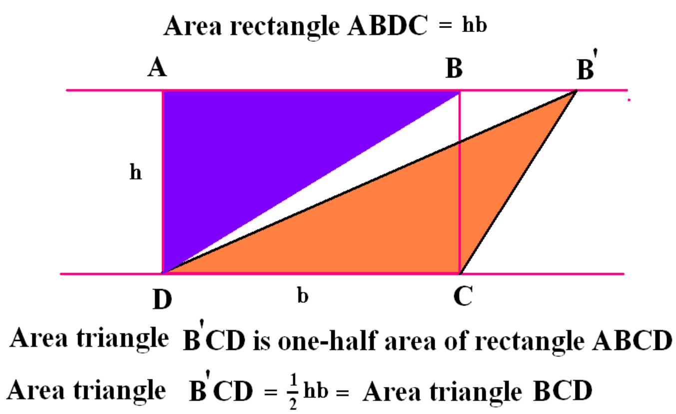


Figure 4-9. **Area of triangle $\triangle B'DC$ and triangle $\triangle BDC$ are equal as B' moves.**

Examine the figure 4-10 which starts with a right triangle $\triangle ABC$ with side a opposite angle A, side b opposite angle B with these sides forming a right angle. The hypotenuse c is opposite the $90°$ angle at vertex C. A square with side a is placed upon side a of the triangle. Similarly, squares of sides b and c are placed upon the respective sides b and c of the triangle, as illustrated in the figure 4-10.

Draw a straight line from point F to point A and make the following observations.

(a) Side $\overline{CA}$ is parallel to side $\overline{FB}$

(b) The line $\overline{FA}$ is like the line $\overline{B'D}$ of figure 4-9.

(c) One can slide point A to point C and conclude that the triangle $\triangle FBA$ has an area of $\frac{1}{2}a^2$ because in sliding point A all triangles have the same base $\overline{FB}$

Do you recognize figure 4-9 within the figure 4-10 ?

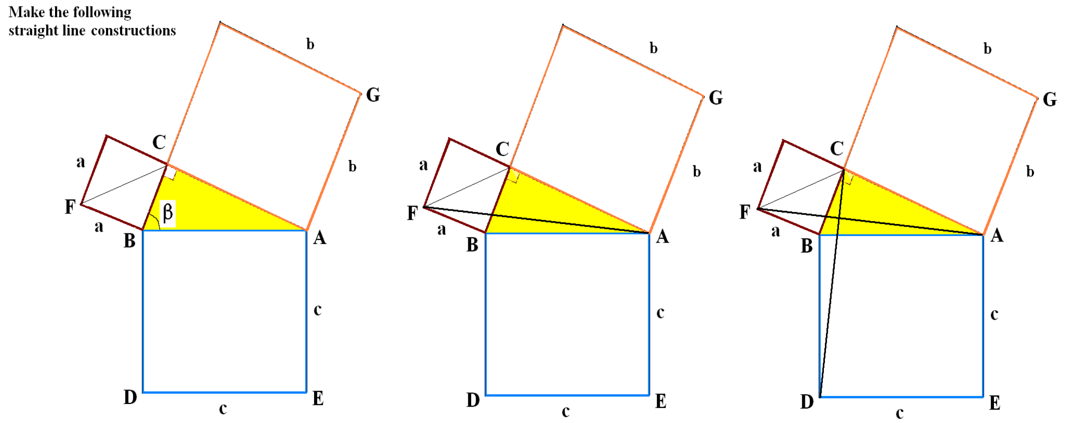


Figure 4-10. **Construction to begin Euclid's proof of Pythagorean theorem**

Draw a straight line from point C to point D and note that triangle $\triangle FBA$ is congruent to triangle $\triangle CBD$ because of SAS (Side-Angle-Side). That is,

(a) $\overline{FB} \cong \overline{BC}$ Both line segments are of length a

(b) $\angle FBA \cong \angle CBD$ Both angles are equal to $(90 + \beta)$ degrees.

(c) $\overline{BA} \cong \overline{BD}$ Both line segments are of length c

Consequently, one can write $\triangle FBA \cong \triangle CBD$ because of SAS.

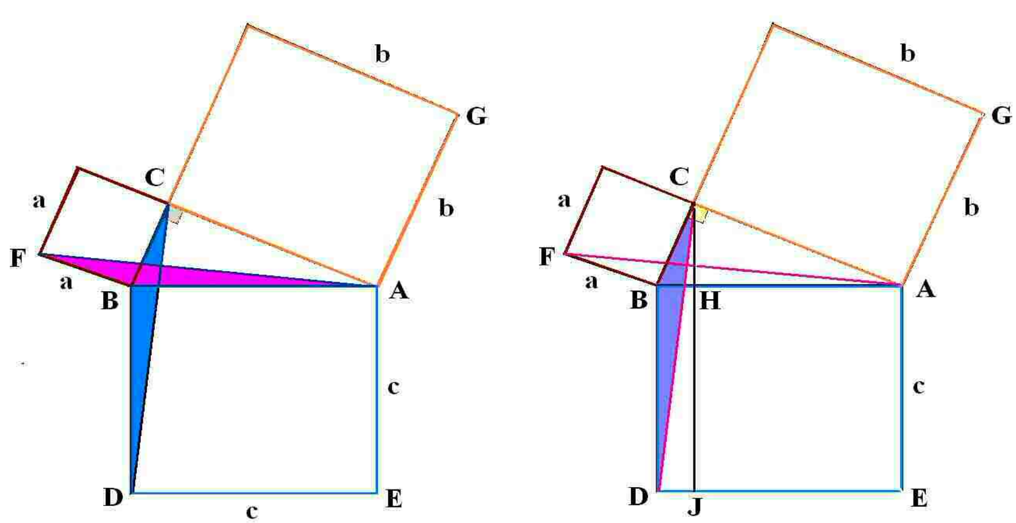


Figure 4-11. **Another construction to begin Euclid's proof of Pythagorean theorem**

The figure 4-11 illustrates the two congruent triangles $\triangle FBA$ and $\triangle CBD$, where the area of each triangle is $\frac{1}{2}a^2$. Draw the straight line $\overline{CJ}$ which is perpendicular to the line segment $\overline{DE}$ and intersects the line segment $\overline{BA}$ at point H. This construction gives the parallel line segments $\overline{CHJ}$ and $\overline{BD}$. By moving the point C to the point H, one can observe that half the area of the rectangle BHJD is given by $\frac{1}{2}a^2$ and so the area of the rectangle BHJD is a^2. Go back to figures 4-8 and 4-9 and observe how one can slide a point along parallel lines to see how an area of the triangles remain constant. Next examine the figures 4-10 and 4-11 to see how this sliding along parallel lines is employed to show two figures have the same area because of a common base associated with a set of parallel lines.

The final steps to complete the proof of the Pythagorean theorem is to show the area b^2 is the same area as the rectangle HAEJ of figure 4-11. Toward this end, draw the straight line $\overline{CE}$ as illustrated in the figure 4-12 and then draw the straight line $\overline{BG}$ followed by the previous straight line $\overline{CHJ}$ which is perpendicular to the line segment $\overline{DE}$. In these figures observe that the line segment $\overline{GA}$ is parallel to the line segment $\overline{CB}$, written $\overline{GA} \parallel \overline{CB}$. Also note that the line segment $\overline{AE}$ is parallel to the line segment $\overline{GA}$, written $\overline{AE} \parallel \overline{CJ}$. Examine these constructions and experiment with the sliding of point B to C one finds that the area of triangle $\triangle BGA$ is $\frac{1}{2}b^2$ and by sliding the point C to point H, one can show the area of triangle $\triangle CAE$ is half the area of rectangle HAEJ. Also observe that $\triangle BGA$ is congruent to $\triangle ECA$, because of SAS, with the angle $(90 + \alpha)$ associated with each triangle.

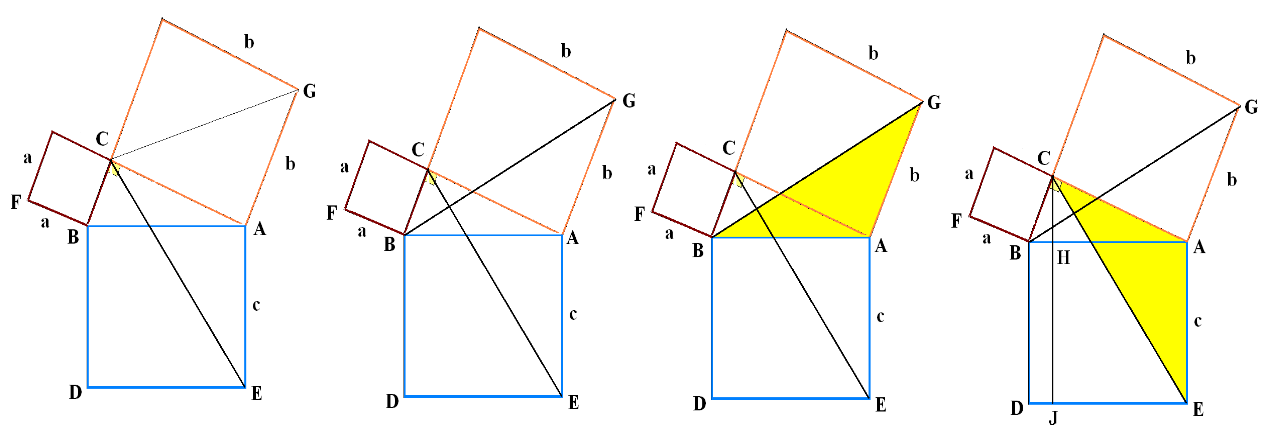


Figure 4-12. **Construction of other straight lines**

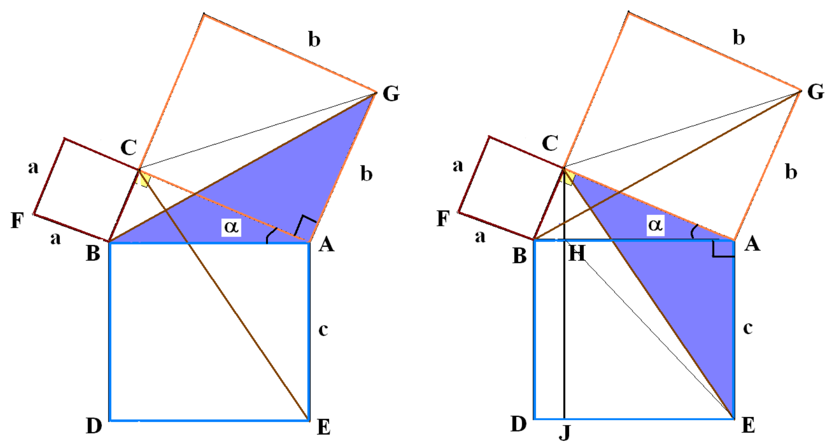


Figure 4-13. **Congruent triangles**

These observations are illustrated in the figure 4-13.

We can formalize Euclid's proof of the Pythagorean theorem by constructing step-by-step statements followed by our reasoning for making the statements. This is known as a (Statements | Reasons) table for a more formal presentation of the above observations.

Euclid's Proof of Pythagorean theorem		
	Statement	Reason
1.	figure 4-9 $\overline{CA} \parallel \overline{FB}$	Square has $\parallel$ opposite sides
2.	Area $\triangle FBA = \frac{1}{2}a^2 =$ Area $\triangle FBC$	Common base $\overline{FB}$
3.	$\overline{FB} \cong \overline{BC} = a$	Equal side of square
4.	$\angle FBA \cong \angle DBC$	Each angel $(90° + \beta)$
5.	$\overline{BA} \cong \overline{BD}$	Sides of square equal
6.	$\triangle FBA \cong \triangle CBD$	Side-Angle-Side (SAS)
7.	Area $\triangle FBA =$Area $\triangle CBD = \frac{1}{2}a^2$	Congruent triangles have same area
8.	figure 4-10 $\overline{BD} \parallel \overline{CHJ}$	By construction
9.	Area $\triangle CBD =$ Area $\triangle BHD$	Common base $\overline{BD}$
10.	Area $\triangle BHD = \frac{1}{2}$ Area rectangle BHJD	Diagonal HD halves rectangle
11.	Area BHJD$= a^2$	2 times area $\triangle BHD$
12.	figure 5-12 $\overline{CB} \parallel \overline{GA}$	Square has $\parallel$ opposite sides
13.	Area $\triangle ABG = \frac{1}{2}b^2 =$ Area $\triangle ACG$	Common base $\overline{AG}$
14.	$\overline{BA} \cong \overline{AE} = c$	Equal side of square
15.	$\angle EAC \cong \angle BAG$	Each angle $(90° + \alpha)$
16.	$\overline{AC} \cong \overline{AG} = b$	Sides of square equal
17.	$\triangle CAE \cong \triangle GAB$	SAS
18.	Area $\triangle GBA =$ Area $\triangle CAE = \frac{1}{2}b^2$	Congruent triangles have same area
19.	$\overline{CHJ} \parallel \overline{AE}$	Square has parallel opposite sides
20.	Area $\triangle CAE = \frac{1}{2}(Area$ HAEJ$)$ =Area $\triangle HAE$	Common base $\overline{AE}$
21.	Area rectangle HAEJ $= b^2$	2 times area $\triangle HAE$
22.	$a^2 + b^2 = c^2$	whole equal to sum of its parts
	rectangle BHJD+ rectangle HAEJ= square BAED	

Still another proof of the Pythagorean theorem

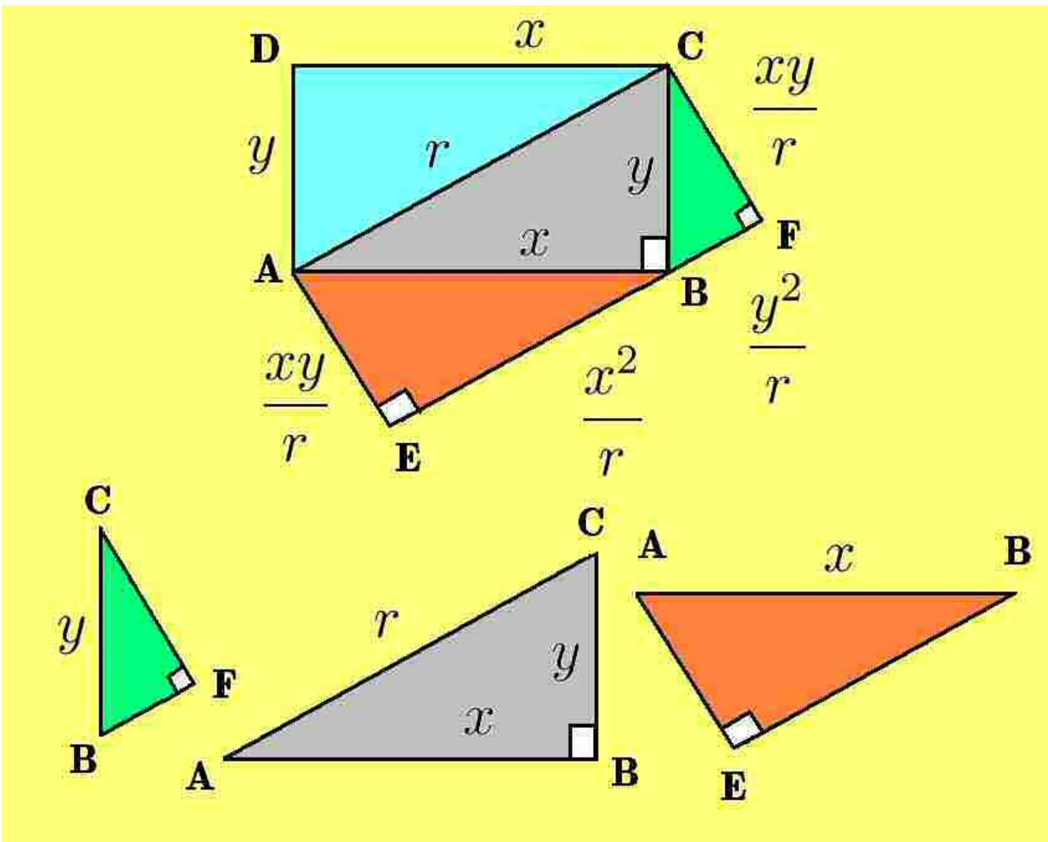


Figure 4-14. Constructions to right triangle $\triangle ABC$

Given the right triangle ABC one can make the following constructions.

(a) The line $\overline{EBF}$ which is parallel to the side $\overline{AC}$

(b) The perpendicular line $\overline{CF} \perp \overline{BF}$

(c) The perpendicular line $\overline{AE} \perp \overline{EB}$

(d) The line $\overline{DC}$ parallel to the line $\overline{AB}$

(e) The line $\overline{DA}$ parallel to the line $\overline{CB}$

This creates the three similar right triangles illustrated in the figure 4-14. Comparing the sides of the triangle $\triangle CBF$ with the triangle $\triangle ABC$ one finds

$$\frac{\overline{CF}}{x} = \frac{y}{r} = \frac{\overline{BF}}{y}$$

which implies that

$$\overline{CF} = \frac{xy}{r} \quad \text{and} \quad \overline{BF} = \frac{y^2}{r}$$

Comparing the sides of triangle $\triangle AEB$ with the triangle $\triangle ABC$ there results

$$\frac{\overline{BE}}{x} = \frac{x}{r} \quad \text{which gives the result} \quad \overline{BE} = \frac{x^2}{r}$$

The area of the figure constructed in figure 4-14 can be viewed in two ways.

$$\text{Area} = \underbrace{\frac{xy}{r}\left(\frac{x^2}{r} + \frac{y^2}{r}\right)}_{\text{area of rectangle ACFE}} + \underbrace{\frac{1}{2}xy}_{\text{area triangle ADC}}$$

$$\text{Area} = \underbrace{xy}_{\text{area rectangle ADCB}} + \underbrace{\frac{1}{2}\frac{x^2}{r}\frac{xy}{r}}_{\text{area triangle AEB}} + \underbrace{\frac{1}{2}\frac{y^2}{r}\frac{xy}{r}}_{\text{area triangle CBF}}$$

Now things equal to the same thing are equal to each other so one can equate the two area representations to obtain the algebraic equation

$$\frac{x^3y}{r^2} + \frac{xy^3}{r^2} + \frac{1}{2}xy = xy + \frac{1}{2}\frac{x^3y}{r^2} + \frac{1}{2}\frac{xy^3}{r^2}$$

$$\text{simplify to find} \quad \frac{1}{2}\frac{x^3y}{r^2} + \frac{1}{2}\frac{xy^3}{r^2} = \frac{1}{2}xy$$

$$\text{or} \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad \Longrightarrow \quad x^2 + y^2 = r^2$$

which states that the square of the hypotenuse must equal the sum of the squares of the other two sides.

Converse of the Pythagorean theorem

From Euclid's Elements, book 1, Proposition 48, one finds that if you are given a triangle with sides of lengths a,b,c such that $a^2 + b^2 = c^2$, then the angle opposite the side of length c must be a right angle. Here one can write

Right triangle with sides of length a,b,c $\Longrightarrow a^2 + b^2 = c^2$, with its converse

$a^2 + b^2 = c^2 \Longrightarrow$ Right triangle with angle C opposite side c $= 90^\circ$ or $\frac{\pi}{2}$ radians

The proof of the converse Pythagorean theorem is as follows.

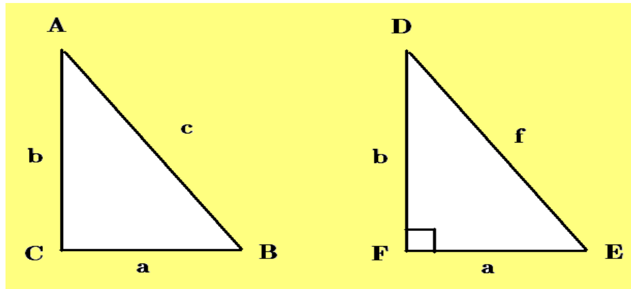


Figure 4-15. Congruent triangles

Construct a **right-angled triangle** $\triangle DEF$ with legs $\overline{\text{DF}} = b$,$\overline{\text{FE}} = a$ and side $\overline{\text{DE}} = f$ as illustrated in the figure 4-15. We wish to show that if $a^2 + b^2 = c^2$, then angle C opposite side c is a right angle.

By construction the triangle $\triangle DEF$ is a right angled triangle and so by the Pythagorean theorem the side opposite the $90°$ angle when squared must equal the sum of the squares of the other two sides or

$$a^2 + b^2 = f^2 \tag{4.13}$$

By hypothesis

$$a^2 + b^2 = c^2 \tag{4.14}$$

Things equal to the same thing are equal to one another so one can conclude

$$f^2 = c^2 \quad \Rightarrow \quad |\, c \,|=|\, f \,| \quad \Rightarrow \quad f = c$$

because both c and f represent positive lengths. Therefore, triangle $\triangle ABC$ is congruent to triangle $\triangle DEF$ ($\triangle ABC \cong \triangle DEF$) because of SSS (Side-Side-Side). In congruent triangles, corresponding angles are equal so that $\angle C = \angle F$ are right angles.

Other triangles

In general, given an acute, right or obtuse triangles with sides a,b,c as illustrated in the figure 4-16, one can show the following inequalities result

obtuse triangle $a^2 + b^2 < c^2$, right triangle $a^2 + b^2 = c^2$, acute triangle $a^2 + b^2 > c^2$

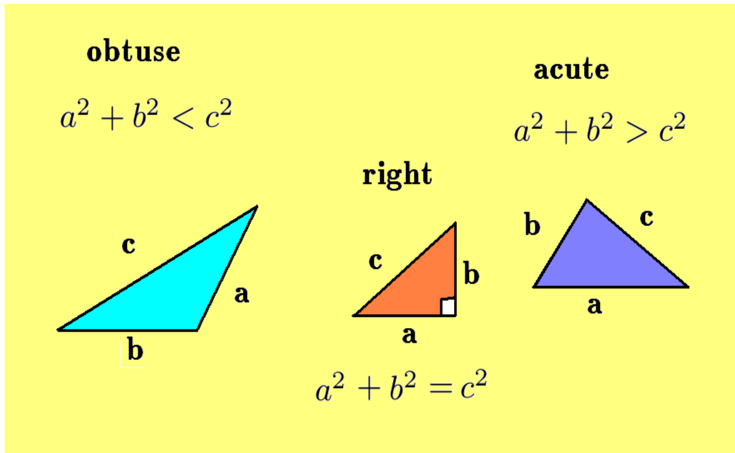


Figure 4-16. Inequalities for acute and obtuse triangles.

Projections

Given two line segments $\overline{\text{AB}}$ and $\overline{\text{CD}}$ as illustrated in the figure 4-17.

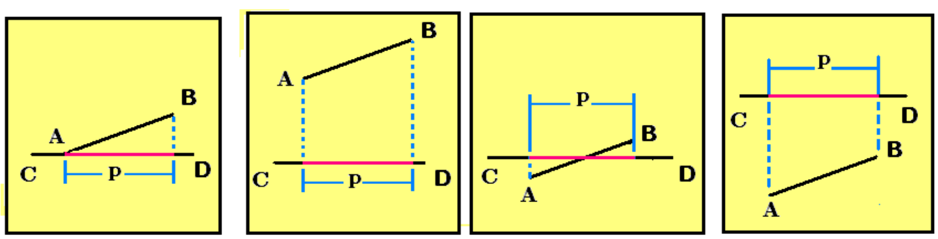


Figure 4-17. Projections of $\overline{\text{AB}}$ onto $\overline{\text{CD}}$

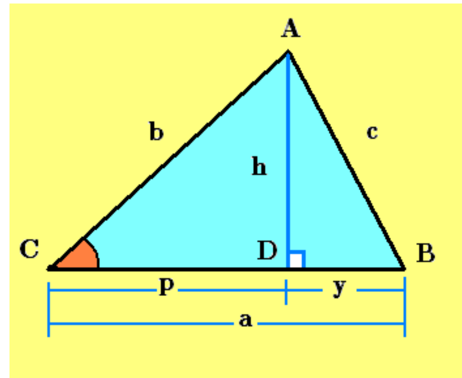
The projection p of $\overline{\text{AB}}$ onto the line $\overline{\text{CD}}$ is obtained by dropping perpendicular lines from the endpoints of the $\overline{\text{AB}}$ line segment onto the $\overline{\text{CD}}$ line as illustrated above. The distance p is called the projection of the line segment $\overline{\text{AB}}$ onto the line $\overline{\text{CD}}$.

Projection and acute triangle

Consider the triangle $\triangle ABC$ with acute angle at vertex C and where p is the projection of side $\overline{\text{AC}}$ onto the side $\overline{\text{CB}}$. Show that the square of the side opposite the acute angle is given by

$$c^2 = a^2 + b^2 - 2ap \tag{4.15}$$

(See Euclid, book 2, proposition, 13)



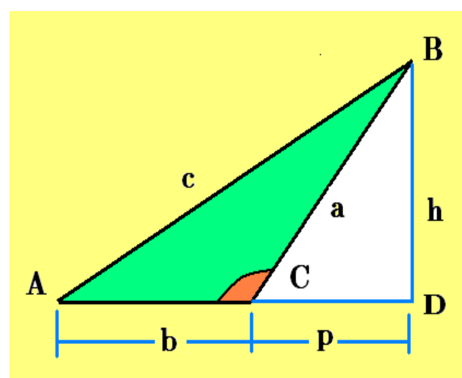
Solution:
Drop a perpendicular line from vertex A to the side $\overline{\text{CB}}$, intersecting at D, and call this altitude h. Let p denote the projection of side $\overline{\text{AC}}$ onto side $\overline{\text{CB}}$.
We employ the Pythagorean theorem to show in $\triangle CDA$ $b^2 = h^2 + p^2$ and in $\triangle ADB$ $h^2 + y^2 = h^2 + (a-p)^2 = c^2$. One can then write

$$c^2 = (b^2 - p^2) + (a-p)^2$$
$$c^2 = b^2 - p^2 + a^2 - 2ap + p^2$$
$$c^2 = a^2 + b^2 - 2ap$$

Projection and obtuse triangle

Consider the triangle $\triangle ABC$ with the vertex C as the obtuse angle and where p is the projection of side $\overline{\text{BC}}$ onto the extended side $\overline{\text{AC}}$. Show that the square of the side opposite the obtuse angle satisfies

$$c^2 = a^2 + b^2 + 2bp \tag{4.16}$$



Solution:
Drop a perpendicular from vertex B to the extended side $\overline{\text{AC}}$ and label the intersection point D and altitude h.

The Pythagorean theorem requires that in $\triangle ABD$ $c^2 = (b+p)^2 + h^2$ and in $\triangle BDC$ $h^2 + p^2 = a^2$. Therefore,

$$c^2 = b^2 + 2bp + p^2 + a^2 - p^2$$
$$c^2 = a^2 + b^2 + 2bp$$

(See Euclid, book 2, proposition, 12)

Pythagorean triples

Three positive integers (a,b,c) such that $a^2+b^2=c^2$ are called a Pythagorean triple. For example, $(3,4,5)$ is a Pythagorean triple because $3^2+4^2=5^2$. Any right triangle having a Pythagorean triple (a,b,c) for its sides is called a Pythagorean triangle.

Euclid's formula Euclid's formula for producing Pythagorean triples begins by selecting two integers p and q which are coprime[3] with $p>q$. If there are three positive integers (a,b,c) such that $a^2+b^2=c^2$, then one can write

$$b^2=c^2-a^2=(c-a)(c+a) \quad \text{or} \quad \frac{b}{c-a}=\frac{c+a}{b}$$

must be a rational number. In this case one can express the rational number as

$$\frac{c}{b}+\frac{a}{b}=\frac{p}{q} \qquad \text{and} \qquad \frac{c}{b}-\frac{a}{b}=\frac{q}{p} \tag{4.17}$$

where p and q are coprime integers. Solve the equations (4.17) for the ratios $\frac{c}{b}$ and $\frac{a}{b}$ as follows. Using addition of the equations (4.17) verify that

$$\frac{c}{b}=\frac{1}{2}\left(\frac{p}{q}+\frac{q}{p}\right)=\frac{1}{2}\left(\frac{p^2+q^2}{qp}\right) \tag{4.18}$$

Using subtraction of the equations (4.17) verify that

$$\frac{a}{b}=\frac{1}{2}\left(\frac{p}{q}-\frac{q}{p}\right)=\frac{1}{2}\left(\frac{p^2-q^2}{qp}\right) \tag{4.19}$$

By equating both the numerators and denominators in the equations (4.18) and (4.19) there results Euclid's formula for Pythagorean triples

$$\begin{aligned} a=&p^2-q^2 \\ b=&2pq \qquad \text{p and q integers with } p>q \\ c=&p^2+q^2 \end{aligned} \tag{4.20}$$

Observe that the results given by equation (4.20) satisfies the Pythagorean theorem because

$$\begin{aligned} a^2+b^2=&c^2 \\ (p^2-q^2)^2+(2pq)^2=&(p^2+q^2)^2 \end{aligned}$$

becomes an identity.

---

[3] Two integers p and q are called coprime if the only integer that divides both p and q is the number 1. One can also say that the greatest common divisor of p and q is 1. The terminology relatively prime, mutually prime and coprime all mean the same thing.

Some well known Pythagorean triples

Pythagorean triples (a,b,c)		
p	q	(a,b,c)
2	1	(3,4,5)
3	1	(8,6,10)=2(4,3,5)
3	2	(5,12,13)
4	1	(15,8,17)
4	2	(12,16,20)=4(3,4,5)
4	3	(7,24,25)
5	1	(24,10,26)=2(12,5,13)
5	2	(21,20,29)
5	3	(16,30,34)=2(8,15,17)
5	4	(9,40,41)

Make note of the following facts concerning Pythagorean triples.

1) If (a,b,c) is a Pythagorean triple, then (ka,kb,kc) is also a Pythagorean triple for all positive integers k.

2) If (a,b,c) forms a Pythagorean triple, then (b,a,c) is also a Pythagorean triple.

3) Pythagorean triples of the form (ka,kb,kc) are called non primitive Pythagorean triples since they are multiples of (a,b,c).

The Euclid formulas given by the equations (4.20) does not generate all of the Pythagorean triples. For example, the Pythagorean triple $(9,12,15)$ or $(12,9,15)$ cannot be generated by using the equations (4.20).

A modified form of the equations (4.20) given by

$$\begin{aligned} a =&m(p^2-q^2) \\ b =&m(2pq) \\ c =&m(p^2+q^2) \end{aligned} \tag{4.21}$$

where m,p,q are positive integers with

$$\begin{aligned} &(i) \quad p>q \\ &(ii) \quad p-q \text{ is an odd number} \\ &(iii) \quad p \text{ and } q \text{ are coprime} \end{aligned}$$

will generate all the Pythagorean triples

Example 4-2. Way back in the year 1643 the French mathematician Fermat[4] asked the following question: Find a Pythagorean triangle with sides a, b and hypotenuse c such that c and $(a+b)$ were perfect squares. The smallest solution turns out to be

$$a = 4,565,486,027,761 \quad b = 1,061,652,293,520 \quad c = 4,687,298,610,289$$

The next larger solution is so large it cannot be written in any book. Note that there were no computers during Fermats lifetime.

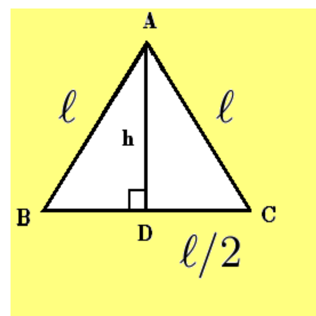
■

Example 4-3.

Find the area and perimeter of an equilateral triangle.

Solution:

Let ℓ denote the length of any side of the equilateral triangle $\triangle ABC$ illustrated. Drop a perpendicular line from the vertex A intersecting the base of the triangle at point D. We know $\triangle ADB \cong \triangle ADC$ with both triangles being right triangles. Consequently, one can show $\overline{\text{BD}} = \overline{\text{DC}} = \ell/2$. Use the Pythagorean theorem on the sides of triangle $\triangle ADC$ to show



$$h^2 + \left(\frac{\ell}{2}\right)^2 = \ell^2 \quad \Rightarrow \quad h = \frac{\sqrt{3}}{2}\ell$$

The area of the equilateral triangle is then $\frac{1}{2}(base)(height)$ or

$$\text{Area} = \frac{1}{2}(\ell)(h) = \frac{1}{2}(\ell)(\frac{\sqrt{3}}{2}\ell) = \frac{\sqrt{3}}{4}\ell^2$$

The perimeter P associated with the equilateral triangle is given by $P = 3\ell$.

■

Example 4-4.

Find the area and perimeter of an isosceles triangle.

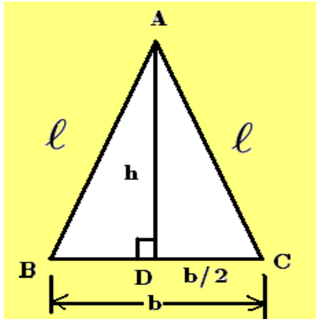
Solution:

Let ℓ denote the length of the equal sides and let b denote the base of the isosceles

[4] Pierre de Fermat (1601-1665)

triangle $\triangle ABC$ illustrated. As in the previous example, drop a perpendicular from vertex A to the base $\overline{BC}$ and show $\triangle ADB \cong \triangle ADC$ with $\overline{BD} = \overline{DC} = b/2$. Apply the Pythagorean theorem to the sides of triangle $\triangle ADC$ and show

$$h^2 + \frac{b^2}{4} = \ell^2 \quad \Rightarrow \quad h = \sqrt{\ell^2 - \frac{b^2}{4}}$$



The area of the isosceles triangle is then $\frac{1}{2}(base)(height)$ or

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}b\sqrt{\ell^2 - \frac{b^2}{4}}$$

The perimeter P is given by $P = 2\ell + b$. ■

Comment on RHS (Right angle-Hypotenuse-Side proposition)

Note that in a right triangle with sides a, b and hypotenuse c one must have $a^2 + b^2 = c^2$. If one knows the hypotenuse c and one side, say a, then the Pythagorean theorem gives you the other side $b = \sqrt{c^2 - a^2}$. Consequently, the RHS proposition for right triangles is really a SSS proposition.

Distance formula

One can use the Pythagorean theorem to find the distance between two points in Cartesian coordinates. Given the two points (x_1, y_1) and (x_2, y_2) one can plot these points in Cartesian coordinates and then using a straight edge they can be connected with a line segment. Let d denote the distance of this line segment which represents the distance between the two given points.

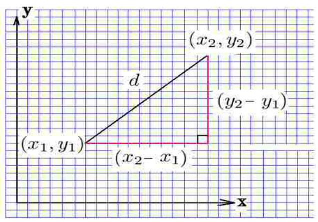


Figure 4-18. Distance d between two given points.

To calculate the distance d first calculate the change in the x-values in moving from (x_1, y_1) to (x_2, y_2). This change in x is $(x_2 - x_1)$ and represents one leg of the right triangle illustrated in figure 4-18. Next calculate the change in the y-values in moving from (x_1, y_1) to (x_2, y_2). This change in y is $(y_2 - y_1)$ and represents the other leg of the right triangle illustrated in the figure 4-18. One can now use the Pythagorean theorem and write

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2 \quad \text{or} \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{4.22}$$

One takes the positive square root in equation (4.22) because the distance d is always considered to be a positive distance.

Equation of a circle

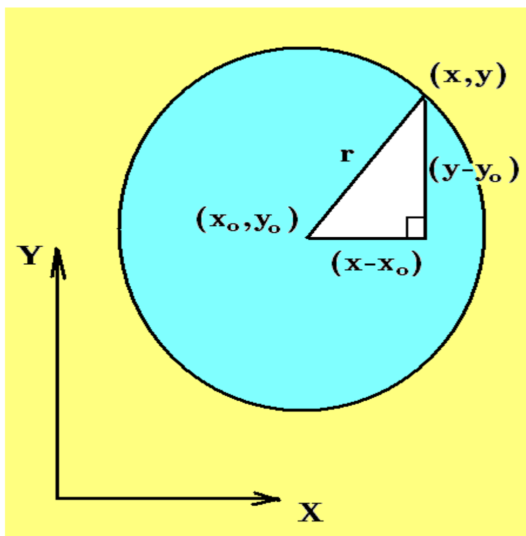


Figure 4-19.

Equation for circle centered at (x_0, y_0) **is** $(x - x_0)^2 + (y - y_0)^2 = r^2$

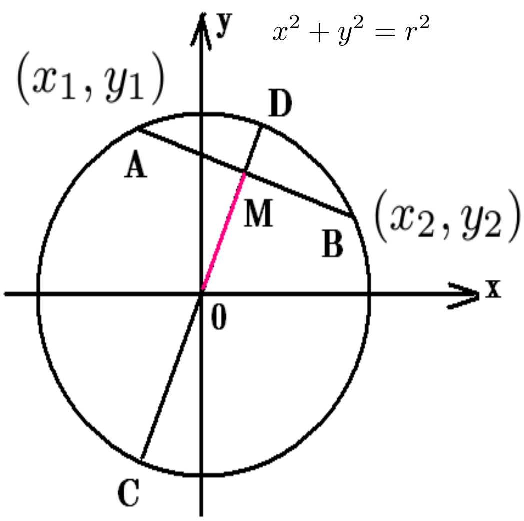
The figure 4-19 can be used to define the equation of a circle. If (x, y) is a variable point restricted to remain a fixed distance r from the point (x_0, y_0), then by the distance formula the coordinates for points on a circle must satisfy

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{4.23}$$

Here (x_0, y_0) is the center of the circle, r is the radius of the circle and (x, y) is a variable point on the circle. The top half of the circle is $y = y_0 + \sqrt{r^2 - (x - x_0)^2}$ and the bottom half of the circle is $y = y_0 - \sqrt{r^2 - (x - x_0)^2}$.

Example 4-5. Show that if A and B are two points on a circle such that the line segment $\overline{\text{AB}}$ does not pass through the center of the circle, then a line $\overline{\text{CD}}$ through the center of the circle which passes through the midpoint M of the line segment $\overline{\text{AB}}$ will be perpendicular to the line segment $\overline{\text{AB}}$. (Euclid, Book 3, proposition 3)

Solution



Let point A have the coordinates (x_1, y_1) and let B have the coordinates (x_2, y_2). These points must lie on the circle, hence these coordinates satisfy the equation of the circle, so that x_1, y_1, x_2, y_2 are restricted so that

$$x_1^2 + y_1^2 = r^2 \qquad \text{and} \qquad x_2^2 + y_2^2 = r^2 \tag{4.24}$$

The line segment $\overline{\text{AB}}$ has the slope m_1 given by

$$m_1 = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{4.25}$$

The point-slope equation for the extended line is given by

$$y - y_1 = m_1(x - x_1) \tag{4.26}$$

The requirement that the line segment $\overline{\text{AB}}$ not pass through the origin is equivalent to the requirement that the point $(0,0)$ is not on the line given by equation (4.26). Hence, it is necessary to assume that $y_1 \neq m_1 x_1$. The midpoint M of the line segment $\overline{\text{AB}}$ is

$$M = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right) \tag{4.27}$$

which represents the average of the x-values and y-values. One can construct the line segment $\overline{0\text{M}}$ and then extend it to intersect the circle at the points C and D. The slope m_2 of the line segment $\overline{0\text{M}}$ is

$$m_2 = \frac{\text{change is } y}{\text{change in } x} = \frac{\frac{1}{2}(y_1 + y_2)}{\frac{1}{2}(x_1 + x_2)} = \frac{y_1 + y_2}{x_1 + x_2} \tag{4.28}$$

The product of these slopes is

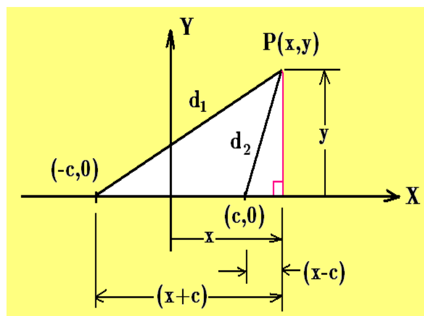
$$m_1m_2 = \frac{(y_2-y_1)}{(x_2-x_1)} \cdot \frac{(y_1+y_2)}{(x_1+x_2)} = \frac{y_1^2-y_2^2}{x_1^2-x_2^2} = \frac{r^2-x_1^2-(r^2-x_2^2)}{x_1^2-x_2^2} = -1 \tag{4.29}$$

where we have used the results from the equations (4.24) to demonstrate that the product of the slopes equals minus 1. This demonstrates that the line segments $\overline{\text{AB}}$ and $\overline{\text{0M}}$ are perpendicular to one another.

■

Equation of an ellipse

In Cartesian coordinates let the points $(c, 0)$ and $(-c, 0)$ denote the foci associated with an ellipse. The equation for the ellipse is defined in terms of the motion of a variable point $P(x, y)$ being a distance d_1 from one focus and a distance d_2 from the other focus and restricted such that the sum of these distances be constant or $d_1 + d_2 = a\ constant$. Let the constant term be denoted by $2a$ with $a > c > 0$, then the equation for the ellipse is obtained using algebra. Using the figure on the right, the distances d_1 and d_2 of point P from the foci are obtained by using the Pythagorean theorem and writing



$$\begin{aligned} d_1^2 =&(x+c)^2+y^2 \quad \Rightarrow \quad d_1 = \sqrt{(x+c)^2+y^2} \\ d_2^2 =&(x-c)^2+y^2 \quad \Rightarrow \quad d_2 = \sqrt{(x-c)^2+y^2} \end{aligned} \tag{4.30}$$

The definition describing an ellipse requires $d_1 + d_2 = 2a$ or

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a \tag{4.31}$$

Express equation (4.31) in the form

$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2} \tag{4.32}$$

and then square both sides to show

$$x^2+2xc+c^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2xc + c^2 + y^2$$

which simplifies to the form

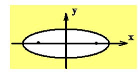
$$a\sqrt{(x-c)^2+y^2} = a^2 - xc \tag{4.33}$$

Square both sides of equation (4.33) and simplify the results to the form

$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2) \tag{4.34}$$

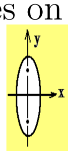
Make the substitution $b^2 = a^2 - c^2$ and express the result in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b, \quad b^2 = a^2 - c^2,$$



This is called the standard form for the equation of an ellipse. Here a is called the semi-major axis of the ellipse and $b < a$ is called the semi-minor axis of the ellipse.

If the foci are on the y-axis, having coordinates $(0, c)$ and $(0, -c)$, then the roles of a and b are reversed with the requirement $d_1 + d_2 = 2b$. In this case the equation of the ellipse takes on the form



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b > a, \quad a^2 = b^2 - c^2$$

Observe that in the special case $a = b$, the equation of an ellipse reduces to the equation of a circle centered at the origin having radius b.

Thales theorem for circle

Thales of Miletus (624-547) BCE, investigated the construction of triangles within a circle. Consider the semicircle illustrated in figure 4-20 where the triangle $\triangle ABC$ has vertex angle A at one end of the circle diameter and the vertex angle B is at the other end of the circle diameter.

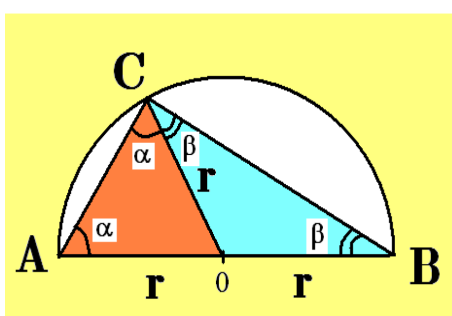


Figure 4-20. Triangle $\triangle ABC$ constructed inside semicircle.

Let 0 denote the center of the circle and let the vertex angle C be anywhere on **the circumference of the circle.** Construct the line segment $\overline{0C} = r$ which is the radius of the semicircle and label the angle $\angle AC0 = \alpha$ and the angle $\angle 0CB = \beta$. The triangle $\triangle A0C$ is now an isosceles triangle with equal sides of length r and hence the angles opposite these sides are equal. That is, $\alpha = \angle 0AC = \angle AC0$. The triangle $\triangle 0CB$ is also an isosceles triangle with equal sides of length r. The angles opposite these sides must also be equal so that $\beta = \angle 0CB = \angle 0BC$. We know that the sum of the angles of any triangle must sum to 180° so in triangle $\triangle ABC$ one can write

$$\alpha + (\alpha + \beta) + \beta = 180^\circ \quad \text{or} \quad 2(\alpha + \beta) = 180^\circ \quad \text{or} \quad \alpha + \beta = 90^\circ \tag{4.35}$$

Thales theorem states that whenever two vertices A,B of a triangle lie at the ends of a circle diameter and the third vertex C lies on the circumference of the circle, then the vertex angle C must be a right angle.

Consider the triangle $\triangle ABC$ illustrated in the figure 4-21 and drop a perpendicular line from vertex C. Use the similar right triangles $\triangle ADC \sim \triangle CDB$ and show the sides are proportional giving

$$\frac{h}{d-a} = \frac{a}{h} \qquad \text{or} \qquad h^2 = a(d-a) \tag{4.36}$$

One can show the triangle in figure 4-21 is a right triangle by using the fact that if two lines are perpendicular, then the product of their slopes is minus 1.

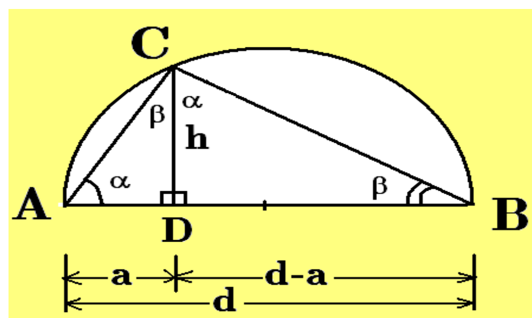


Figure 4-21.
Use slopes to verify intersection

$$\text{The slope of line segment } \overline{AC} \text{ is } m_{AC} = \frac{\text{change in y}}{\text{change in x}} = \frac{h}{a}$$

$$\text{and the slope of the line segment } \overline{CB} \text{ is } m_{CB} = \frac{\text{change in y}}{\text{change in x}} = \frac{-h}{d-a}$$

Using the result from equation (4.36) note that the product of these slopes produces

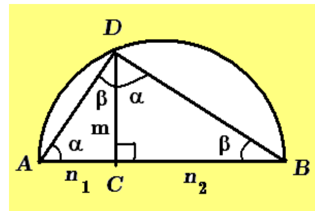
$$m_{AC} \cdot m_{CB} = \frac{-h^2}{a(d-a)} = \frac{-h^2}{h^2} = -1 \tag{4.37}$$

which demonstrates the line segment $\overline{AC}$ is perpendicular to the line segment $\overline{CB}$, verifying our earlier result concerning perpendicular lines and the fact that angle C is 90° or $\frac{\pi}{2}$ radians.

Example 4-6. (Mean proportion) Given two numbers n_1 and n_2. The mean proportion of n_1 and n_2 is defined

$$m = \sqrt{n_1 n_2} \text{ or } m^2 = n_1 n_2 \text{ or } \frac{m}{n_2} = \frac{n_1}{m} \text{ or } \frac{m}{n_1} = \frac{n_2}{m}$$

This can be illustrated graphically by constructing a line of length $\overline{\text{AB}} = n_1 + n_2$ with the segments n_1 and n_2 meeting at point C. One can now calculate the midpoint of $\overline{\text{AB}}$ and then construct a semicircle as illustrated. This is followed by the construction of the line segments $\overline{\text{AD}}$,$\overline{\text{DB}}$ as well as the line $\overline{\text{CD}}$ of length m which is perpendicular to $\overline{\text{AB}}$. Here m represents the mean proportion of n_1 and n_2.

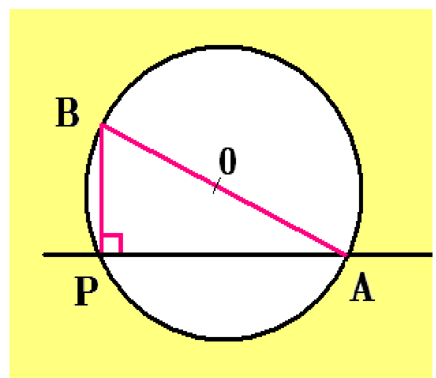


To prove this assertion examine the angles in the triangles $\triangle ACD$ and $\triangle DCB$ and show $\triangle ACD \sim \triangle DCB$ with proportional sides giving $\dfrac{m}{n_2} = \dfrac{n_1}{m}$.

■

Example 4-7.

One can make use of Thales theorem for the circle and devise a method for constructing a perpendicular at a specified point on a line. Let P denote a given point on a line. Select a point 0 not on the line and then use a drawing compass with radius $\overline{\text{0P}}$ to draw a circle which intersects the given line at the points P and A. One can then construct the diameter of the circle by drawing a line through point A and the center point 0 to intersect the circle at point B. The line $\overline{\text{PB}}$ will then be perpendicular to the given line because of Thales theorem for the circle.

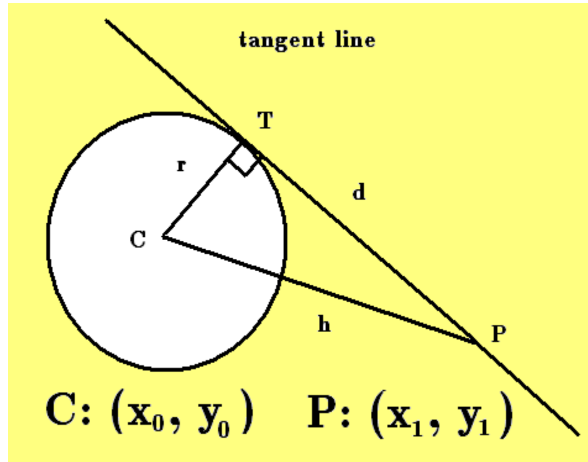


■

Example 4-8. Let $P:\ (x_1, y_1)$ denote any point outside a given circle with center $C:\ (x_0, y_0)$ and radius r. Construct the tangent line to the circle which passes through the point P and label the point of tangency T. Find the distance of the line segment $\overline{\text{PT}} = d$.

Solution

The triangle $\triangle CTP$ is a right triangle with sides r, the given radius, h the hypotenuse distance $\overline{\text{PC}}$ and the distance $\overline{\text{PT}} = d$. By the Pythagorean theorem $d^2 + r^2 = h^2$ The distance $\overline{\text{PC}} = h$ is determined from the distance formula



$$h^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$$

which is also a result of using the Pythagorean theorem.

Consequently,

$$d^2 = h^2 - r^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 - r^2 \tag{4.38}$$

Take the square root of both sides to solve for d.

■

Heron's formula for area of a triangle

The Heron's formula for the area of a triangle is obtained **by knowing only the sides of the triangle.** This area formula was developed by Heron of Alexandria (10-70)CE and uses the Pythagorean theorem to aid in developing the formula. It is derived as follows.

Given a general triangle $\triangle ABC$ as illustrated in the figure 4-23 one can construct a perpendicular line from the vertex C to the base c of the triangle. This construction forms two right triangles and divides the side c into two parts labeled c_1 and c_2 such that $c_1 + c_2 = c$. The Pythagorean theorem requires

$$\begin{aligned} b^2 =& c_1^2 + h^2 \quad \text{and} \\ a^2 =& h^2 + c_2^2 \quad \text{where } c_2 = c - c_1 \end{aligned} \tag{4.39}$$

Subtract the upper equation from the lower equation (4.39) and show

$$a^2 - b^2 = c_2^2 - c_1^2 = (c - c_1)^2 - c_1^2 = c^2 - 2cc_1 \tag{4.40}$$

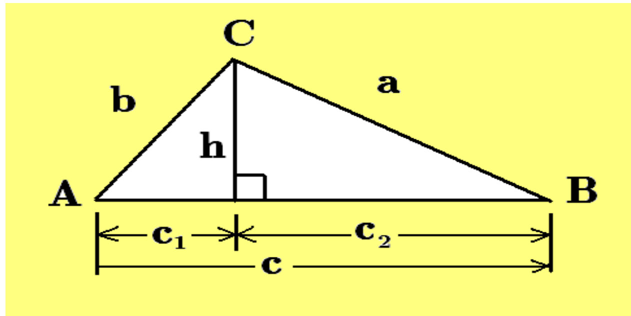


Figure 4-23. Constructions to triangle $\triangle ABC$

Solve equation (4.40) for c_1 and show

$$c_1 = \frac{c^2 + b^2 - a^2}{2c} \tag{4.41}$$

One can now do some algebra on the Pythagorean equation $h^2 = b^2 - c_1^2$ from equation (4.39). Substitute for c_1 from equation (4.41) and show

$$\begin{aligned} h^2 =& b^2 - c_1^2 = b^2 - \left(\frac{c^2 + b^2 - a^2}{2c}\right)^2 \\ h^2 =& \frac{4b^2c^2 - (c^2 + b^2 - a^2)^2}{4c^2} \end{aligned} \tag{4.42}$$

We will now use the algebraic equation $(A^2 - B^2) = (A - B)(A + B)$ for different selections for the quantities A and B. Observe that with $A = 2bc$ and $B = c^2 + b^2 - a^2$ the equation (4.42) can be expressed

$$h^2 = \frac{(2bc + a^2 - b^2 - c^2)(2bc - a^2 + b^2 + c^2)}{4c^2} \tag{4.43}$$

The terms in equation (4.43) can be rearranged into the form

$$\begin{aligned} 4c^2h^2 &= \left(a^2 - [b^2 - 2bc + c^2]\right)\left([b^2 + 2bc + c^2] - a^2\right) \\ 4c^2h^2 &= \left(a^2 - [b - c]^2\right)\left([b + c]^2 - a^2\right) \end{aligned} \tag{4.44}$$

Note the terms in equation (4.44) have the form $A^2 - B^2$ for different values of A and B. Therefore, equation (4.44) can be factored into the form

$$4c^2h^2 = (a - (b - c))\,(a + (b + c))\,((b + c) - a)\,((b + c) + a) \tag{4.45}$$

The equation (4.45) can be expressed in a more symmetric form by defining the quantity

$$s = \frac{a+b+c}{2} = \text{the semi-perimeter of triangle ABC} \tag{4.46}$$

One can then write

$$\begin{aligned} 2s =& a+b+c \\ 2(s-a) =& b+c-a \\ 2(s-b) =& a+c-b \\ 2(s-c) =& a+b-c \end{aligned}$$

These equations can now be substituted in the equation (4.45) to obtain

$$c^2h^2 = 4s(s-a)(s-b)(s-c) \quad \text{where} \quad s = \frac{a+b+c}{2} \tag{4.47}$$

The area $[ABC]$ of the triangle $\triangle ABC$ is half the base c times the height h or $[ABC] = \frac{1}{2}ch$. Therefore, the equation (4.47) becomes

$$\begin{aligned} 4[ABC]^2 =& c^2h^2 = 4s(s-a)(s-b)(s-c) \\ \text{or} \quad [ABC] =& \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \frac{a+b+c}{2} \end{aligned} \tag{4.48}$$

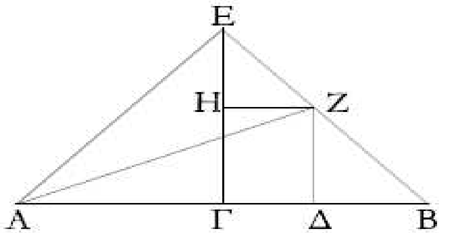
The equation (4.48) is known as Heron's formula for the area of a triangle where $s = \frac{a+b+c}{2}$ is the average perimeter of the triangle $\triangle ABC$ having the sides a,b and c. It is required that the quantity under the square root symbol must be positive in order for the area to be positive. It turns out that this is equivalent to requiring that the triangle inequality be satisfied in order for a triangle to exist.

Then and now

The following presentation is a comparison of the way things were done thousands of years ago compared to modern day methods. The presentation selected is the proposition 9, from Euclid's Elements, Book 2. A picture of Euclid's presentation of proposition 9, is given in the figure 4-24, which comes from the Richard Fitzpatrick translation, "Euclid's Elements of Geometry", found on the internet. The figure 4-24 is given as an illustration of ideas presented thousands of years ago to be used for comparison with a more modern presentation of the same ideas.

θ΄.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου.



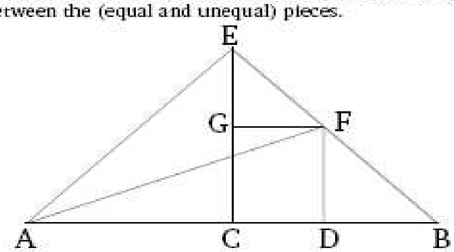
Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἤχθω γὰρ ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΕ, καὶ κείσθω ἴση ἑκατέρᾳ τῶν ΑΓ, ΓΒ, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ, καὶ διὰ μὲν τοῦ Δ τῇ ΕΓ παράλληλος ἤχθω ἡ ΔΖ, διὰ δὲ τοῦ Ζ τῇ ΑΒ ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΑΖ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΕΑΓ γωνία τῇ ὑπὸ ΑΕΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ πρὸς τῷ Γ, λοιπαὶ ἄρα αἱ ὑπὸ ΕΑΓ, ΑΕΓ μιᾷ ὀρθῇ ἴσαι εἰσίν· καί εἰσιν ἴσαι· ἡμίσεια ἄρα ὀρθῆς ἐστιν ἑκατέρα τῶν ὑπὸ ΓΕΑ, ΓΑΕ. δια τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΕΒ, ΕΒΓ ἡμίσειά ἐστιν ὀρθῆς· ὅλη ἄρα ἡ ὑπὸ ΑΕΒ ὀρθή ἐστιν. καὶ ἐπεὶ ἡ ὑπὸ ΗΕΖ ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ΕΗΖ· ἴση γάρ ἐστι τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ἡ ὑπὸ ΕΖΗ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα [ἐστὶν] ἡ ὑπὸ ΗΕΖ γωνία τῇ ὑπὸ ΕΖΗ· ὥστε καὶ πλευρὰ ἡ ΕΗ τῇ ΗΖ ἐστιν ἴση. πάλιν ἐπεὶ ἡ πρὸς τῷ Β γωνία ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ΖΔΒ· ἴση γὰρ πάλιν ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ἡ ὑπὸ ΒΖΔ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα ἡ πρὸς τῷ Β γωνία τῇ ὑπὸ ΔΖΒ· ὥστε καὶ πλευρὰ ἡ ΖΔ πλευρᾷ τῇ ΔΒ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴσον ἐστὶ καὶ τὸ ἀπὸ ΑΓ τῷ ἀπὸ ΓΕ· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΕ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ ΑΓ. τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ τετράγωνον· ὀρθὴ γὰρ ἡ ὑπὸ ΑΓΕ γωνία· τὸ ἄρα ἀπὸ τῆς ΕΑ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΕΗ τῇ ΗΖ, ἴσον καὶ τὸ ἀπὸ τῆς ΕΗ τῷ ἀπὸ τῆς ΗΖ· τὰ ἄρα ἀπὸ τῶν ΕΗ, ΗΖ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΗΖ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΗ, ΗΖ τετραγώνοις ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΖ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΕΖ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΗΖ. ἴση δὲ ἡ ΗΖ τῇ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΖ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΖ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΖ τετράγωνον· ὀρθὴ γάρ ἐστιν ἡ ὑπὸ ΑΕΖ γωνία· τὸ ἄρα ἀπὸ τῆς ΑΖ τετράγωνον διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΖ ἴσα τὰ ἀπὸ τῶν ΑΔ, ΔΖ· ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΖ διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. ἴση δὲ ἡ ΔΖ τῇ ΔΒ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου· ὅπερ ἔδει δεῖξαι.

Proposition 9†

If a straight-line is cut into equal and unequal (pieces), then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line), and (the square) on the difference between the (equal and unequal) pieces.



For let any straight-line AB have been cut—equally at C, and unequally at D. I say that the (sum of the) squares on AD and DB is double the (sum of the squares) on AC and CD.

For let CE have been drawn from (point) C, at right-angles to AB [Prop. 1.11], and let it be made equal to each of AC and CB [Prop. 1.3], and let EA and EB have been joined. And let DF have been drawn through (point) D, parallel to EC [Prop. 1.31], and (let) FG (have been drawn) through (point) F, (parallel) to AB [Prop. 1.31]. And let AF have been joined. And since AC is equal to CE, the angle EAC is also equal to the (angle) AEC [Prop. 1.5]. And since the (angle) at C is a right-angle, the (sum of the) remaining angles (of triangle AEC), EAC and AEC, is thus equal to one right-angle [Prop. 1.32]. And they are equal. Thus, (angles) CEA and CAE are each half a right-angle. So, for the same (reasons), (angles) CEB and EBC are also each half a right-angle. Thus, the whole (angle) AEB is a right-angle. And since GEF is half a right-angle, and EGF (is) a right-angle—for it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) EFG is thus half a right-angle [Prop. 1.32]. Thus, angle GEF [is] equal to EFG. So the side EG is also equal to the (side) GF [Prop. 1.6]. Again, since the angle at B is half a right-angle, and (angle) FDB (is) a right-angle—for again it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) BFD is half a right-angle [Prop. 1.32]. Thus, the angle at B (is) equal to DFB. So the side FD is also equal to the side DB [Prop. 1.6]. And since AC is equal to CE, the (square) on AC (is) also equal to the (square) on CE. Thus, the (sum of the) squares on AC and CE is double the (square) on AC. And the square on EA is equal to the (sum of the) squares on AC and CE. For angle ACE (is) a right-angle [Prop. 1.47]. Thus, the (square) on EA is double the (square) on AC. Again, since EG is equal to GF, the (square) on EG (is) also equal to the (square) on GF. Thus, the (sum of the squares) on EG and GF is double the square on GF. And the square on EF is equal to the (sum of the) squares on EG and GF [Prop. 1.47]. Thus, the square on EF is double the (square) on GF. And GF (is) equal to CD [Prop. 1.34]. Thus, the (square) on EF is double the (square) on CD. And the (square) on EA is also double the (square) on AC. Thus, the (sum of the) squares on AE and EF is double the (sum of the) squares on AC and CD. And the square on AF is equal to the (sum of the squares) on AE and EF. For the angle AEF is a right-angle [Prop. 1.47]. Thus, the square on AF is double the (sum of the squares) on AC and CD. And the (sum of the squares) on AD and DF (is) equal to the (square) on AF. For the angle at D is a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AD and DF is double the (sum of the) squares on AC and CD. And DF (is) equal to DB. Thus, the (sum of the) squares on AD and DB is double the (sum of the) squares on AC and CD.

Thus, if a straight-line is cut into equal and unequal (pieces), then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line), and (the square) on the difference between the (equal and unequal) pieces. (Which is) the very thing it was required to show.

Figure 4-24. Euclid's Proposition 9, from Richard Fitzpatrick translation.

Then

To simplify the proposition 9 of Euclid's book, I will resort to modern day methods to represent the ideas presented.

Proposition 9, book 2, compares the division of a line segment $\overline{\text{AB}}$ (i) into two equal parts by point C and (ii) into two unequal parts by point D. Euclid claims that

$$\overline{\text{AD}}^2 + \overline{\text{DB}}^2 = 2\left[\overline{\text{AC}}^2 + \overline{\text{CD}}^2\right] \tag{4.49}$$

This situation is illustrated in the figure 4-25.

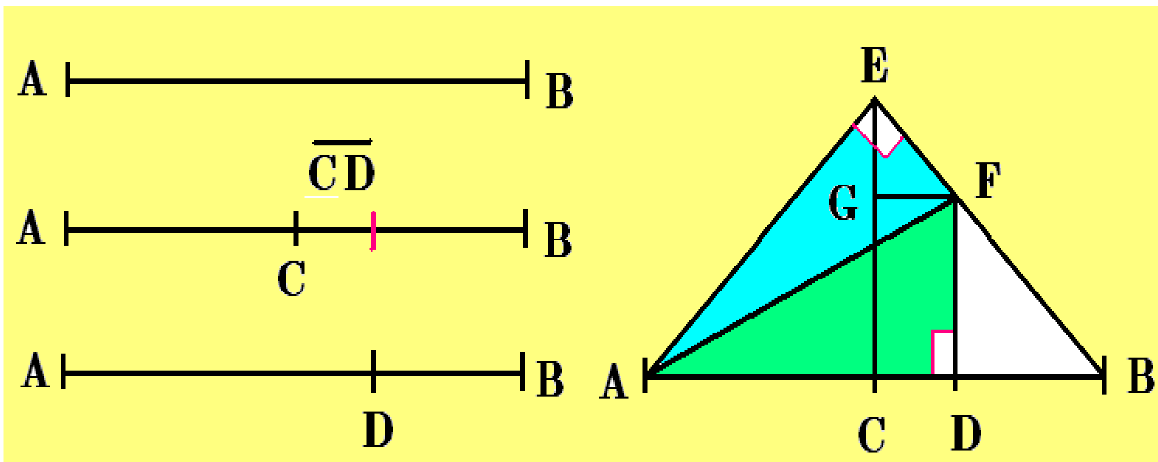


Figure 4-25. Line segment divided into equal and unequal parts.

Euclid's Proof

Construct the line $\overline{\text{CE}}$ which is perpendicular to $\overline{\text{AB}}$ such that $\overline{\text{CE}} = \overline{\text{AC}} = \overline{\text{CB}}$. Now construct the lines $\overline{\text{EA}}$ and $\overline{\text{EB}}$ followed by the lines $\overline{\text{DF}}$ perpendicular to $\overline{\text{AB}}$ and parallel to $\overline{\text{EC}}$. Another construction $\overline{\text{GF}}$ perpendicular to $\overline{\text{CE}}$ and parallel to $\overline{\text{AB}}$ followed by the construction of line $\overline{\text{AF}}$.

We know $\overline{\text{AC}} = \overline{\text{CE}}$ so that the angles $\angle CAE = \angle CEA = \frac{\pi}{4}$ are equal. Similarly, $\overline{\text{CE}} = \overline{\text{CB}}$ implies the angles $\angle CBE = \angle CEB = \frac{\pi}{4}$. Therefore, $\angle AEB$ is a right angle. Observe that $\angle GEF = \angle GFE = \frac{\pi}{4}$ so that $\overline{\text{EG}} = \overline{\text{GF}}$. By the same reasoning one can show $\angle DFB = \angle DBF = \frac{\pi}{4}$ and therefore $\overline{\text{FD}} = \overline{\text{DB}}$.

We now have several equalities

$$\overline{\text{AC}} = \overline{\text{CE}} = \overline{\text{CB}}, \quad \angle AEB = \frac{\pi}{2}, \quad \overline{\text{EG}} = \overline{\text{GF}} = \overline{\text{CD}}, \quad \overline{\text{FD}} = \overline{\text{DB}}$$

and two right triangles to work with, namely $\triangle AEF$ and $\triangle ADF$. Using the above identities and the Pythagorean theorem one can discern the following results.

$$(i)\ \overline{\mathrm{AC}}^2 + \overline{\mathrm{CE}}^2 = 2\overline{\mathrm{AC}}^2$$

$$(ii)\ \overline{\mathrm{AE}}^2 = \overline{\mathrm{AC}}^2 + \overline{\mathrm{CE}}^2 = 2\overline{\mathrm{AC}}^2$$

$$(iii)\ \overline{\mathrm{EG}}^2 = \overline{\mathrm{GF}}^2 \quad \text{and} \quad \overline{\mathrm{GF}} = \overline{\mathrm{CD}}$$

$$(iv)\ \overline{\mathrm{EF}}^2 = \overline{\mathrm{EG}}^2 + \overline{\mathrm{GF}}^2 = 2\overline{\mathrm{GF}}^2 = 2\overline{\mathrm{CD}}^2$$

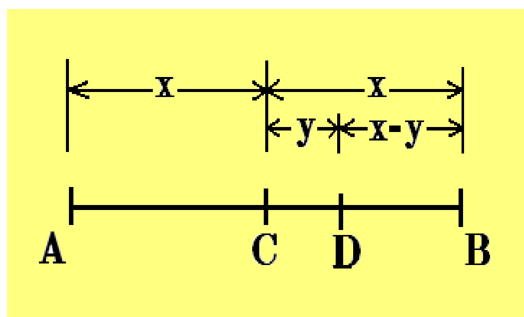
$$(v)\ \overline{\mathrm{EF}}^2 + \overline{\mathrm{AE}}^2 = \overline{\mathrm{AF}}^2 = 2\overline{\mathrm{CD}}^2 + 2\overline{\mathrm{AC}}^2 = 2\left[\overline{\mathrm{AC}}^2 + \overline{\mathrm{CD}}^2\right]$$

$$(vi)\ \overline{\mathrm{AF}}^2 = \overline{\mathrm{FD}}^2 + \overline{\mathrm{AD}}^2 = \overline{\mathrm{AD}}^2 + \overline{\mathrm{DB}}^2$$

Therefore, by using (v) and (vi) with things equal to the same thing are equal to one another, one finds the truth of Euclid's claim. In Euclid's proposition 9 all of the above statements (i) through (vi) are written out in sentences with reasons for each statement. This makes the proof of the claim quite long.

Now

For a more modern approach to the above problem. Let a symbol represent an idea!



Let $\overline{\mathrm{AC}} = \overline{\mathrm{CB}} = x$

Let $\overline{\mathrm{CD}} = y$

Let $\overline{\mathrm{DB}} = x - y$

Euclid wanted to show $(x+y)^2 + (x-y)^2 = 2\left[x^2 + y^2\right]$

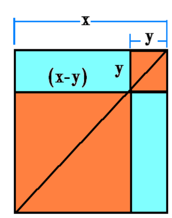
Use algebra and expand the left-hand side of the above equation to show

$$x^2 + 2xy + y^2 + x^2 - 2xy + y^2 = 2[x^2 + y^2]$$

which establishes Euclid's claim.

As an exercise examine other propositions in Euclid's Book 2 and give an algebraic interpretation to the results. For example, the proposition 4, book 2 of the Elements states: If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments. This is equivalent to the algebraic statement

$$x^2 = y^2 + (x-y)^2 + 2(x-y)y$$

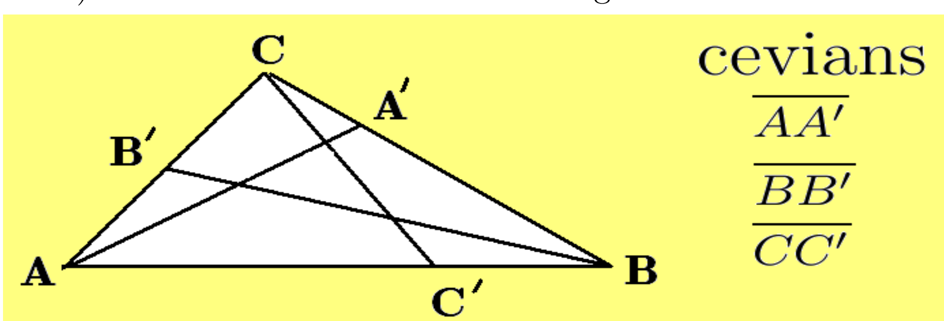


as being an identity.

The above proof from Euclid's Elements shows that thousands of years ago they used geometry to prove algebraic identities. Make note of the fact that algebra[5] wasn't used much until about the 16th century when it started to gain acceptance. We have demonstrated that by defining symbols one can use algebra to produce some of the results found in Euclids Elements. In dealing with mathematical symbols in equations, you must learn to understand the meaning of the symbols as well as being able to write out in words what the symbols are trying to tell you. For example, if you are given a right triangle with sides a, b, c and you know $a^2 + b^2 = c^2$, write out in words all that this equation is telling you and what is implied by knowing the equation is true. Too many students memorize equations without having any understanding of what the equations represent. The memorization of equations is not what mathematics is about. The secret to success in mathematics is definitions, understanding them and recognizing them when they arise in the application of mathematics and employing them to make new discoveries, prove theorems and expand the knowledge about various subjects.

Cevians of a triangle

By definition any line segment in a triangle running from a vertex to a point on the side opposite the vertex is called a **cevian**. Named after Giovanni Benedetto Ceva (1647-1734) an Italian mathematician and engineer.



Stewart's theorem

Matthew Stewart[6] investigated the length d of a cevian from the vertex A to side $\overline{\text{BC}}$ of a general triangle $\triangle ABC$. The cevian $\overline{AA'}$ divides the side $a = \overline{\text{BC}}$ into two parts labeled αa and $(1-\alpha)a$, where $0 \le \alpha \le 1$. The situation is illustrated in the

[5] The name "algebra" comes from the Arabic "Al-jabr". The beginnings of algebra were developed around 700 CE by a Persian mathematician by the name of *Abu Jáfar Muhammad ibn Mūsā al-Khwārizmī*.

[6] Matthew Stewart (1717-1785) A Scottish mathematician.

figure 4-26 which illustrates the cases for an acute and obtuse triangle. Stewart's theorem states that the cevian $d = \overline{AA'}$ is obtained from the relation

$$(1-\alpha)b^2 + \alpha c^2 = d^2 + \alpha(1-\alpha)a^2 \tag{4.50}$$

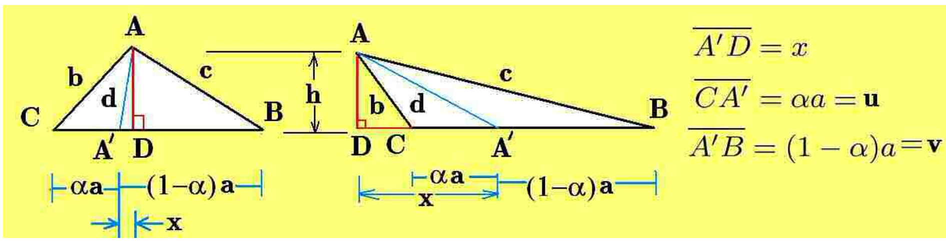


Figure 4-26. Cevian $\overline{AA'}$ divides $\overline{\text{BC}}$ into two parts.

We use the Pythagorean theorem together with some algebra to prove Stewart's result. First drop a perpendicular from vertex A to the triangle base and label the point of intersection D, then

	acute		obtuse
From $\triangle ADA'$	$h^2 + x^2 = d^2$	From $\triangle ADA'$	$h^2 + x^2 = d^2$
From $\triangle ADC$	$h^2 + (\alpha a + x)^2 = b^2$	From $\triangle ADC$	$h^2 + (x - \alpha a)^2 = b^2$
From $\triangle ADB$	$h^2 + ((1-\alpha)a - x)^2 = c^2$	From $\triangle ADB$	$h^2 + (x + (1-\alpha)a))^2 = c^2$

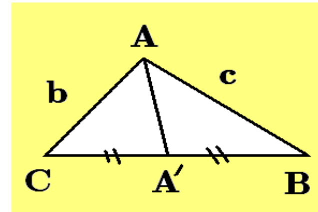
Multiply the middle equation by $v = (1-\alpha)a$ and the bottom equation by $u = \alpha a$ to show

acute	obtuse
$vb^2 = vh^2 + v\left(u^2 + 2ux + x^2\right)$	$vb^2 = vh^2 + v\left(x^2 - 2xu + u^2\right)$
$uc^2 = uh^2 + u\left(v^2 - 2vx + x^2\right)$	$uc^2 = uh^2 + u\left(x^2 + 2xv + v^2\right)$

Addition of these equations followed by simplification using the results $u + v = a$ and $h^2 + x^2 = d^2$, there results the Stewart formula

$$vb^2 + uc^2 = a\left(d^2 + uv\right) \quad \text{or} \quad (1-\alpha)b^2 + \alpha c^2 = d^2 + \alpha(1-\alpha)a^2 \tag{4.51}$$

Special case cevian is a median

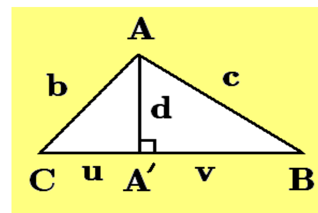


In the special case the cevian is a median, then the side a of triangle $\triangle ABC$ is divided into two equal parts. In this special case $u = v$ or $\alpha = \frac{1}{2}$ or $v = \frac{a}{2}$, so that Stewart's formula becomes

$$v(b^2 + c^2) = a\left(d^2 + v^2\right) \;\Rightarrow\; 2(b^2 + c^2) = 4d^2 + a^2 \;\Rightarrow\; d^2 = \frac{2b^2 + 2c^2 - a^2}{4} \tag{4.52}$$

Special case cevian is an altitude

In the special case the cevian is an altitude, then the cevian $\overline{AA'}$ is perpendicular to the side $\overline{\text{CB}}$ associated with triangle $\triangle ABC$ and there is created two right triangles so that the length of the cevian can be obtained from either of the Pythagorean relations



$$d^2 + v^2 = c^2 \quad \Rightarrow \quad d^2 = c^2 - v^2$$
$$d^2 + u^2 = b^2 \quad \Rightarrow \quad d^2 = b^2 - u^2$$

In this case the Stewart formula for the length of the cevian becomes an identity. The length of the cevian can be obtained in terms of the semiperimeter $s = \frac{a+b+c}{2}$ as follow. The area of the triangle squared is written

$$[ABC]^2 = \left(\frac{1}{2}ad\right)^2 = \frac{1}{4}a^2d^2 \tag{4.53}$$

By Heron's formula the area squared is written

$$[ABC]^2 = s(s-a)(s-b)(s-c) \tag{4.55}$$

Things equal to the same thing are equal to one another so that one finds the length of the cevian, when it is an altitude, can be obtained from the alternative formula

$$d = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} \tag{4.55}$$

Ceva's theorem

Ceva's theorem is a condition that must be satisfied if the three cevians associated with the three vertices of a triangle meet at a point of concurrency somewhere inside the triangle as illustrated in the accompanying figure. Ceva's condition is that the cevians are concurrent at point 0 if the product of the ratios associated with each side of the triangle equals unity or

$$\frac{\overline{AD}}{\overline{DB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$$

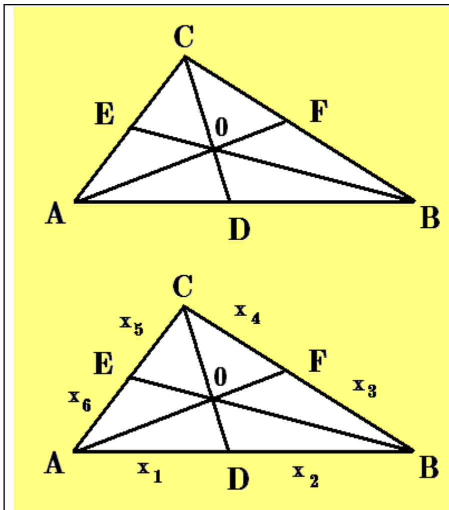


Figure 4-27. Ceva's triangle

Note the cevians divide the sides of the triangle into six parts as illustrated in the accompanying figure. The Ceva's theorem can then be expressed in the alternate form

$$\frac{x_1}{x_2}\frac{x_3}{x_4}\frac{x_5}{x_6} = 1$$

Knowledge of the following two mathematical facts makes the proof of Ceva's theorem easy.

Fact 1:

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}$ for $b \neq d$.

Proof:

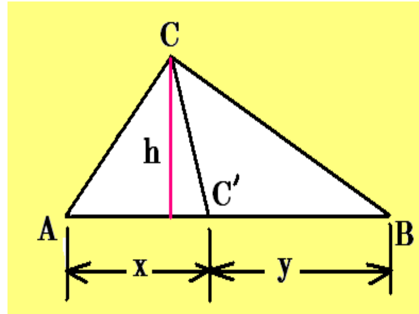
If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{a+\lambda c}{b+\lambda d}$ for any constant λ because the cross ratio produces an identity. That is,

$$ab + \lambda ad = ab + \lambda bc \;\Rightarrow\; ad = bc \;\Rightarrow\; \frac{a}{b} = \frac{c}{d}$$

In the special case $\lambda = -1$, the above ratio results.

Fact 2:

If $\overline{CC'}$ is a cevian of triangle $\triangle ABC$, then it divides the side $\overline{\text{AB}}$ into two parts x and y where the ratio $\frac{x}{y}$ equals the ratio of the areas associated with triangles $\triangle ACC'$ and $\triangle CC'B$.



$$\frac{x}{y} = \frac{[ACC']}{[CC'B]}$$

Recall the notation $[ABC]$ = Area of triangle $\triangle ABC$ as we will be using this notation in proving Ceva's theorem. For example, the cevian CC' divides the given triangle into two triangles where the area of each triangle is given by $\frac{1}{2}(base)(height)$ so that their ratio becomes

$$\frac{[ACC']}{[CC'B]} = \frac{\frac{1}{2}hx}{\frac{1}{2}hy} = \frac{x}{y}$$

Proof of Ceva's theorem

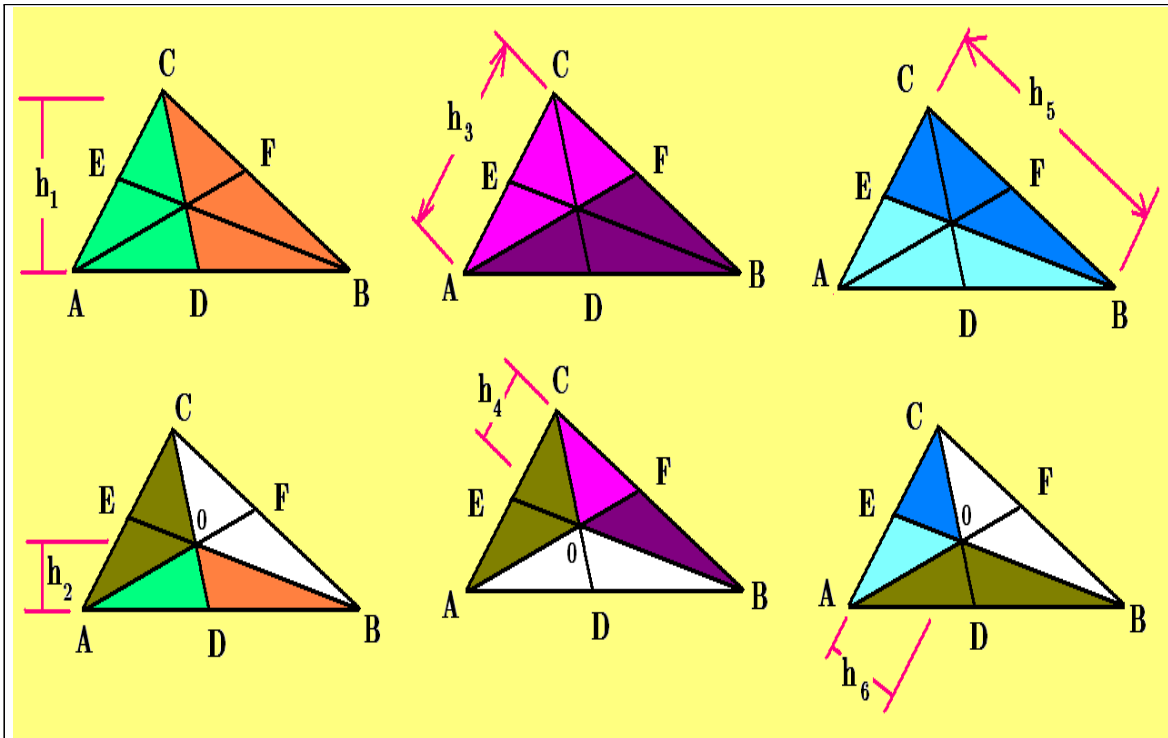


Figure 4-28. Triangles within triangles for determining ratios.

Refer to the figure 4-28 to verify the following ratio of areas

$$\frac{[ACD]}{[CDB]} = \frac{\frac{1}{2}\overline{\text{AD}}\,h_1}{\frac{1}{2}\overline{\text{DB}}\,h_1} = \frac{\overline{\text{AD}}}{\overline{\text{DB}}}$$

$$\frac{[AD0]}{[DB0]} = \frac{\frac{1}{2}\overline{\text{AD}}\,h_2}{\frac{1}{2}\overline{\text{DB}}\,h_2} = \frac{\overline{\text{AD}}}{\overline{\text{DB}}}$$

$$\frac{[A0C]}{[B0C]} = \frac{[ACD]-[AD0]}{[CDB]-[DB0]} = \frac{\overline{\text{AD}}}{\overline{\text{DB}}}$$

$$\frac{[ABF]}{[AFC]} = \frac{\frac{1}{2}\overline{\text{BF}}\,h_3}{\frac{1}{2}\overline{\text{FC}}\,h_3} = \frac{\overline{\text{BF}}}{\overline{\text{FC}}}$$

$$\frac{[0BF]}{[0FC]} = \frac{\frac{1}{2}\overline{\text{BF}}\,h_4}{\frac{1}{2}\overline{\text{FC}}\,h_4} = \frac{\overline{\text{BF}}}{\overline{\text{FC}}}$$

$$\frac{[B0A]}{[A0C]} = \frac{[ABF]-[0BF]}{[AFC]-[0FC]} = \frac{\overline{\text{BC}}}{\overline{\text{FC}}}$$

$$\frac{[CEB]}{[EAB]} = \frac{\frac{1}{2}\overline{\text{CE}}\,h_5}{\frac{1}{2}\overline{\text{EA}}\,h_5} = \frac{\overline{\text{CE}}}{\overline{\text{EA}}}$$

$$\frac{[CE0]}{[EA0]} = \frac{\frac{1}{2}\overline{\text{CE}}\,h_6}{\frac{1}{2}\overline{\text{EA}}\,h_6} = \frac{\overline{\text{CE}}}{\overline{\text{EA}}}$$

$$\frac{[B0C]}{[B0A]} = \frac{[CEB]-[CE0]}{[EAB]-[EA0]} = \frac{\overline{\text{CE}}}{\overline{\text{EA}}}$$

Consequently,

$$\frac{\overline{\text{AD}}}{\overline{\text{DB}}} \cdot \frac{\overline{\text{BF}}}{\overline{\text{FC}}} \cdot \frac{\overline{\text{CE}}}{\overline{\text{EA}}} = \frac{[A0C]}{[B0C]} \cdot \frac{[B0A]}{[A0C]} \cdot \frac{[B0C]}{[B0A]} = 1 \tag{4.56}$$

because the ratio of areas divide out leaving unity.

Converse of Ceva's theorem

Assume in triangle $\triangle ABC$ one can find points D, E, F on the triangle sides $\overline{\text{AB}}, \overline{\text{AC}}, \overline{\text{BC}}$ such that

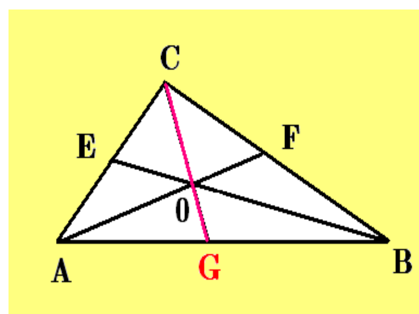
$$\frac{\overline{\text{AD}}}{\overline{\text{DB}}} \cdot \frac{\overline{\text{BF}}}{\overline{\text{FC}}} \cdot \frac{\overline{\text{CE}}}{\overline{\text{EA}}} = 1 \tag{4.57}$$

then this implies the cevians $\overline{\text{AF}}, \overline{\text{BE}}, \overline{\text{CD}}$ meet at a point of concurrency.

Proof:

Assume the cevians $\overline{\text{AF}}$ and $\overline{\text{BE}}$ meet at a point 0 as illustrated in the accompanying figure. Construct the line $\overline{\text{C0}}$ and extend it to intersect the side $\overline{\text{AB}}$ at a point G. By Ceva's theorem one must have

$$\frac{\overline{\text{AG}}}{\overline{\text{GB}}} \cdot \frac{\overline{\text{BF}}}{\overline{\text{FC}}} \cdot \frac{\overline{\text{CE}}}{\overline{\text{EA}}} = 1 \tag{4.58}$$



Equate equations (4.57) and (4.58) to obtain

$$\frac{\overline{\text{AD}}}{\overline{\text{DB}}} \cdot \frac{\overline{\text{BF}}}{\overline{\text{FC}}} \cdot \frac{\overline{\text{CE}}}{\overline{\text{EA}}} = \frac{\overline{\text{AG}}}{\overline{\text{GB}}} \cdot \frac{\overline{\text{BF}}}{\overline{\text{FC}}} \cdot \frac{\overline{\text{CE}}}{\overline{\text{EA}}} \quad \Rightarrow \quad \frac{\overline{\text{AD}}}{\overline{\text{DB}}} = \frac{\overline{\text{AG}}}{\overline{\text{GB}}}$$

Add 1 to both sides of this equation and show

$$\frac{\overline{\text{AD}}}{\overline{\text{DB}}} + 1 = \frac{\overline{\text{AG}}}{\overline{\text{GB}}} + 1 \quad \Rightarrow \quad \frac{\overline{\text{AD}} + \overline{\text{DB}}}{\overline{\text{DB}}} = \frac{\overline{\text{AG}} + \overline{\text{GB}}}{\overline{\text{GB}}}$$

which simplifies to the requirement

$$\frac{\overline{\text{AB}}}{\overline{\text{DB}}} = \frac{\overline{\text{AB}}}{\overline{\text{GB}}}$$

This equation can only hold true if the points G and D are concurrent. This requires $\overline{\text{AD}} = \overline{\text{AG}}$ for the requirement that the three cevians meet at a point of concurrency.

Ratios of lengths created by cevians of a triangle

Whenever three cevians all pass through an interior point of a triangle, various ratios of the lengths are created. Associated with the figure 4-29 are the following ratio properties.

$$\begin{aligned}
&\frac{\overline{\text{0F}}}{\overline{\text{CF}}} + \frac{\overline{\text{0D}}}{\overline{\text{AD}}} + \frac{\overline{\text{0E}}}{\overline{\text{BE}}} = 1\\
&\frac{\overline{\text{C0}}}{\overline{\text{CF}}} + \frac{\overline{\text{A0}}}{\overline{\text{AD}}} + \frac{\overline{\text{B0}}}{\overline{\text{BE}}} = 2\\
&\frac{\overline{\text{C0}}}{\overline{\text{0F}}} = \frac{\overline{\text{CD}}}{\overline{\text{DB}}} + \frac{\overline{\text{CE}}}{\overline{\text{EA}}}\\
&\frac{\overline{\text{A0}}}{\overline{\text{0D}}} = \frac{\overline{\text{AE}}}{\overline{\text{EC}}} + \frac{\overline{\text{AF}}}{\overline{\text{FB}}}\\
&\frac{\overline{\text{B0}}}{\overline{\text{0E}}} = \frac{\overline{\text{BD}}}{\overline{\text{DC}}} + \frac{\overline{\text{BF}}}{\overline{\text{FA}}}
\end{aligned} \tag{4.59}$$

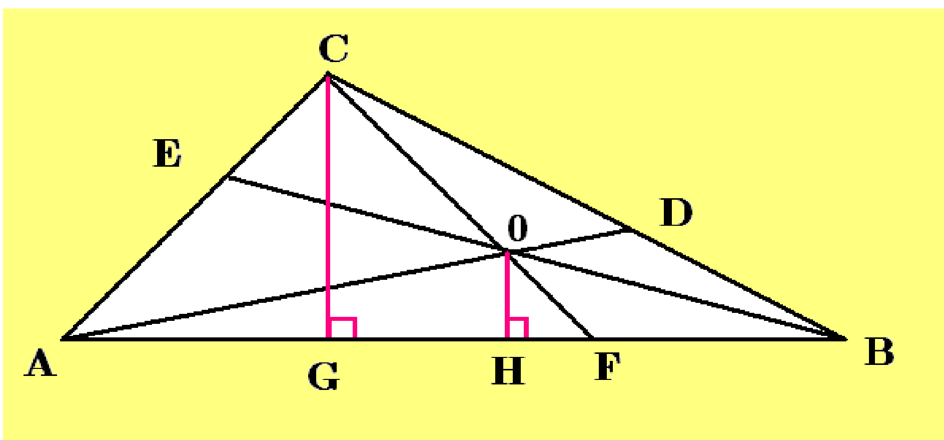


Figure 4-29. Triangle with concurrent point associated with three cevians.

Proof:

The following is an example of, "What has been done once can be done again," which is used quite often in mathematics.

Procedure A

In the figure 4-29 examine the triangles $\triangle A0B$ and $\triangle ABC$ and construct the altitudes $\overline{\text{CG}}$ and $\overline{\text{0H}}$ as illustrated and note that $\overline{\text{CG}}$ is parallel to $\overline{\text{0H}}$ ($\overline{\text{CG}} \parallel \overline{\text{0H}}$). The triangles $\triangle CGF$ and $\triangle 0HF$ are right triangles and they are similar ($\triangle CGH \sim \triangle 0HF$). Therefore, one can write the ratio

$$\frac{\overline{0\mathrm{H}}}{\overline{\mathrm{CG}}} = \frac{\overline{0\mathrm{F}}}{\overline{\mathrm{CF}}} \tag{4.60}$$

One can also examine the ratio of triangle areas associated with triangles having the same base. For example, using the notation

$$[ABC] = [BCA] = [CAB] = [ACB] = [BAC] = [CBA] = \text{Area of the triangle } \triangle ABC$$

one finds

$$\frac{[0AB]}{[CAB]} = \frac{\frac{1}{2}(\overline{\mathrm{AB}})(\overline{0\mathrm{H}})}{\frac{1}{2}(\overline{\mathrm{AB}})(\overline{\mathrm{CG}})} = \frac{\overline{0\mathrm{H}}}{\overline{\mathrm{CG}}} = \frac{\overline{0\mathrm{F}}}{\overline{\mathrm{CF}}} \tag{4.61}$$

End of procedure A

By rotating triangle $\triangle ABC$ so that $\overline{\mathrm{BC}}$ is the base and doing exactly the same steps as was done in procedure A, with the symbols changed, one can show

$$\frac{[0CA]}{[ABC]} = \frac{\overline{0\mathrm{D}}}{\overline{\mathrm{AD}}} \tag{4.62}$$

Rotate the triangle again so that $\overline{\mathrm{AC}}$ is the base and again use the procedure A, with the symbols changed, to show

$$\frac{[0CA]}{[BCA]} = \frac{\overline{0\mathrm{E}}}{\overline{\mathrm{BE}}} \tag{4.63}$$

By addition of the equations (4.61), (4.62), (4.63) there results

$$\frac{\overline{0\mathrm{F}}}{\overline{\mathrm{CF}}} + \frac{\overline{0\mathrm{D}}}{\overline{\mathrm{AD}}} + \frac{\overline{0\mathrm{E}}}{\overline{\mathrm{BE}}} = \frac{[0AB]}{[CAB]} + \frac{[0BC]}{[ABC]} + \frac{[0CA]}{[BCA]} = 1 \tag{4.64}$$

Note that in the equation (4.64) the denominators are all equal to the area of triangle $\triangle ABC$ and the summation of the numerators in equation (4.64) represents the sum of the triangles interior to $\triangle ABC$. The whole being the sum of its parts.

Using the subtraction of line segments, which are all positive in length, one can write

$$\frac{\overline{\mathrm{C}0}}{\overline{\mathrm{CF}}} = \frac{\overline{\mathrm{CF}} - \overline{0\mathrm{F}}}{\overline{\mathrm{CF}}} = 1 - \frac{\overline{0\mathrm{F}}}{\overline{\mathrm{CF}}} \tag{4.65}$$

$$\frac{\overline{\mathrm{A}0}}{\overline{\mathrm{AD}}} = \frac{\overline{\mathrm{AD}} - \overline{0\mathrm{D}}}{\overline{\mathrm{AD}}} = 1 - \frac{\overline{0\mathrm{D}}}{\overline{\mathrm{AD}}} \tag{4.66}$$

$$\frac{\overline{\mathrm{B}0}}{\overline{\mathrm{BE}}} = \frac{\overline{\mathrm{BE}} - \overline{0\mathrm{E}}}{\overline{\mathrm{BE}}} = 1 - \frac{\overline{0\mathrm{E}}}{\overline{\mathrm{BE}}} \tag{4.67}$$

Adding the equations (4.65), (4.66), (4.67) there results

$$\frac{\overline{\mathrm{C0}}}{\overline{\mathrm{CF}}}+\frac{\overline{\mathrm{A0}}}{\overline{\mathrm{AD}}}+\frac{\overline{\mathrm{B0}}}{\overline{\mathrm{BE}}}=3-\left[\frac{\overline{\mathrm{0F}}}{\overline{\mathrm{CF}}}+\frac{\overline{\mathrm{0D}}}{\overline{\mathrm{AD}}}+\frac{\overline{\mathrm{0E}}}{\overline{\mathrm{BE}}}\right]$$

Using the results from the equation (4.64) the above result becomes

$$\frac{\overline{\mathrm{C0}}}{\overline{\mathrm{CF}}}+\frac{\overline{\mathrm{A0}}}{\overline{\mathrm{AD}}}+\frac{\overline{\mathrm{B0}}}{\overline{\mathrm{BE}}}=2 \tag{4.68}$$

Using the results from Ceva's theorem, given by equation (4.56) which is written in terms of the notation of triangle $\triangle ABC$ of figure 4-29 one can verify that

$$\frac{\overline{\mathrm{AF}}}{\overline{\mathrm{FB}}}\cdot\frac{\overline{\mathrm{DB}}}{\overline{\mathrm{CD}}}\cdot\frac{\overline{\mathrm{CE}}}{\overline{\mathrm{EA}}}=\frac{[A0C]}{[CB0]}\cdot\frac{[A0B]}{[AC0]}\cdot\frac{[BC0]}{[A0B]}=1 \tag{4.69}$$

which implies

$$\frac{\overline{\mathrm{AF}}}{\overline{\mathrm{FB}}}=\frac{[A0C]}{[CB0]},\qquad \frac{\overline{\mathrm{DB}}}{\overline{\mathrm{CD}}}=\frac{[A0B]}{[AC0]},\qquad \frac{\overline{\mathrm{CE}}}{\overline{\mathrm{EA}}}=\frac{[BC0]}{[A0B]} \tag{4.70}$$

Recall that the ratio of triangle areas, using triangles having the same height, produces the ratio of the bases. Then an examination of the triangles to the left and right of the cevian $\overline{\mathrm{CF}}$ one can verify

$$\frac{\overline{\mathrm{C0}}}{\overline{\mathrm{0F}}}=\frac{[AC0]}{[A0F]}=\frac{[BC0]}{[B0F]} \tag{4.71}$$

Also note the theory of proportions produces the ratio

$$\frac{\overline{\mathrm{C0}}}{\overline{\mathrm{0F}}}=\frac{[AC0]+[BC0]}{[A0F]+[B0F]}=\frac{[AC0]+[BC0]}{[A0B]}=\frac{[AC0]}{[A0B]}+\frac{[BC0]}{[A0B]} \tag{4.72}$$

where now one can use the results from equation (4.70) to show equation (4.72) reduces to the desired equation

$$\frac{\overline{\mathrm{C0}}}{\overline{\mathrm{0F}}}=\frac{\overline{\mathrm{CD}}}{\overline{\mathrm{DB}}}+\frac{\overline{\mathrm{CE}}}{\overline{\mathrm{EA}}} \tag{4.73}$$

The above result is cyclic in nature so it follows that the following ratios are also true.

$$\frac{\overline{\mathrm{A0}}}{\overline{\mathrm{0D}}}=\frac{\overline{\mathrm{AE}}}{\overline{\mathrm{EC}}}+\frac{\overline{\mathrm{AF}}}{\overline{\mathrm{FB}}}\quad\text{and}\quad \frac{\overline{\mathrm{B0}}}{\overline{\mathrm{0E}}}=\frac{\overline{\mathrm{BD}}}{\overline{\mathrm{DC}}}+\frac{\overline{\mathrm{BF}}}{\overline{\mathrm{FA}}} \tag{4.74}$$

Example 4-9. Prove the medians of a triangle meet at a point of concurrency.

Solution:

This is a special case where $x_1=x_2$, $x_3=x_4$ and $x_5=x_6$ (See figure 4-27) and Ceva's theorem holds showing the medians of a triangle produce a point of concurrency called the centroid.

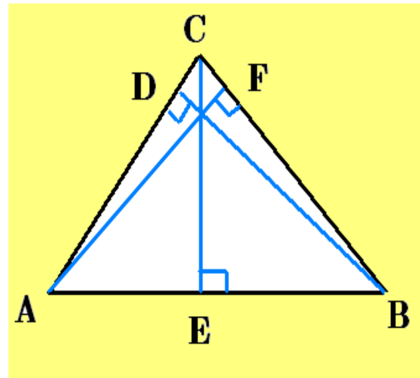
■

Example 4-10. Prove the altitudes of a triangle meet at a point of concurrency.

Solution

Use similar triangles $\triangle AFC \sim \triangle BDC$ to show

$$\frac{\overline{CD}}{\overline{FC}} = \frac{\overline{BC}}{\overline{AC}} \tag{4.75}$$



Use similar triangles $\triangle ABD \sim \triangle AEC$ to show

$$\frac{\overline{AE}}{\overline{DA}} = \frac{\overline{AC}}{\overline{AB}} \tag{4.76}$$

Use similar triangles $\triangle ABF \sim \triangle BCE$ to show

$$\frac{\overline{BF}}{\overline{EB}} = \frac{\overline{AB}}{\overline{BC}} \tag{4.77}$$

Take the products of equations (4.75), (4.76), (4.77) to show

$$\frac{\overline{CD}}{\overline{FC}} \cdot \frac{\overline{AE}}{\overline{DA}} \cdot \frac{\overline{BF}}{\overline{EB}} = \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AB}} \cdot \frac{\overline{AB}}{\overline{BC}} = 1 \tag{4.78}$$

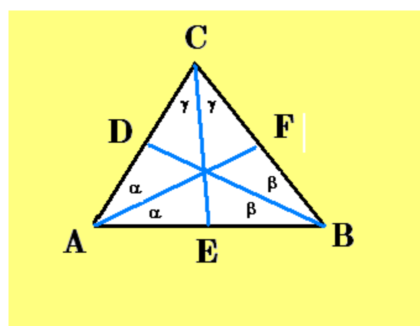
Changing the numerator of the left hand side of equation (4.78) gives Ceva's theorem and demonstrates the altitudes must meet at a point of concurrency called the orthocenter.

■

Example 4-11. Prove the angle bisectors in a triangle meet at a point of concurrency called the incenter.

Solution:

Each cevian divides the triangle into two parts. Use Thales theorem of proportion associated with each angle bisector to show the following.



$$\frac{\overline{BF}}{\overline{FC}} = \frac{\overline{AB}}{\overline{AC}}, \qquad \frac{\overline{CD}}{\overline{DA}} = \frac{\overline{BC}}{\overline{AB}}, \qquad \frac{\overline{AE}}{\overline{EB}} = \frac{\overline{AC}}{\overline{BC}}$$

Multiply these ratios and show

$$\frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CD}}{\overline{DA}} \cdot \frac{\overline{AE}}{\overline{EB}} = \frac{\overline{AB}}{\overline{AC}} \cdot \frac{\overline{BC}}{\overline{AB}} \cdot \frac{\overline{AC}}{\overline{BC}} = 1$$

which is Ceva's theorem and demonstrating that the angle bisectors meet at a point of concurrency.

■

Exercises

▶ **4-1.** Write down two numbers which are coprime.

▶ **4-2.** Find the distance between the given points.

(a) $(1,2)$ $(5,5)$ (d) $(1,2)$ $(25,9)$

(b) $(1,2)$ $(13,7)$ (e) $(1,2)$ $(2,3)$

(c) $(1,2)$ $(9,8)$ (f) $(1,2)$ $(1+a,2+b)$

▶ **4-3.** Sketch the given ellipse

$$(a) \quad \frac{x^2}{9}+\frac{y^2}{16}=1 \qquad\qquad (b) \quad \frac{x^2}{16}+\frac{y^2}{9}=1$$

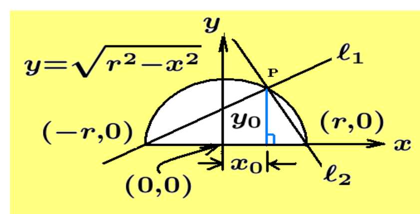
▶ **4-4.**

Given the semicircle $y=\sqrt{r^2-x^2}$, $-r\le x\le r$ as illustrated. Let P denote the point (x_0,y_0) on the semicircle where $y_0=\sqrt{r^2-x_0^2}$.

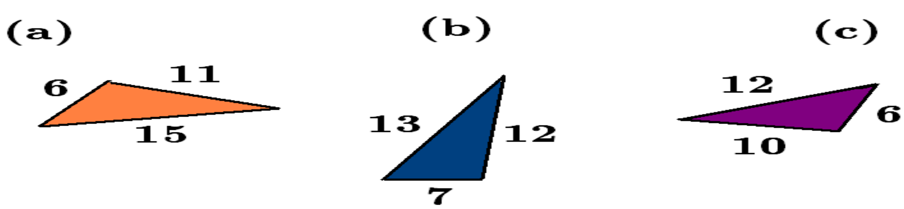
(a) Find the equation of the line ℓ_1 passing through $(-r,0)$ and P.

(b) Find the equation of the line ℓ_2 passing through $(r,0)$ and P.

(c) Show $\ell_1 \perp \ell_2$



▶ **4-5.** Find the area of the given triangles.



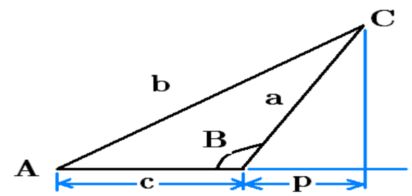
▶ **4-6.** Given triangle $\triangle ABC$ with sides a,b,c. What kind of triangle results if

$$(a) \quad a^2+b^2<c^2 \qquad (b) \quad a^2+b^2=c^2 \qquad (c) \quad a^2+b^2>c^2$$

▶ **4-7.** Find the length of the diagonal of a square with side of length s.

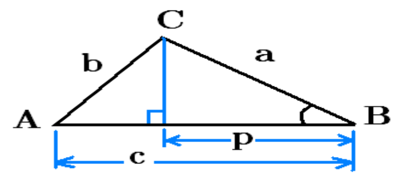
▶ **4-8.**

Given the obtuse triangle $\triangle ABC$ with p the projection of side a onto the side $\overline{\text{AB}}$ extended.
Show that $b^2 = a^2 + c^2 + 2cp$
(Euclid, Book 2, Proposition 12)
Here b equals the side opposite the obtuse angle, a, c are the other two sides and $2pc$ is twice the product of the projection length and c the length of the triangle side used for the projection.



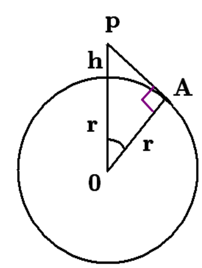
▶ **4-9.**

Given an acute triangle $\triangle ABC$ with side a being projected onto the side $\overline{\text{AB}}$ giving a length p.
Show that $b^2 = a^2 + c^2 - 2cp$
(Euclid, Book 2, Proposition 13)
Here b is the side opposite the acute angle, a, c are the other two sides and $2pc$ is twice the product of the projection length and c the side used for the projection.



▶ **4-10.**

Assume the earth is a sphere and point p is a distance h above the earth. Let 0 denote the center of the earth, $r \approx 3960$ miles, the radius of the earth and point A a point on the horizon as viewed from point p. Assume all units are in miles and answer the following questions.



(a) Find a formula to calculate the distance $\overline{\text{PA}}$ to the horizon.
(b) If h is very small compared to the radius of the earth and h is converted to units of feet, show that

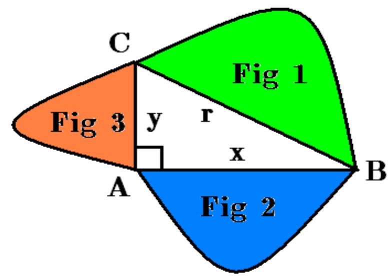
$$\overline{\text{PA}} \approx \sqrt{\frac{(2)(3960)\,h}{5280}} \qquad h \text{ in units of feet and } \overline{\text{PA}} \text{ in units of miles.}$$

▶ **4-11.** Find the length of the diagonal of a rectangle having sides of length 5 and 10 meters.

▶ **4-12.** A room in a certain house for sale measures 24 by 34 by 8 feet. Find the distance from one corner on the floor to the opposite corner on the ceiling.

▶ **4-13.**

Given the right triangle $\triangle ABC$ with sides x, y and hypotenuse r. There is some type of figure attached to each side of the triangle. The attached figures have the attachment lengths $\overline{\text{BC}}, \overline{\text{AB}}$ and $\overline{\text{CA}}$ respectively. Examine the area of figure 1 in comparison to the the sum of the areas from figures 2 and 3 in the following cases.



(a) Each figure is a square.
(b) Each figure is a pentagon.
(c) Each figure is a hexagon.
(d) Each figure is an n-gon.
(e) Each figure is a semicircle.
(f) Each figure is an equilateral triangle.

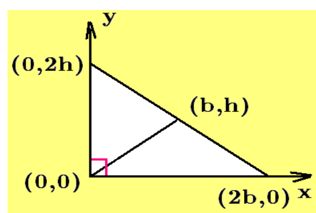
▶ **4-14.** Given a circle of radius R meters. Find the length of the chord $\overline{\text{AB}}$ which is $(R-a)$ meters from the center of the circle.

▶ **4-15.** Given a circle of radius R meters. A chord $\overline{\text{AB}}$ is constructed having a length $(2R-a)$ meters. How far from the circle center is the chord?

▶ **4-16.** Given a chord $\overline{\text{AB}}$ of length ℓ_1 meters which is ℓ_2 meters from the center of the circle. Find the radius of the circle.

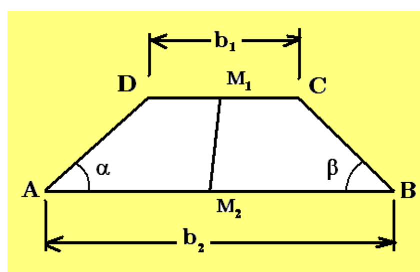
▶ **4-17.**

Use the distance formula to show that in a right triangle the length of the median from the right angle to the hypotenuse will always equal one-half the length of the hypotenuse.



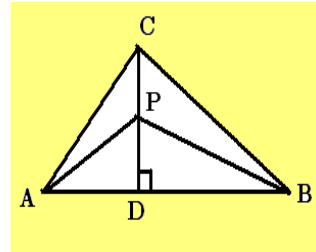
▶ **4-18.**

Given a trapezoid ABCD with M_1, M_2 the midpoints of the bases b_1 and b_2. If the base angles α, β satisfy $\alpha+\beta=\frac{\pi}{2}$ show that $\overline{M_1M_2}=\frac{1}{2}(b_2-b_1)$.
Hint: See the previous problem.



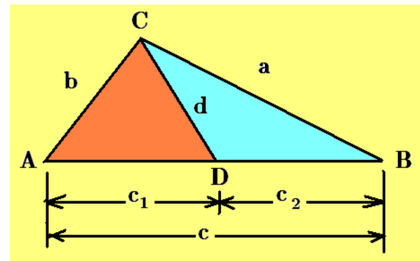
▶ **4-19.**

Let $\overline{\text{CD}}$ denote the altitude of triangle $\triangle ABC$ and let P denote an arbitrary point on $\overline{\text{CD}}$. Use the Pythagorean theorem to show



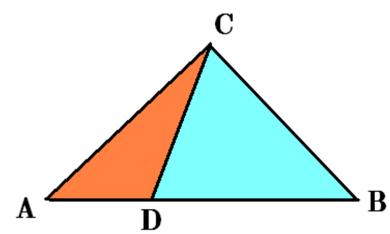
$$\overline{\text{AC}}^2 - \overline{\text{AP}}^2 = \overline{\text{CB}}^2 - \overline{\text{PB}}^2$$

▶ **4-20.**
Given the triangle $\triangle ABC$ with cevian $\overline{\text{CD}}$. Express Stewarts theorem in terms of the lengths a, b, c, d, c_1, c_2.

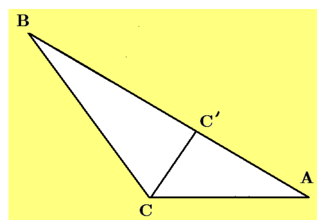


▶ **4-21.**

Given the triangle $\triangle ABC$ with $\overline{\text{AC}} = 5$, $\overline{\text{BC}} = 8$, $\overline{\text{AB}} = 16$ and $\overline{\text{DB}} = 3\overline{\text{AD}}$. Find the length of the cevian $\overline{\text{CD}}$.



▶ **4-22.**
Given the triangle $\triangle ABC$ with $\overline{\text{CA}} = 8$, $\overline{\text{BC}} = 10$, $\overline{C'A} = 2$ and $\overline{\text{BA}} = 13 + \sqrt{205}$. Find the length of the cevian $\overline{CC'}$.



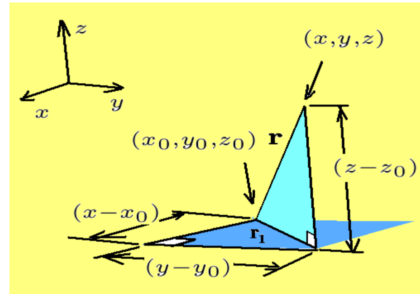
▶ **4-23.** Given an isosceles triangle with vertex angle C and cevian $\overline{CC'} = 19$. The base $\overline{\text{AB}}$ of the isosceles triangle is divided into two parts by the cevian such that $\overline{AC'} = 3$ and $\overline{C'B} = 13$. Find the length of the sides associated with the isosceles triangle.

▶ **4-24.** Apply a cyclic rotation of symbols to the equation (4.52) to find the length of the medians $\overline{BB'}$ and $\overline{CC'}$ associated with triangle $\triangle ABC$.

▶ **4-25.** Apply a cyclic rotation of symbols to the equation (4.55) to find the length of the altitudes $\overline{BB'}$ and $\overline{CC'}$ associated with triangle $\triangle ABC$.

▶ **4-26. Equation of a sphere**

A sphere is defined as the locus of points equidistant from a fixed point. If (x, y, z) denotes a variable point which is always a distance r from a fixed point (x_0, y_0, z_0), then use the Pythagorean theorem to show the equation of the sphere centered at (x_0, y_0, z_0) is given by $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$



▶ **4-27.** Find the distance between the given points.

$(0, 0, 0)$ to $(3, 4, 5)$ $\qquad$ $(3, 4, 5)$ to $(20, 32, 18)$

$(1, 2, 3)$ to $(3, 4, 5)$ $\qquad$ $(12, 35, 8)$ to $(20, 45, 58)$

▶ **4-28.**

Consider a regular pentagon with unit sides inscribed within a circle as illustrated. Extend the unit sides of the pentagon to form a pentagram and then connect the points AGBCH with straight lines to form a larger pentagon. Let $x = \overline{\text{AD}}$ and demonstrate that

(a) the radius of the circumscribe circle is $r = \sqrt{\frac{5+\sqrt{5}}{10}}$

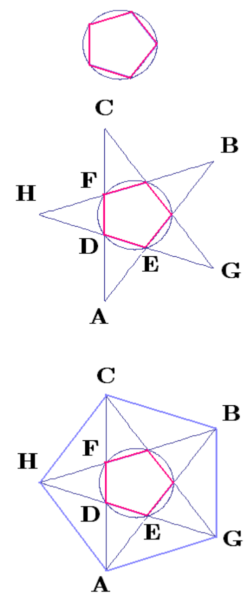
(b) the quadrilateral $AFBG$ is a rhombus.

(c) one side of the larger pentagon is $\overline{\text{BG}} = 1 + x$

(d) the triangles $\triangle ADE$ and $\triangle ACB$ are similar.

(e) $\dfrac{x}{1} = \dfrac{2x + 1}{x + 1}$

(f) $x = \phi$ is the golden ratio.



▶ **4-29.**

Given the equilateral triangle $\triangle ABC$ with side s inscribed in a circle of radius r.

(a) Show the altitude is $h = \frac{\sqrt{3}}{2}s$

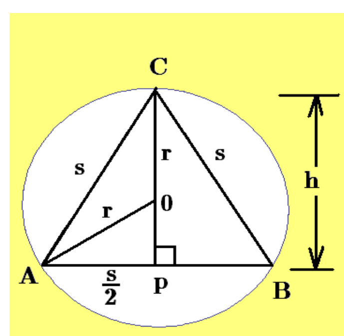
(b) Show the area of the triangle is $[ABC] = \frac{\sqrt{3}}{4}s^2$

(c) Show the perimeter of the triangle is $p = 3s$

(d) Show the centroid gives $r = \frac{2}{3}h$

(e) Show the apothem of the equilateral triangle is $\overline{\text{0P}} = \frac{\sqrt{3}}{6}s$

(f) Show $s = \sqrt{3}r$



Geometry

Chapter 5

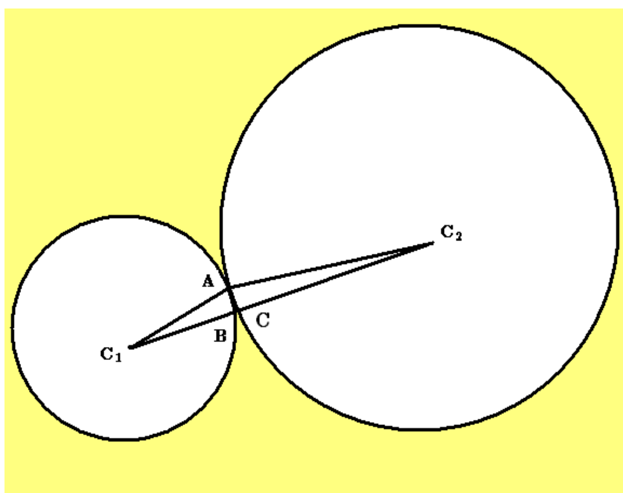
Properties of geometric figures

A proof by contradiction is an indirect proof where one begins with a hypothesis which is contrary to that which you want to prove. One then shows that this assumption leads to a contradiction. This type of proof is known as **reductio ad absurdum** which is Latin for reduction to absurdity.

Example 5-1. **Theorem** If two circles touch one another externally, the straight line joining the centers will pass through the point of contact. (Euclid, book 3, proposition 12)

Proof:

Let A denote the point of contact of the two circles and assume that the line connecting the centers c_1, c_2 of the circles does not pass through the point A.



If this hypothesis is true, then the line connecting the centers is divided into the three segments

$$\overline{c_1B} + \overline{BC} + \overline{Cc_2} = \overline{c_1c_2} \tag{5.1}$$

If c_1 is the center of the first circle, then $\overline{c_1A} = \overline{c_1B}$ because both line segments are radii of circle 1. Similarly, for circle 2 one can say, $\overline{c_2A} = \overline{c_2C}$ since both segments are radii of circle 2. Observe that from these equalities and the equation (5.1) one can write the inequality

$$\overline{c_1A} + \overline{c_2A} < \overline{c_1c_2} \tag{5.2}$$

By proposition 20, book 1 of Euclid's Elements, the sum of two sides of a triangle is always greater than the third side so that one can write

$$\overline{c_1 A} + \overline{c_2 A} > \overline{c_1 c_2} \tag{5.3}$$

Both the inequalities (5.2) and (5.3) cannot hold simultaneously, so the original assumption is false and the line segment $\overline{c_1 c_2}$ does pass through the point of contact of the two circles.

■

Construct line through given point P which is parallel to a given line ℓ
(Euclid, Book 1, Proposition 31)

Here it is assumed that the given point P is non collinear with the given line ℓ.

1) Construct a line L through point P which intersects the given line ℓ at some point A and making a positive angle with line ℓ, as illustrated in the figure 5-1.

2) Place the needle of the compass at point A and draw an arc which intersect line ℓ at point C and intersects line L at point B.

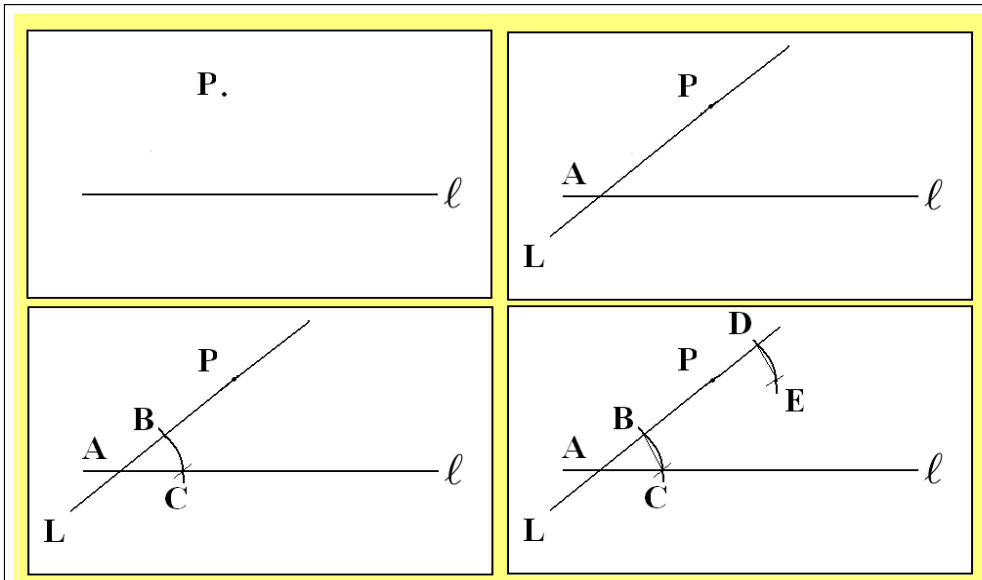


Figure 5-1.
Steps for constructing parallel line.

3) Using the same compass distance, place the compass needle at point P and make the same type of arc that you made in step 2 above and call the point of intersection with line L the point D.

4) Adjust the compass to measure the distance $\overline{BC}$ and then put the needle of the compass at point D and make an arc which intersects the previous arc at a point labeled E. Note that the distance $\overline{DE} = \overline{BC}$ in figure 5-1.

5) Use a straight edge to construct line through the points P and E and call this line ℓ'.

6) The line ℓ and ℓ' are parallel because the angles $\angle DPE$ and $\angle BAC$ are corresponding angles and are therefore equal. See figure 5-2 for details.

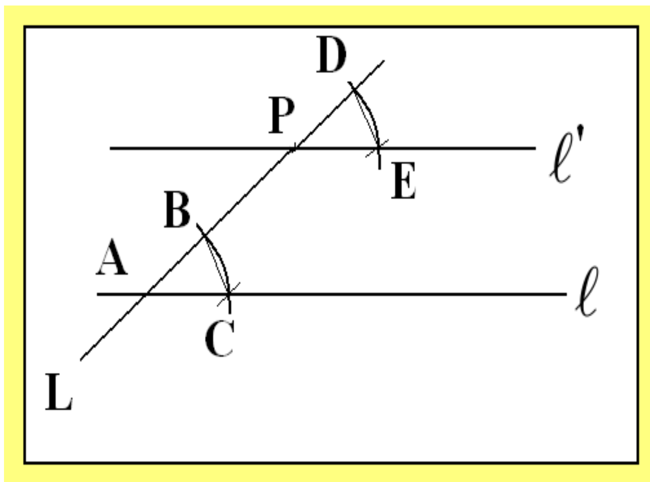


Figure 5-2.
Constructing parallel line to given line ℓ through given point P.

Divide a line segment into three equal parts
(Euclid, Book 6, Proposition 9)

1) Given the line segment $\overline{AB}$ one can construct a line ℓ which passes through point A as illustrated in the figure 5-3.

2) Set compass distance greater than what you think is 1/3 of the length $\overline{AB}$. Then place the needle of the compass at point A and make an arc on line ℓ and label it point C. Without changing the distance setting on the compass place the needle of the compass at point C and make a mark on line ℓ labeling it point D. Now place needle of compass at point D and make another mark on line ℓ labeling the mark point E. Examine figure 5-3 and note that the lengths $\overline{AC} = \overline{CD} = \overline{DE}$ are all equal if you haven't changed the compass spacing.

3) Put the needle of the compass at point E and widen the compass to make an arc which passes through the point B. Then without changing the compass spacing move the needle to point A and make a circular arc as illustrated in the figure 5-3.

4) Reset compass length by placing the needle at point A and the pencil point at point E so that the compass radius is set equal to the length $\overline{AE}$. After having set the compass distance, move the needle point to B and make an arc which intersect the arc constructed from 3) above and label the intersection point F as illustrated in figure 5-3.

5) Using a straight edge construct the line through points F and B and label this line ℓ'. Note that ℓ' is parallel to line ℓ.

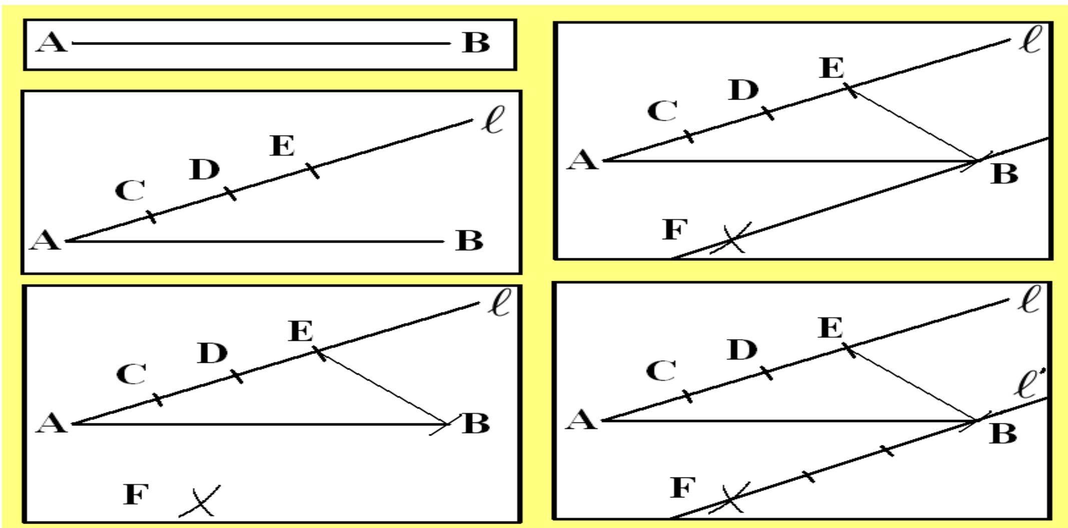


Figure 5-3.
Constructions for dividing a line segment.

6) Reset compass distance to the length $\overline{AC}$ and the place the compass needle at point F and make the marks G,H,I so that the distances $\overline{AC}, \overline{FG}, \overline{CD}, \overline{GH}, \overline{DE}, \overline{HB}$ are all equal to one another.

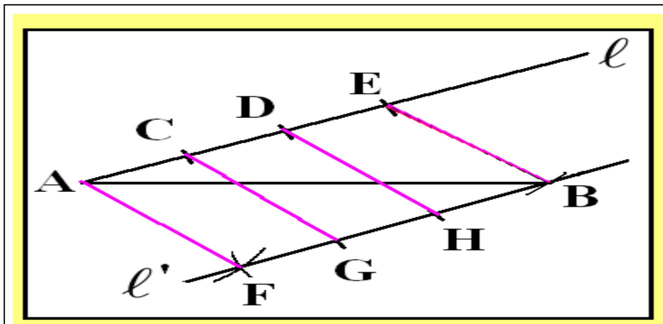


Figure 5-4.
Final steps for dividing line segment $\overline{AB}$.

7) Use a straight edge and construct the line segments $\overline{AF}, \overline{CG}, \overline{DH}, \overline{EB}$. These line segments intersect the line segment $\overline{AB}$ and divide it into three equal parts.

Note that this construction can be generalized so that the line segment $\overline{AB}$ can be divided into as many parts as you wish.

Example 5-2. Given a line segment $\overline{AB}$, find the point P between the points A and B such that the ratio $\dfrac{\overline{AP}}{\overline{PB}} = \dfrac{3}{2}$

Solution

Start at the point A and construct a line making an acute angle with the segment $\overline{AB}$ as illustrated in the figure 5-5. Then use a drawing compass with fixed distance between needle point and pencil point to mark off the 5 equal distances $\overline{AA_1} = \overline{A_1A_2} = \overline{A_2A_3} = \overline{A_3A_4} = \overline{A_4A_5}$.

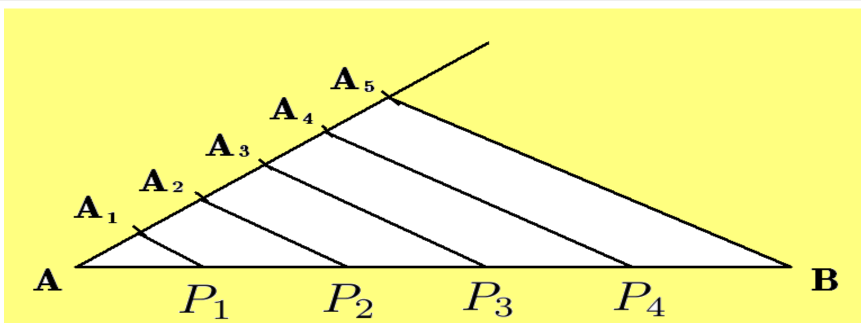


Figure 5-5.
Dividing line segment $\overline{AB}$ into two parts with ratio 3/2.

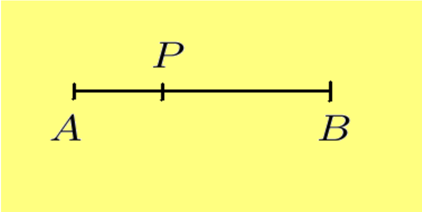
Connect the point A_5 with point B using a straight edge and then construct the parallel lines $\overline{A_4P_4} \parallel \overline{A_5B}$, $\overline{A_3P_3} \parallel \overline{A_5B}$, $\overline{A_2P_2} \parallel \overline{A_5B}$, $\overline{A_1P_1} \parallel \overline{A_5B}$. The point $P = P_3$ then divides the line segment $\overline{AB}$ into the 3/2 ratio requested.

■

Section formula

The previous example can be generalized into the problem of **dividing a line segment $\overline{AB}$ into two parts having a specified ratio r_1/r_2.** That is, find the coordinates of a point P between the endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ of a line segment, such that the ratio $\dfrac{\overline{AP}}{\overline{PB}} = \dfrac{r_1}{r_2}$ is satisfied, where r_1 and r_2 are specified numbers. The resulting formula for the coordinates of the point P is called **a section formula**, because it divides the given line segment into specified section lengths.

A section formula is also known as a ratio of division. A point P is selected **between two known points** A **and** B, then one can form the ratio of the positive line segments $\overline{\text{AP}}$ and $\overline{\text{PB}}$ to obtain the ratio $\frac{\overline{\text{AP}}}{\overline{\text{PB}}} = \frac{r_1}{r_2}$ which is called the ratio of division and the point P is called the point of division.



The ratio of division will always be positive if P is between A and B.

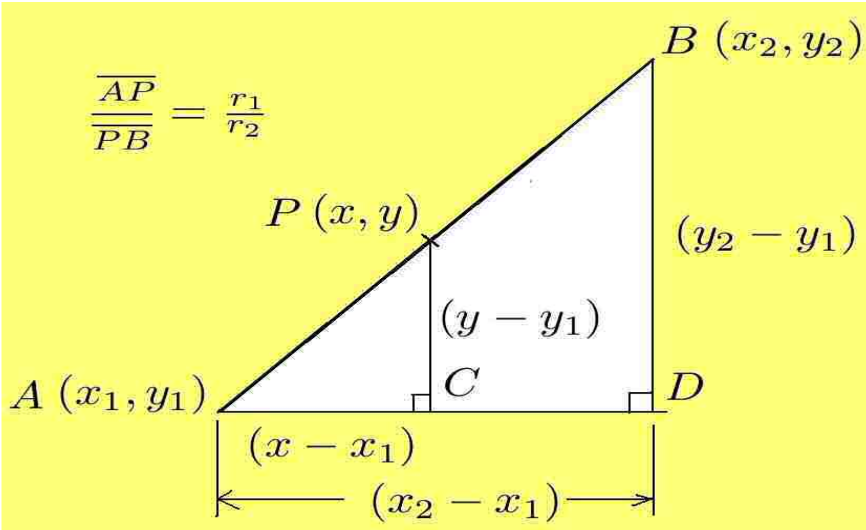


Figure 5-6. Dividing line segment $\overline{AB}$ into two parts with ratio $\frac{\overline{AP}}{\overline{PB}} = \frac{r_1}{r_2}$

As illustrated in the figure 5-6, construct the right triangles $\triangle ABD$ and $\triangle APC$ which are similar triangles with proportional sides. Examine these triangles and show

$$\frac{x - x_1}{x_2 - x_1} = \frac{\overline{AP}}{\overline{AB}} \qquad \text{and} \qquad \frac{y - y_1}{y_2 - y_1} = \frac{\overline{AP}}{\overline{AB}} \tag{5.4}$$

It is required that $\frac{\overline{AP}}{\overline{PB}} = \frac{r_1}{r_2}$. This can be expressed in the form

$$\frac{\overline{\text{AP}}}{\overline{\text{PB}}} + \frac{\overline{\text{PB}}}{\overline{\text{PB}}} = \frac{r_1}{r_2} + 1 \;\Rightarrow\; \frac{\overline{\text{AB}}}{\overline{\text{PB}}} = \frac{r_1 + r_2}{r_2} = \frac{\overline{\text{AB}}}{\overline{\text{AB}} - \overline{\text{AP}}} = \frac{1}{1 - \frac{\overline{\text{AP}}}{\overline{\text{AB}}}}$$

or

$$\frac{\overline{\text{AP}}}{\overline{\text{AB}}} = 1 - \frac{r_2}{r_1 + r_2} = \frac{r_1}{r_1 + r_2}$$

so that one can rewrite the equations (5.4) as

$$\frac{x - x_1}{x_2 - x_1} = \frac{r_1}{r_1 + r_2} \qquad \text{and} \qquad \frac{y - y_1}{y_2 - y_1} = \frac{r_1}{r_1 + r_2} \tag{5.5}$$

Use algebra to solve for x and y and show

$$x = \frac{r_2 x_1 + r_1 x_2}{r_1 + r_2}, \qquad y = \frac{r_2 y_1 + r_1 y_2}{r_1 + r_2} \tag{5.6}$$

The equations (5.6) are called the **section formulas** for determining the coordinates of the point P given the values $(x_1, y_1), (x_2, y_2)$ together with the specified ratio r_1/r_2.

Note the special case where $r_2 = r_1 = 1$ so that the point P is the midpoint of the line segment $\overline{AB}$. In this special case the equation (5.6) reduces to the midpoint formula produced in chapter 1.

$$x = \frac{1}{2}(x_1 + x_2), \qquad y = \frac{1}{2}(y_1 + y_2) \tag{5.7}$$

Example 5-3. The median of a triangle is a line segment that joins a vertex of the triangle to the midpoint of the opposite side of the triangle. Every triangle has three medians. **(Theorem) Show that the median of a triangle divides the area of the triangle into equal halves.**

Solution

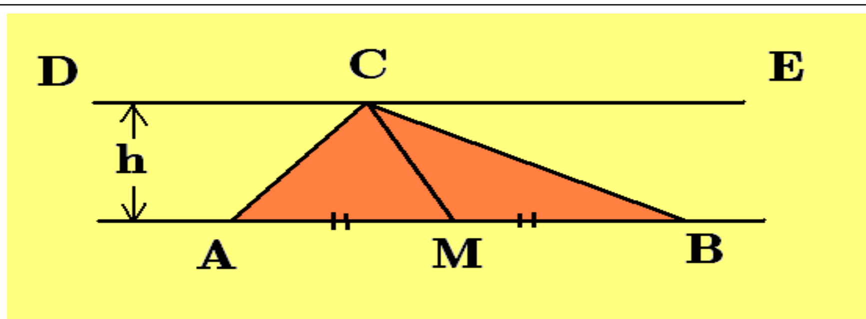


Figure 5-7. Median line from vertex C to midpoint M of $\overline{AB}$

Given the triangle ABC, find the midpoint M of $\overline{AB}$ and construct the median line $\overline{CM}$. Next construct a line through the vertex C which is parallel to the segment $\overline{AB}$ as illustrated in the above figure. If h is the distance between the parallel line segments $\overline{DE}$ and $\overline{AB}$, then one can write

$$\begin{aligned}
\text{Area } \triangle ABC = [ABC] =& \frac{1}{2}\overline{AB} \cdot h = \frac{1}{2}\left(\overline{AM} + \overline{MB}\right) \cdot h \\
=& \frac{1}{2}\overline{AM} \cdot h + \frac{1}{2}\overline{MB} \cdot h \\
\text{Area } \triangle CAM = [CAM] =& \frac{1}{2}\overline{AM} \cdot h \\
\text{Area } \triangle CBM = [CBM] =& \frac{1}{2}\overline{MB} \cdot h
\end{aligned}$$

Note that $\overline{AM} = \overline{MB}$ because M is the midpoint of the segment $\overline{AB}$ by construction. Therefore,

$$\text{Area } \triangle CAM = \text{Area } \triangle CBM = \frac{1}{2}\text{Area } \triangle ABC$$

This demonstrates that the median of a triangle divides the triangle into two equal areas. These equal areas have equal bases $\overline{AM} = \overline{MB}$ and the same height h.

■

Example 5-4. (Theorem) Show the three medians of a triangle meet at a point of concurrency. This point of common intersections is called the **centroid of the triangle** or center of gravity and is sometimes referred to as the barycenter of the triangle.

Solution

Use Cartesian coordinates and represent the triangle as illustrated.

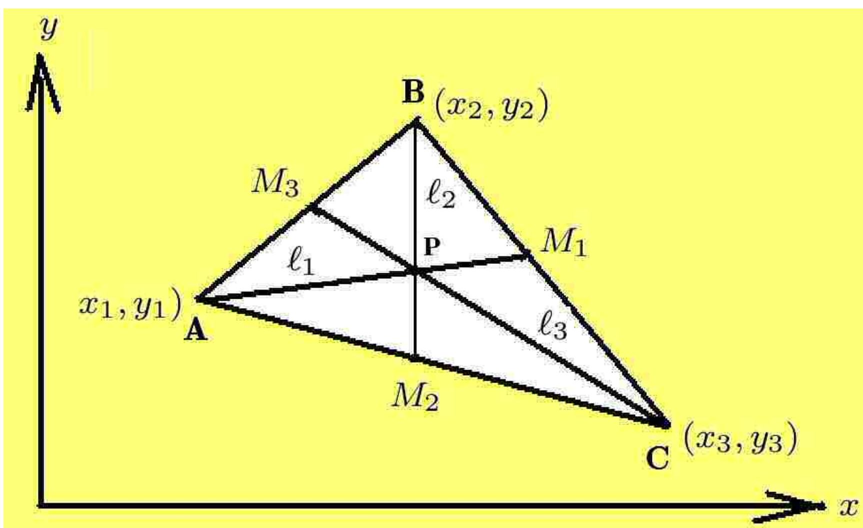


Figure 5-8. Median lines ℓ_1, ℓ_2, ℓ_3 from each vertex to midpoint of opposite sides.

Construct a general triangle $\triangle ABC$ as illustrated in the figure 5-8 having Cartesian coordinates (x_1, y_1) for vertex A, (x_2, y_2) for vertex B and (x_3, y_3) for the vertex C. Recall that the midpoint of a line segment is determined by taking the average of the endpoint x-values and average of the endpoint y-values. This gives the coordinates of the midpoints as

$$M_1 = \left(\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3)\right)$$
$$M_2 = \left(\frac{1}{2}(x_1 + x_3), \frac{1}{2}(y_1 + y_3)\right)$$
$$M_3 = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$$

Using the point-slope formula one can determine the equations of the median lines ℓ_1, ℓ_2 and ℓ_3 as follows.

$$\text{Line } \ell_1: \text{ slope of line } \ell_1 = m_1 = \frac{\text{change in y}}{\text{change in x}} = \left[\frac{\frac{1}{2}(y_2 + y_3) - y_1}{\frac{1}{2}(x_2 + x_3) - x_1}\right] \tag{5.8}$$

point-slope formula for equation of line ℓ_1

$$y - y_1 = \left[\frac{\frac{1}{2}(y_2 + y_3) - y_1}{\frac{1}{2}(x_2 + x_3) - x_1}\right](x - x_1) \tag{5.9}$$

$$\text{Line } \ell_2: \text{ slope of line } \ell_2 = m_2 = \frac{\text{change in y}}{\text{change in x}} = \left[\frac{\frac{1}{2}(y_1 + y_3) - y_2}{\frac{1}{2}(x_1 + x_3) - x_2}\right] \tag{5.10}$$

point-slope formula for equation of line ℓ_2

$$y - y_2 = \left[\frac{\frac{1}{2}(y_1 + y_3) - y_2}{\frac{1}{2}(x_1 + x_3) - x_2}\right](x - x_2) \tag{5.11}$$

$$\text{Line } \ell_3: \text{ slope of line } \ell_3 = m_3 = \frac{\text{change in y}}{\text{change in x}} = \left[\frac{y_3 - \frac{1}{2}(y_1 + y_2)}{x_3 - \frac{1}{2}(x_1 + x_2)}\right] \tag{5.12}$$

point-slope formula for equation of line ℓ_3

$$y - y_3 = \left[\frac{y_3 - \frac{1}{2}(y_1 + y_2)}{x_3 - \frac{1}{2}(x_1 + x_2)}\right](x - x_3) \tag{5.13}$$

To show that the median lines meet at a point of concurrency P, first find where the lines ℓ_1 and ℓ_2 intersect by solving the simultaneous equations (5.9) and (5.11).

Next, determine the intersection of the lines ℓ_1 and ℓ_3. This requires one to solve the simultaneous equations (5.9) and (5.13).

One can now verify that the intersection of the lines ℓ_2 and ℓ_3 occurs at the point satisfying the simultaneous equations (5.11) and (5.13).

One finds, after some algebra, that the above simultaneous equations all have the same solution for the coordinates of the point of concurrency as

$$x_G = \frac{1}{3}(x_1 + x_2 + x_3), \qquad y_G = \frac{1}{3}(y_1 + y_2 + y_3) \tag{5.14}$$

This demonstrates that the point of concurrency of the three lines ℓ_1, ℓ_2 and ℓ_3 occurs at the point P having the coordinates (x_G, y_G), where the x_G and y_G values are obtained from equation (5.14). This point of concurrency is called **the centroid** of the triangle ABC.

Let points D and E denote the midpoints of the line segments $\overline{\text{AP}}$ and $\overline{\text{CP}}$. The coordinates of these points are obtained from the midpoint formula as

$$D: \left[\frac{1}{2}\left(x_1 + \frac{1}{3}(x_1+x_2+x_3)\right), \frac{1}{2}\left(y_1 + \frac{1}{3}(y_1+y_2+y_3)\right)\right)$$

and

$$E: \left[\frac{1}{2}\left(x_3 + \frac{1}{3}(x_1+x_2+x_3)\right), \frac{1}{2}\left(y_3 + \frac{1}{3}(y_1+y_2+y_3)\right)\right]$$

Knowing these coordinates we can calculate the slopes of the line segments $\overline{M_1E}$ and $\overline{M_1D}$ to show these line segments are parallel. One finds

$$\text{Slope } \overline{M_1E} = \frac{\text{change in y-values}}{\text{change in x-values}} = \frac{y_1 - 2y_2 + y_3}{x_1 - 2x_2 + x_3}$$
$$\text{Slope } \overline{M_3D} = \frac{\text{change in y-values}}{\text{change in x-values}} = \frac{y_1 - 2y_2 + y_3}{x_1 - 2x_2 + x_3}$$

provided $x_1 - 2x_2 + x_3 \neq 0$. If it is zero, then rotate the triangle to remove this problem.

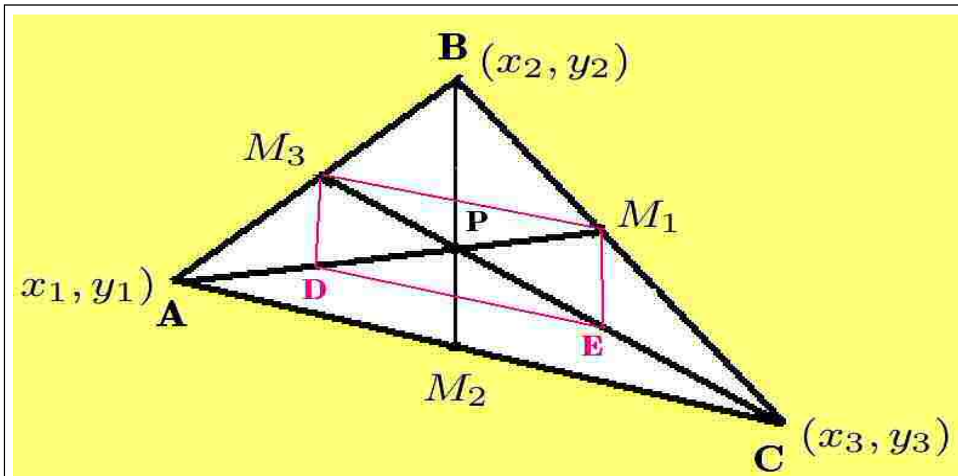


Figure 5-9.
Quadrilateral M_3DEM_1 is a parallelogram

The above calculations demonstrate the line segments $\overline{M_1E}$ and $\overline{M_3D}$ are parallel. By the midpoint parallel line theorem $\overline{M_3M_1} \parallel \overline{\text{AC}}$. Also the line segment $\overline{\text{DE}}$ is parallel to $\overline{\text{AC}}$ and 1/2 its length by the mid segment theorem. Therefore $\overline{\text{DE}} \parallel \overline{M_3M_1}$ and consequently the quadrilateral M_3DEM_1 is a parallelogram where we know the diagonals of a parallelogram meet at a point and bisect one another.

Therefore one can write

$$\overline{M_3P} = \overline{\text{PE}} = \overline{\text{EC}} = \frac{1}{3}\overline{CM_3} \quad \text{or} \quad \overline{\text{AP}} = \frac{2}{3}\overline{AM_1}$$

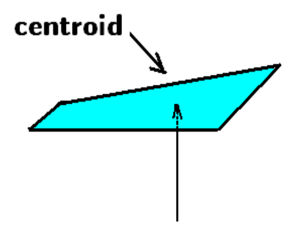
By rotating the triangle one can repeat what has just been done and verify that

$$\overline{M_2P} = \overline{\text{PE}} = \overline{\text{FB}} = \frac{1}{3}\overline{BM_2} \quad \text{or} \quad \overline{\text{BP}} = \frac{2}{3}\overline{BM_2}$$
$$\overline{M_1P} = \overline{\text{PD}} = \overline{\text{DA}} = \frac{1}{3}\overline{AM_1} \quad \text{or} \quad \overline{\text{CP}} = \frac{2}{3}\overline{CM_3}$$

This demonstrates the medians of a triangle will always meet at a point of concurrency, called the centroid, which is located **two thirds of the distance along the lines from each vertex to the midpoints of the opposite sides.** ∎

Centroid of any figure

The centroid of any plane figure is the average position of all the points of the figure. For example, the centroid of a quadrilateral is the average of all points within the quadrilateral and so one could balance the quadrilateral on a pin placed at the centroid.



Thale's intercept theorem

Given two parallel lines which are intersected by two lines ℓ_1 and ℓ_2 intersecting at point P as illustrated in the figure 5-10. Thales theorem of intercepts consists of three ratio properties associated with the given situation.

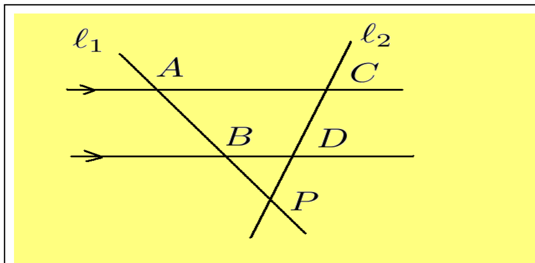


Figure 5-10.
Two lines intersecting parallel lines.

(i) The ratio of any two line segments on line ℓ_1 must equal the corresponding ratio on line ℓ_2. For example,

$$\frac{\overline{PB}}{\overline{PA}} = \frac{\overline{PD}}{\overline{PC}}, \qquad \frac{\overline{PA}}{\overline{BA}} = \frac{\overline{PC}}{\overline{DC}}, \qquad \frac{\overline{PB}}{\overline{BA}} = \frac{\overline{PD}}{\overline{DC}}$$

(ii) The ratio of any two line segments on line ℓ_1 starting at P, equals the ratio of the segments on the parallels.

$$\frac{\overline{PB}}{\overline{PA}} = \frac{\overline{PD}}{\overline{PC}} = \frac{\overline{BD}}{\overline{AC}}$$

(iii) If more than two lines cross the parallel lines and intersect at a point P, then the ratio of any two segments on one parallel line is in the same ratio as the corresponding segment on the other parallel line.

$$\frac{\overline{BA}}{\overline{EF}} = \frac{\overline{BC}}{\overline{ED}}, \qquad \frac{\overline{BA}}{\overline{BC}} = \frac{\overline{EF}}{\overline{ED}}$$

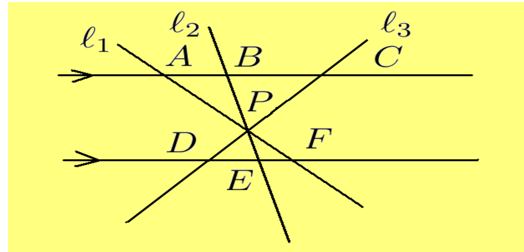


Figure 5-11.
Three lines intersecting parallel lines.

Proof of the above theorem is based upon producing similar triangles from which all of the above proportions are obtained.

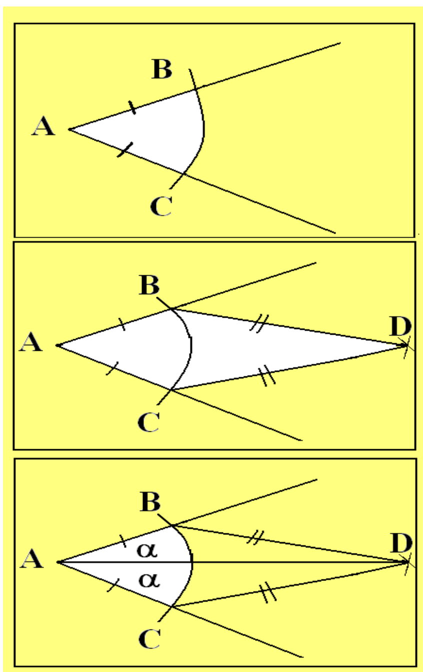


Figure 5-12.
Bisection of angle

Bisect a given angle
(Euclid, Book 1, Proposition 9)

1) Place the needle of the compass at the vertex A and sketch the arc $\overset{\frown}{\text{CB}}$ where the points B and C are where the arc intersects with the initial and terminal sides of the given angle.

2) Select a constant compass distance and place the needle point at B and construct a small arc followed by placing the needle point at C and constructing an intersecting arc as illustrated in the figure 5-12. Label the point where the arcs intersect as point D.

3) Construct line segment $\overline{AD}$ which bisects the angle $\angle BAC$

The reason why the above construction bisects the angle is because the line segments $\overline{AB}, \overline{BC}, \overline{DC}, \overline{CA}$ form a kite with diagonal $\overline{AD}$. Then triangle ABD is congruent to triangle ACD ($\triangle ABD \cong \triangle ACD$) because of side-side-side, where $\overline{AD}$ is a common side to both triangles. Therefore, angle $\alpha = \angle DAC = \angle DAB$ which implies $2\alpha = \angle BAC$ or $\alpha = \frac{1}{2}\angle BAC$.

Example 5-5. (Theorem) **Show that in an isosceles triangle the angles opposite the equal sides are equal. These angles are called the base angles associated with an isosceles triangle.** (Euclid, Book 1, Proposition 5)

Solution

Consider the isosceles triangles sketched in the figure 5-13. Construct the bisector of the vertex angle A of the isosceles triangle illustrated in the figure 5-13. This bisector intersects the base line $\overline{BC}$ at the point M as illustrated in the figure 5-13. The two triangles formed by the angle bisection are congruent. That is, $\triangle AMB \cong \triangle AMC$ because of side-angle-side, with $\overline{\text{AM}}$ a common side to both right triangles. Here $\overline{AB} = \overline{AC}$ because the given triangle is isosceles. The angles $\alpha = \angle BAM = \angle CAM$ are equal because $\overline{\text{AM}}$ is an angle bisector.

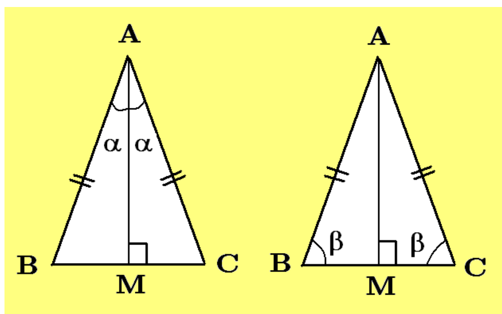


Figure 5-13. Isosceles triangles

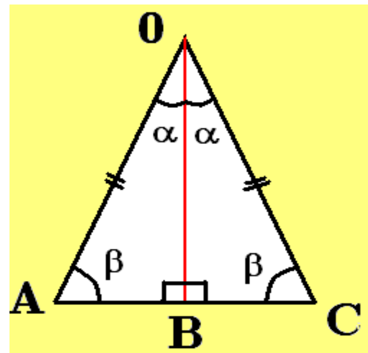
Since the triangles are congruent all the corresponding interior angles are equal so the base angles $\angle ABM = \angle ACM = \beta$. Note also that the angle bisector intersects the base line $\overline{BC}$ perpendicularly. This is because the angles $\angle AMB$ and $\angle AMC$ are equal and they are supplementary. Let $\gamma = \angle AMB = \angle AMC$, then one can write $\gamma + \gamma = 180^\circ$ or $\gamma = 90^\circ$.

Conversely, if the base angles $\angle ABM$ and $\angle ACM$ are equal, then the triangle must be an isosceles triangle. (Euclid, Book 1, Proposition 6) Let us assume these base angles are equal and show that this implies that $\overline{AB} = \overline{AC}$ or the sides opposite the base angles are equal. To prove this, construct the line through vertex A which is perpendicular to the triangle base $\overline{BC}$. One can then say that the triangles $\triangle ABM$ and $\triangle ACM$ are congruent because of angle-side-angle with $\overline{\text{AM}}$ the common side and the angle α are known since there has to be 180° in every triangle. If $\triangle ABM \cong \triangle ACM$, the side $\overline{AB} = \overline{AC}$ or the sides opposite the equal base angles are equal.

Note also that the line $\overline{\text{AM}}$ is called the **altitude of the isosceles triangle.** This altitude is the perpendicular bisector of the base $\overline{BC}$ and also is the angle bisector of the vertex angle A. ■

Two column proof of isosceles triangle properties

The construction of an altitude from the vertex angle $\angle A0C$ of the isosceles triangle illustrated produces two right triangles $\triangle 0BA$ and $\triangle 0BC$. One can state that these triangles are congruent $(\triangle 0AB \cong \triangle 0BC)$ because of the RHS (Right triangle-Hypotenuse-Side) proposition, where the altitude $\overline{OB}$ is a shared side of the two right triangles.



One can then conclude that for an isosceles triangle

1) The angles $\alpha = \angle A0B = \angle C0B$, so that the altitude bisects the vertex angle $\angle A0C$

2) The base of the isosceles triangle is bisected $\overline{\text{AB}} \cong \overline{\text{BC}}$

3) The base angles are equal $\beta = \angle 0AB = \angle 0BC$

The following is a (Statement | Reason) two column proof of the above facts.

	Statements	Reasons
1.	$0B$ is altitude of $\triangle 0AC$	By construction
2.	$0B$ is common to two right triangles	By construction
3.	$\overline{\text{OA}} = \overline{\text{0C}}$	Given $\triangle 0AC$ is isosceles
4.	$\triangle A0B \cong \triangle C0B$	SAS (side-angle-side) postulate
5.	$\overline{\text{AB}} = \overline{\text{BC}}$	Corresponding parts of congruent triangles are equal
6.	$\angle A0B = \angle C0B = \alpha$	Corresponding parts of congruent triangles are equal
7.	$\angle BA0 = \angle BC0 = \beta$	Corresponding parts of congruent triangles are equal

Construction of an equilateral triangle

An equilateral triangle has three equal sides and three equal vertex angles To construct an equilateral triangle having all sides of some given length $\overline{BC} = \ell$, set the drawing compass needle at point B and widen the compass until the pencil point is at the endpoint C, then make a circular arc with radius ℓ above the line. Next move the needle point of the compass to point C and construct another circular arc of radius ℓ which intersects the first arc at the point labeled A. Finally construct the line segments $\overline{\text{AC}}$ and $\overline{\text{AB}}$ to produce the equilateral triangle.

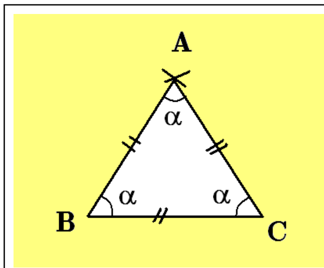


Figure 5-14.
Equilateral triangle

Consider the equilateral triangle $\triangle ABC$ illustrated in the figure 5-15. Let the sides have length s and denote the interior angles by the Greek letter α. We know that the interior angles of the equilateral triangle are equal because they are all opposite equal sides. We also know the sum of the interior angles of a triangle is $180°$, so $3\alpha = 180°$ or $\alpha = 60°$. We also know that if an angle bisector $\overline{AD}$ is constructed, then triangles $\triangle ABD$ and $\triangle ACD$ are congruent $(\triangle ABD \cong \triangle ACD)$ because of SAS (side-angle-side) with $\overline{AD}$ the common side. Therefore, $\overline{BD} = \overline{DC} = \frac{s}{2}$ and the angles $\angle ADB = \angle ADC = \frac{\pi}{2}$ so the line $\overline{AD}$ is perpendicular to $\overline{BC}$.

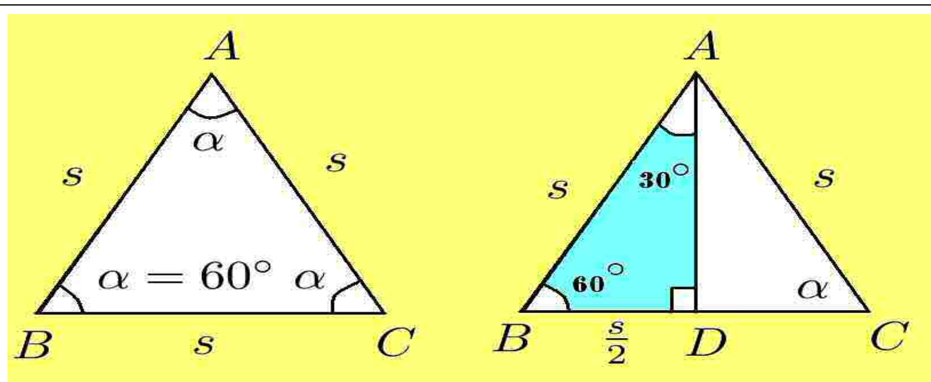


Figure 5-15.
Equilateral triangle $\triangle ABC$

Examine the figure 5-15 and remember the following fact.

In a $30° - 60°$ right triangle, the side opposite the $30°$ angle will always be half the hypotenuse.

The Viviani theorem

Vincenzo Viviani (1622-1703) was a contemporary of Galileo who studied equilateral triangles and came up with the following observation. In an equilateral triangle with all sides of length s, one can select **any interior point** P and then construct the perpendicular distances from point P to each side of the triangle as illustrated in the figure 5-16. Label the lengths of these perpendicular distances as h_1, h_2 and h_3 and call the height of the equilateral triangle h. **The Viviani theorem states that** $h_1 + h_2 + h_3 = h$ or the height of the equilateral triangle must equal the sum of the perpendicular distances from point P. To prove this theorem, construct the line segments $\overline{PA}, \overline{PB}, \overline{PC}$ which represent the lines from point P to each vertex of the equilateral triangle. Note that

$$\text{Area triangle } \triangle ABC = A = \frac{1}{2}hs$$

This area is the sum of the three smaller triangles having the common vertex at point P.

$$A = \text{Areas of } \triangle APC + \triangle PBC + \triangle BPA$$
$$A = \frac{1}{2}h_1 s + \frac{1}{2}h_2 s + \frac{1}{2}h_3 s = \frac{1}{2}(h_1 + h_2 + h_3)\, s$$

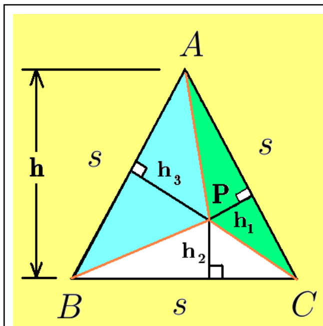


Figure 5-16.
Equilateral triangle

Things equal to the same thing are equal to one another or

$$\frac{1}{2}hs = \frac{1}{2}(h_1 + h_2 + h_3)s$$

which simplifies to the result

$$h = h_1 + h_2 + h_3$$

proving the Viviani theorem.

Angle bisector theorem

(Euclid, Book 6, Proposition 3)

The angle bisector theorem involves **a line bisecting an interior angle of a triangle** and cutting **the side opposite the angle** at a point such that the triangle is divided into two parts where **the sides of the two parts are proportional to one another.** To illustrate where these ratios come from, consider the triangle ABC illustrated in the figure 5-17.

If the line which bisects the angle $\angle CAB$ intersects side $\overline{CB}$ at the point D, then the following ratio must hold true

$$\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{BD}}{\overline{CD}}$$

Proof

Construct a line through the vertex C which is parallel to the triangle base $\overline{\text{AB}}$ and then extend the line $\overline{\text{AD}}$ to intersect this line at point E. The line $\overline{AE}$ is a transversal line cutting two parallel lines and so angle $\angle DEC = \alpha$. Similarly, the line $\overline{BC}$ is a transversal line cutting two parallel lines and so angle $\angle ECD = \angle ABD = \gamma$.

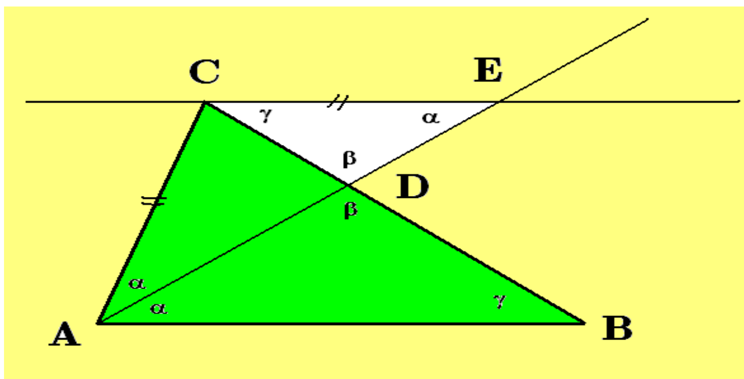


Figure 5-17.
Line segment $\overline{AE}$ bisects angle $\angle CAB$

The triangle $\triangle AEC$ is an isosceles triangle with base angles α so one can say that the sides opposite these angles are equal or $\overline{AC} = \overline{CE}$. Also the triangles $\triangle CED$ and $\triangle ABD$ are similar triangles because all three interior angles are the same. Consequently, the sides of these triangles are proportional giving the ratios

$$\frac{\overline{AB}}{\overline{CE}} = \frac{\overline{BD}}{\overline{CD}}$$

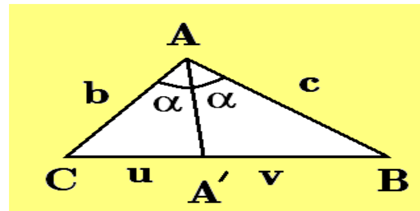
However, $\overline{CE} = \overline{AC}$ giving the ratio

$$\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{BD}}{\overline{CD}}$$

Make note that this theorem is employed in many problems where an angle bisector is involved.

Example 5-6. Special case cevian is angle bisector

In the special case the cevian of a triangle is also an angle bisector of the vertex angle A, then one can simplify the Stewart's formula for the length of the cevian $\overline{AA'}$. One can employ the angle bisector theorem that states $\frac{b}{u} = \frac{c}{v}$. Write Stewart's formula as



$$b^2 + \frac{u}{v}c^2 = \frac{a}{v}\left(d^2 + uv\right)$$

then use $\frac{u}{v}c = b$ to obtain

$$b^2 + bc = \frac{a}{v}\left(d^2 + uv\right) \tag{5.15}$$

Modify the angle bisector theorem

$$\frac{c}{b} + 1 = \frac{v}{u} + 1 \quad\Rightarrow\quad \frac{b+c}{b} = \frac{u+v}{u} \quad\Rightarrow\quad (b+c)^2 = \frac{a^2b^2}{u^2} \tag{5.16}$$

to simplify equation (5.15) as follows.

$$\begin{aligned} b(b+c) =& \frac{a}{v}\left(d^2 + uv\right) \\ (b+c)^2 =& \frac{a}{bv}\frac{b(u+v)}{u}\left(d^2 + uv\right) \\ (b+c)^2 =& \frac{a^2}{uv}\left(d^2 + uv\right) \end{aligned} \tag{5.17}$$

The results from equation (5.16) can now be used to show

$$\begin{aligned} \frac{a^2b^2}{u^2} =& \frac{a^2}{uv}\left(d^2 + uv\right) \\ \frac{b^2v}{u} =& bc = d^2 + uv \end{aligned}$$

This gives the angle bisector form of Stewart's formula as

$$d^2 = bc - uv = bc - a^2\alpha(1-\alpha), \qquad u = \alpha a,\ v = (1-\alpha)a \tag{5.18}$$

The equation (5.18) can be expressed in terms of the semiperimeter of the triangle $\triangle ABC$. Let $s = \frac{a+b+c}{2}$ denote the semiperimeter and show that

$$s(s-a) = \frac{(b+c)^2 - a^2}{4} \tag{5.19}$$

Note that the equation (5.17) can be rearranged into the form

$$(b+c)^2 - a^2 = \frac{a^2d^2}{uv} \tag{5.20}$$

where now the equation (5.19) can be used to obtain a semiperimeter form for the distance of the cevian $d = \overline{AA'}$ as

$$d = \frac{2\sqrt{bc\, s(s-a)}}{b+c} \tag{5.21}$$

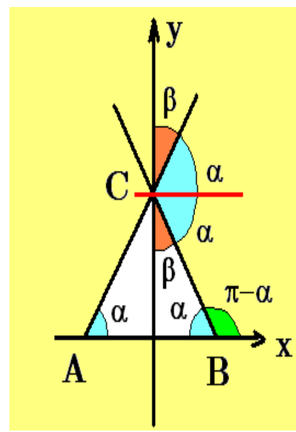
As a final note in this special case, we find from the angle bisector theorem

$$\frac{u}{v} = \frac{b}{c} = \frac{\alpha a}{(1-\alpha)a} = \frac{\alpha}{(1-\alpha)}$$

which shows that the side a of the triangle is split by the cevian into two parts having the ratio $\frac{b}{c}$.

■

Exterior angle bisectors



Examine the exterior angle bisector associated with the exterior angle C of the isosceles triangle $\triangle ABC$ illustrated. Note that the angle bisector of this exterior angle is parallel to the base $\overline{\text{AB}}$. Consequently the exterior angle bisector will never intersect the extended base $\overline{\text{AB}}$.

One can examine the angle bisectors of the exterior angles of any given non-isosceles triangle $\triangle ABC$ as illustrated in the figure 5-18. One can show that when the exterior angles of a non-isosceles triangle are bisected, the bisector lines will intersect with an extended side of the triangle. These points of intersection are illustrated by the points P, Q, R in figure 5-18. Note that any exterior angle bisector associated with a given triangle will always intersect with an extended opposite side of the triangle and create two line segments. For example, the bisector of angle $(\pi - A)$ intersects the extended side $\overline{\text{BC}}$ at point P to create the two segments $\overline{\text{CB}}$ and $\overline{\text{CP}}$. The ratio

$$\frac{\overline{\text{CP}}}{\overline{\text{CP}} + \overline{\text{CB}}} = \frac{\overline{\text{CP}}}{\overline{\text{BP}}}$$

is proportional to the lengths of the other two sides of the triangle or $\dfrac{\overline{\text{AC}}}{\overline{\text{AB}}}$.

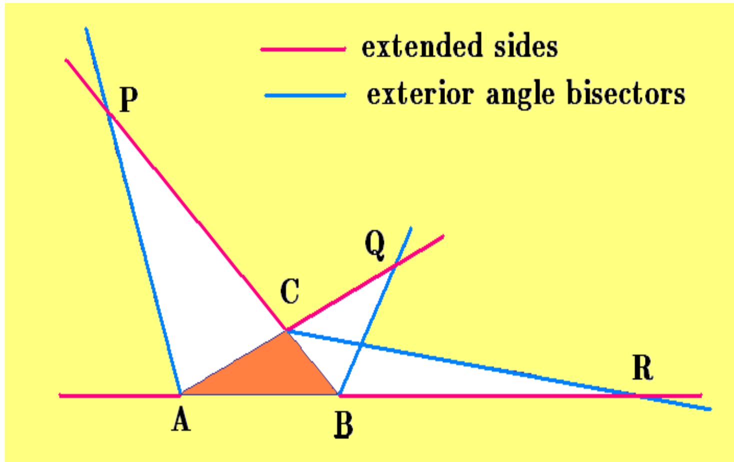
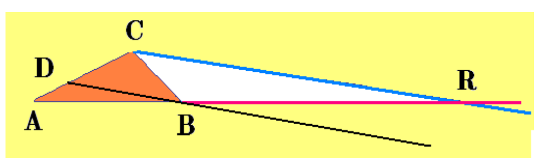


Figure 5-18.
$\triangle ABC$ with extended sides and exterior angle bisectors

One can express the external angle bisector theorem for the exterior angles as producing the ratios

$$\begin{array}{lll}
\text{bisector of angle } (\pi - A) \text{ creates the ratio} & \dfrac{\overline{CP}}{\overline{BP}} = \dfrac{\overline{AC}}{\overline{AB}} & \triangle PAB \\
\text{bisector of angle } (\pi - B) \text{ creates the ratio} & \dfrac{\overline{CQ}}{\overline{AQ}} = \dfrac{\overline{CB}}{\overline{AB}} & \triangle AQB \\
\text{bisector of angle } (\pi - C) \text{ creates the ratio} & \dfrac{\overline{BR}}{\overline{AR}} = \dfrac{\overline{CB}}{\overline{AC}} & \triangle ACR
\end{array}$$

To prove the last ratio construct a line through point B which is parallel to the angle bisector $\overline{CR}$ and intersects side $\overline{AC}$ at point D as illustrated below.



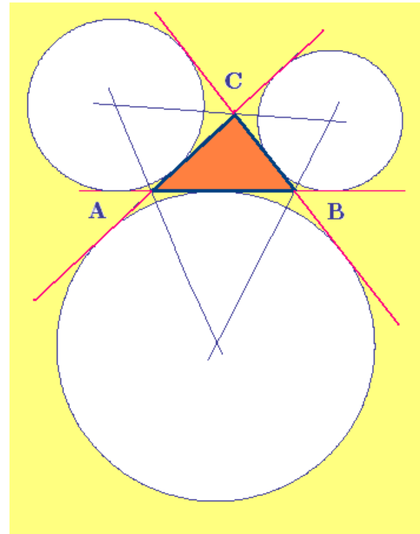
Observe that $\overline{CB}$ becomes a transversal so that one can show the triangle $\triangle DCB$ is isosceles. This implies $\overline{CD} = \overline{CB}$. The parallel lines cut the triangle $\triangle ACR$ so that the sides are proportional giving

$$\frac{\overline{CA}}{\overline{CD}} = \frac{\overline{AR}}{\overline{BR}} \quad \Rightarrow \quad \frac{\overline{CA}}{\overline{CB}} = \frac{\overline{AR}}{\overline{BR}}$$

since $\overline{CD} = \overline{CB}$.

Similar arguments produce the other ratios associated with the bisection of the external angles.

Excircles



Any circle which is tangent to both the **extended sides of a triangle** and a side of the triangle is called an **excircle**. The centers of excircles occur at the intersection of the exterior angle bisectors and are called **excenters**. The radius of an excircle is called an exradius. Every non-isosceles triangle has three excircles.

Example 5-7. (Theorem) Show that a point P which lies on an angle bisector must be equidistant from the sides defining the angle.

(Illustrated in Euclid, Book 4, Proposition 4)

Solution

Construct the lines $\overline{PE}$ and $\overline{PF}$ in figure 5-19 which are perpendicular to the sides defining the angle A. The resulting triangles $\triangle APE$ and $\triangle APF$ are right angle triangles and are such that

$$\angle APE = \angle APF = 90° - \alpha = \beta$$

since there must be $180°$ in every triangle. Note also the line segment $\overline{AP}$ is common to the triangles $\triangle APF$ and $\triangle APE$ and consequently one can state that these triangles are congruent.

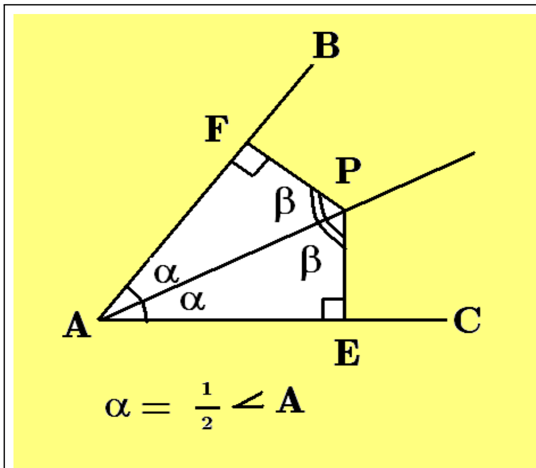


Figure 5-19.
Point P on angle bisector.

The statement $\triangle APE \cong \triangle APF$ is true because of the angle-side-angle (ASA) postulate. Therefore, $\overline{PE} = \overline{PF}$ demonstrating that the point P is equidistant from the angle sides.

Note also that the converse of the above theorem also holds true.

If a point P on a line through angle A is equidistant from the sides defining the angle, then P must lie on an angle bisector.

In figure 5-19 construct the line $\overline{EF}$ as illustrated in the figure 5-20. This construction forms an isosceles triangle $\triangle PFE$ with equal base angles $\angle PFE = \angle PEF = \gamma$. The complementary angles $90° - \gamma = \delta$ associated with the angle γ are also equal to one another so that

$$\angle AFE = \angle AEF = \delta = 90° - \gamma$$

The sides opposite these angles must be equal. This produces another isosceles triangle $\triangle AFE$ with sides opposite the base angles being equal ($\overline{AF} = \overline{AE}$). The side $\overline{AP}$ is a common side and consequently one can write $\triangle APF \cong \triangle APE$ because of SSS (side-side-side postulate). Therefore, the angles $\angle FAP$ and $\angle EAP$ are equal and so the line $\overline{AP}$ is the angle bisector of the vertex angle A.

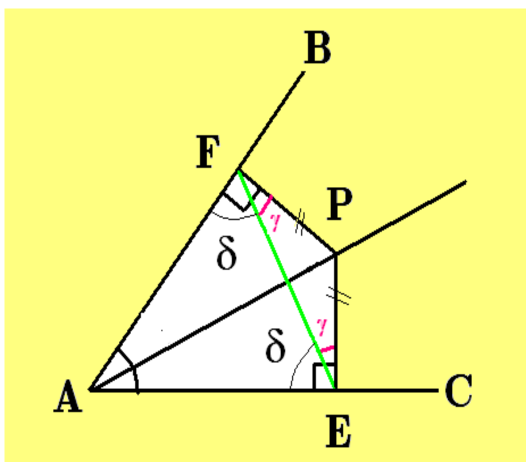


Figure 5-20.
Sides $\overline{PE} = \overline{PF}$

■

Example 5-8. **(Theorem) Show the angle bisectors of a triangle meet at point of concurrency, called the incenter, which is equidistant from the sides of the triangle.**

Solution

Given the triangle ABC construct the angle bisectors $\overline{AF}, \overline{BE}$ and $\overline{CH}$. The angle bisectors $\overline{AF}$ and $\overline{BE}$ intersect at a point P. We have shown that the point P, lying on an angle bisector, must be equidistant from the angle sides. Construct the perpendicular segments $\overline{PG}, \overline{PI}, \overline{PD}$ from the point P to the sides of the triangle. The point P lying on the angle bisector $\overline{AF}$ produces the equality $\overline{PI} = \overline{PD}$. The point P lying on the angle bisector $\overline{BE}$ produces the equality $\overline{PI} = \overline{PG}$. Therefore, $\overline{PD} = \overline{PI}$ because things equal to the same thing are equal to each other. This shows the point P is equidistant from the sides CB and CA and so it must lie on the angle bisector $\overline{\text{CH}}$.

The point P is a point of concurrency associated with the angular bisectors of triangle ABC and is called **the incenter.** Drawing a circle with center P having the radius $\overline{\text{PI}} = \overline{\text{PD}} = \overline{\text{PG}}$ produces **an inscribed circle** within the given triangle ABC. This result is related to Euclid, Book 4, Proposition 4.

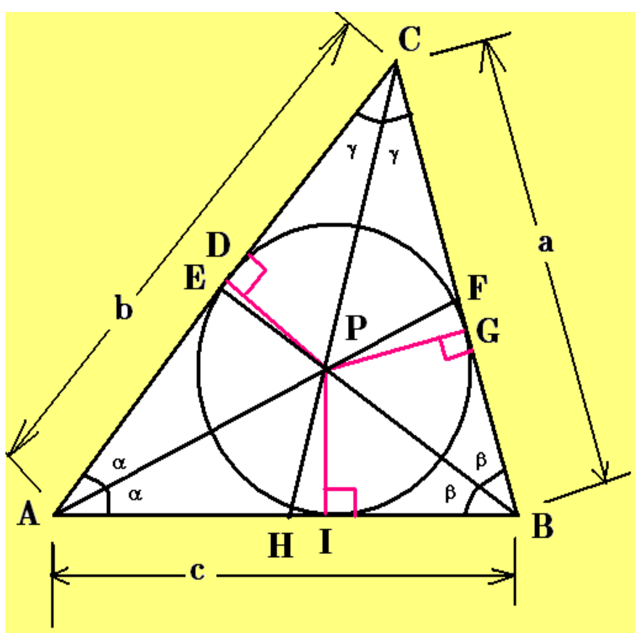


Figure 5-21.
Triangle ABC with point of concurrency.

Let $r = \overline{\text{PI}} = \overline{\text{PD}} = \overline{\text{PG}}$ denote the radius of the inscribed circle, then observe that the three triangles $\triangle PAB$, $\triangle PBC$, $\triangle PCA$ have the respective areas

$$[PAB] = \frac{1}{2}rc, \qquad [PBC] = \frac{1}{2}ra, \qquad [PCA] = \frac{1}{2}rb \tag{5.22}$$

where a, b, c are the lengths of the triangle sides. The total area of the triangle is then the sum of these smaller areas giving the area of triangle $\triangle ABC$ as

$$[ABC] = \frac{1}{2}r(a+b+c) = r\,s \quad \text{where } s = \frac{1}{2}(a+b+c) \tag{5.23}$$

is the semiperimeter of the triangle and r is the radius of the inscribed circle. The equation (5.23) represents the area of a general triangle as a product of the radius of the inscribed circle times the semiperimeter of the triangle.

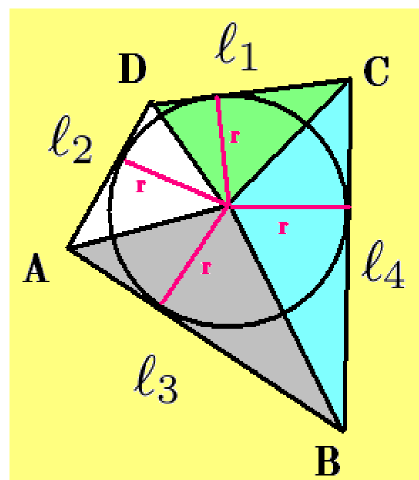
■

Example 5-9. As a corollary to the previous example, show that the area of any polygon circumscribed about a circle is given by the formula

$$\text{Area of circumscribed polygon} = \frac{1}{2}(\text{perimeter of polygon}) \cdot (\text{radius of circle}) \tag{5.24}$$

Solution:

Consider the quadrilateral $ABCD$ circumscribing a circle as illustrated in the accompanying figure. One can construct line segments from the center of the circle to each vertex of the polygon. This creates four triangles as illustrated. By using the results from the previous example, the area of each triangle is given the formula



$$\text{Triangle Area} = \frac{1}{2}(\text{base})(\text{height})$$

The total area of the quadrilateral is then the sum of the areas of each triangle or

$$\text{Area quadrilateral} = \frac{1}{2}\ell_1 r + \frac{1}{2}\ell_2 r + \frac{1}{2}\ell_3 r + \frac{1}{2}\ell_4 r = \frac{1}{2}(\ell_1+\ell_2+\ell_3+\ell_4)r = \frac{1}{2}rp \tag{5.25}$$

where p is the perimeter of the quadrilateral circumscribing the circle.

If you replace the quadrilateral with an n-gon then there results n-triangles and the perimeter of the n-gon becomes

$$p = \ell_1 + \ell_2 + \ell_3 + \cdots + \ell_n$$

with the area of the n-gon still given by the formula $\frac{1}{2}rp$, where r is the radius of the circle.

Example 5-10. Using Cartesian coordinates show the angle bisectors of the vertex angles A,B,C of a general triangle $\triangle ABC$ meet at a point of concurrency called the incenter. Find the coordinates of this point.

Solution

Assume the vertices A, B, C have the Cartesian coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and the angle bisector of the vertex angle A intersects the side $\overline{\text{CB}}$ at the point D with coordinates (x_4, y_4). The situation is illustrated in the figure 5-22. Using the **angle bisector theorem** one knows that the ratio

$$\frac{\overline{BD}}{\overline{DC}} = \frac{c}{b} \tag{5.26}$$

must be satisfied. The coordinates of the point D can be determined by using the **section formula** (3.15) from chapter 3, with $r_2 = b$ and $r_1 = c$. One finds the coordinates of the point (x_4, y_4) are given by

$$x_4 = \frac{bx_2 + cx_3}{b + c} \qquad \text{and} \qquad y_4 = \frac{by_2 + cy_3}{b + c} \tag{5.27}$$

The line bisecting the vertex angle B intersects the line segment $\overline{AD}$ at the point P called the incenter of triangle $\triangle ABC$.

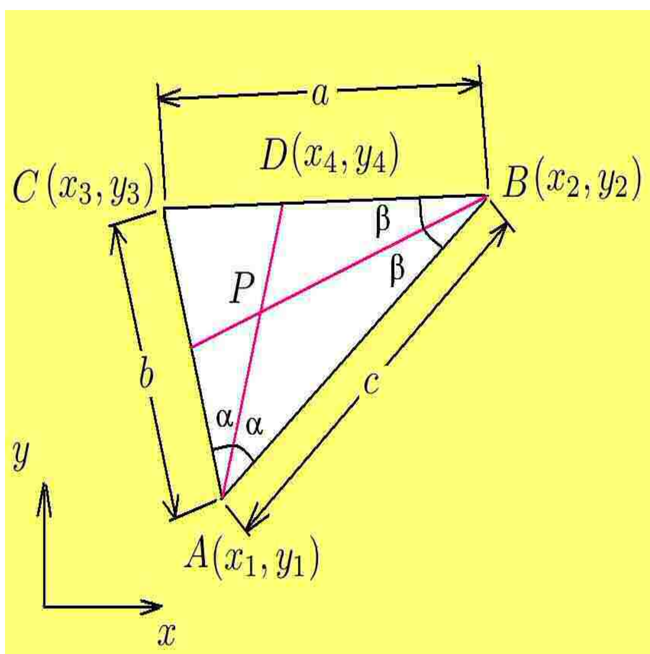


Figure 5-22. Angle bisectors of a triangle meet at a point P of concurrency.

This bisector of angle B divides the line $\overline{AD}$ into two parts $\overline{AP}$ and $\overline{PD}$. Employing the **angle bisector theorem** again, these two parts must be in the ratio

$$\frac{\overline{AP}}{\overline{PD}} = \frac{c}{\overline{BD}} \tag{5.28}$$

Write equation (5.26) in the form

$$\frac{\overline{DC}}{\overline{BD}} = \frac{b}{c}$$

and note that

$$\frac{\overline{DC}}{\overline{BD}} + 1 = \frac{b}{c} + 1 \qquad \text{equals added to equals the results are equal}$$
$$\frac{\overline{DC} + \overline{BD}}{\overline{BD}} = \frac{\overline{BC}}{\overline{BD}} = \frac{a}{\overline{BD}} = \frac{b+c}{c}$$

which simplifies to

$$\overline{BD} = \frac{ac}{b+c} \tag{5.29}$$

Substituting the results from equation (5.29) into the equation (5.28) one finds that

$$\frac{\overline{AP}}{\overline{PD}} = \frac{b+c}{a} \tag{5.30}$$

One can now employ the **section formula** with $r_2 = a$ and $r_1 = (b+c)$ to find the coordinates (x, y) of the point P is given by

$$x = \frac{ax_1 + (b+c)x_4}{a+b+c}, \quad y = \frac{ay_1 + (b+c)y_4}{a+b+c} \tag{5.31}$$

Substitute the values for (x_4, y_4) from the equations (5.27) into the equations (5.31) and show the incenter coordinates (x_I, y_I) of the point P are

$$x_I = \frac{ax_1 + bx_2 + cx_3}{a+b+c}, \qquad y_I = \frac{ay_1 + by_2 + cy_3}{a+b+c} \tag{5.32}$$

■

Cavalieri's principle for the plane

Given two plane figures located between two parallel lines ℓ_a and ℓ_b a distance h apart, as illustrated in the figure 5-24. Consider a third line ℓ between and parallel to both ℓ_a and ℓ_b. The line ℓ cuts the figures to form the line segments $\overline{\text{AB}}$ and $\overline{\text{CD}}$ as illustrated.

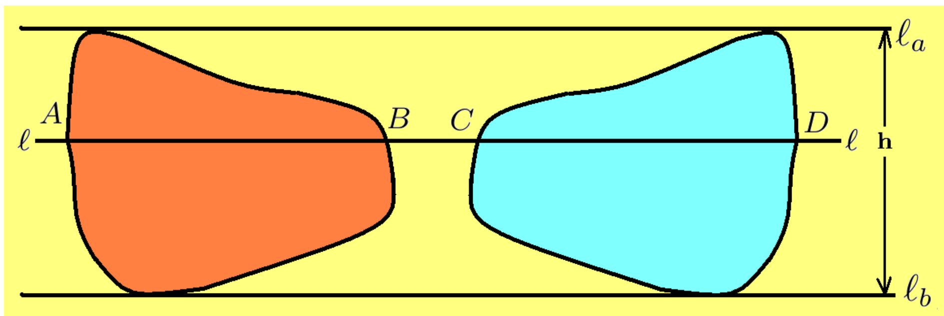


Figure 5-24.
Figures between two parallel lines cut by a third parallel line.

If for every parallel line ℓ, between ℓ_a and ℓ_b, the line segments $\overline{\text{AB}}$ and $\overline{\text{CD}}$ are equal, then the two regions have equal areas. This is known as Cavalieri's principle, developed by Bonaventura Francesco Cavalieri (1598-1647) an Italian mathematician.

The reason why Cavalieri's principle holds is that one can divide the distant h into n-parts of length $\Delta h = \frac{h}{n}$. At each height $h_i = i\frac{h}{n}$, for i ranging from 1 to n, one can calculate the cross-sections $\overline{AB_i}$ and $\overline{CD_i}$ at height h_i. Next calculate the sandwich type elements of area given by $A_i = \overline{AB_i} \cdot \Delta h$ and $\mathcal{A}_i = \overline{CD_i} \cdot \Delta h$ associated with the given figures. A summation of these area elements from $i = 1$ to $i = n$, for both figures, will produce equal areas even in the limit as $n \to \infty$ and Δh becomes very small, since by hypothesis $\overline{CD_i} = \overline{AB_i}$ for all values of the index i.

Example 5-11. Consider the two triangles illustrated in the figure 5-25. We have demonstrated earlier that these triangles have the same area. We can now demonstrate by Cavalier's principle that these areas are the same.

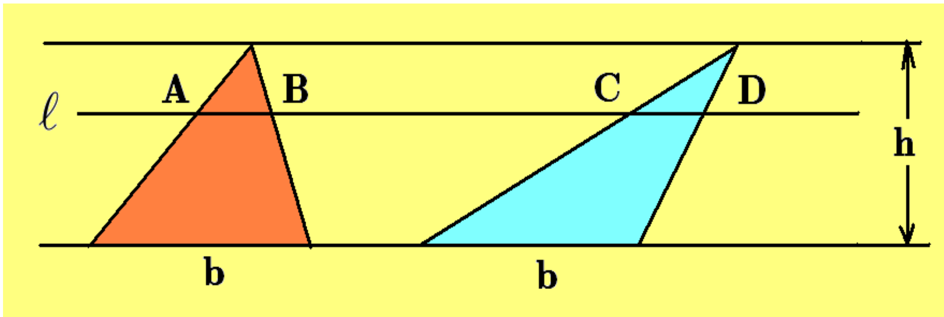


Figure 5-25.
Triangles cut by parallel line.

We employ Cartesian coordinates and some simple algebra to show that for all lines $y = y_0$, $0 \le y_0 \le h$, that $\overline{\text{AB}} = \overline{\text{CD}}$ which satisfies Cavalier's condition for equal areas associated with the figures given. Begin by moving both triangles with equal bases to the origin.

Consider the Cartesian representation of the lines $\ell_1, \ell_2, \ell_3, \ell_4$ which form the sides of the given triangles. Assume lines ℓ_1 and ℓ_2 intersect at the point (x_0, h) and the lines ℓ_3 and ℓ_4 intersect at the point (x_1, h) as illustrated in the figure 5-26. Using the point-slope formula for the equation of a line, the above lines can be shown to have the equations

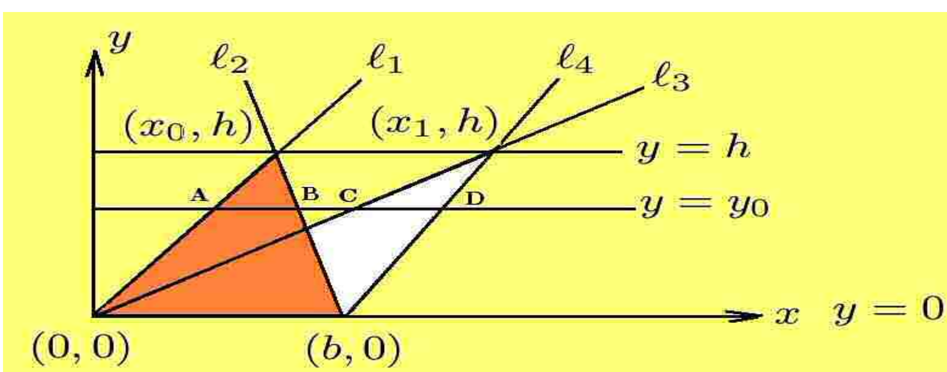


Figure 5-26.
Triangles formed by lines $\ell_1, \ell_2, \ell_3, \ell_4$

$$
\begin{aligned}
\ell_1 : &\quad y = \frac{h}{x_0} x \\
\ell_2 : &\quad y = \frac{-h}{(b - x_0)}(x - b) \\
\ell_3 : &\quad y = \frac{h}{x_1} x \\
\ell_4 : &\quad y = \frac{h}{(x_1 - b)}(x - b)
\end{aligned}
\tag{5.33}
$$

The line $y = y_0$, $0 \le y_0 \le h$, y_0 constant, cuts the two triangles to form the line segments $\overline{\text{AB}}$ and $\overline{\text{CD}}$. Substituting the value $y = y_0$ into the equations (5.33) one obtains

$$
\begin{aligned}
\ell_1 : &\quad y_0 = \frac{h}{x_0} x \quad \Rightarrow \quad x = \frac{x_0 y_0}{h} \\
\ell_2 : &\quad y_0 = \frac{-h}{(b - x_0)}(x - b) \quad \Rightarrow \quad x = b - \frac{y_0}{h}(b - x_0)
\end{aligned}
$$

The difference in these x-values gives

$$
\overline{\text{AB}} = b - \frac{y_0}{h}(b - x_0) - \frac{x_0 y_0}{h} = b\left(1 - \frac{y_0}{h}\right) \tag{5.34}
$$

and for the lines ℓ_3, ℓ_4 one finds that when $y = y_0$ there results

$$
\begin{aligned}
\ell_3 : &\quad y_0 = \frac{h}{x_1} x \quad \Rightarrow \quad x = \frac{x_1 y_0}{h} \\
\ell_4 : &\quad y_0 = \frac{h}{x_1 - b}(x - b) \quad \Rightarrow \quad x = b + \frac{y_0}{h}(x_1 - b)
\end{aligned}
$$

The difference in these x-values gives

$$
\overline{\text{CD}} = b + \frac{y_0}{h}(x_1 - b) - \frac{x_1 y_0}{h} = b\left(1 - \frac{y_0}{h}\right) \tag{5.35}
$$

Comparing the equations (5.34) and (5.35) one finds $\overline{\text{AB}} = \overline{\text{CD}}$ for all values y_0 satisfying $0 \le y_0 \le h$. Therefore, by Cavalier's principle the areas of the triangles are the same.

■

Example 5-12.

For the rectangle and parallelogram illustrated in the figure 5-27 one can say that if the line segments $\overline{\mathrm{AB}} = \overline{\mathrm{CD}}$ for all lines $y = \ell$ between $y = 0$ and $y = h$, then by Cavalier's principle the area of the two figures are the same.

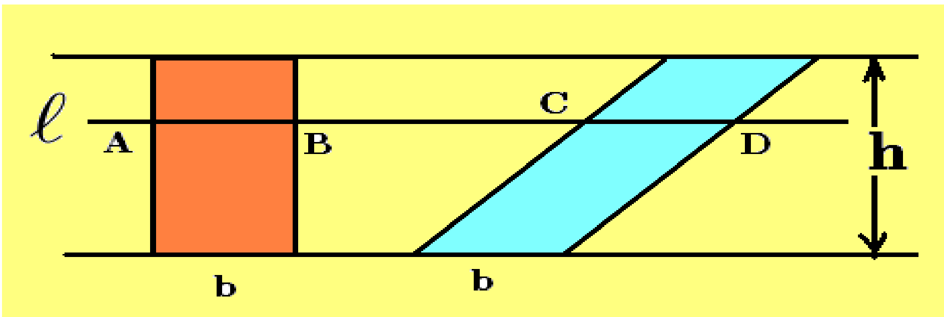


Figure 5-27.
Rectangle and parallelogram with same base and height.

Sketching these figure on a Cartesian axes one obtains the figure 5-28 where the bases of both figures are made to coincide.

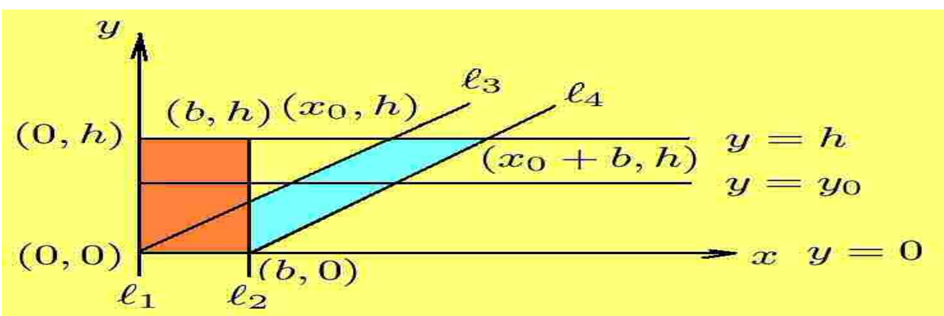


Figure 5-28.
Cartesian coordinates for rectangle and parallelogram.

In this figure the rectangle has sides given by the lines ℓ_1 and ℓ_2 while the parallelogram has sides described by the lines ℓ_3 and ℓ_4. The equations of these lines are

$$
\begin{aligned}
\ell_1 :&\quad x = 0 \\
\ell_2 :&\quad x = b \\
\ell_3 :&\quad y = \frac{h}{x_0} x \\
\ell_4 :&\quad y = \frac{h}{x_0}(x - b)
\end{aligned}
\tag{5.36}
$$

The line $y = y_0$, where y_0 is a constant value between 0 and h, cuts the figures at the x-values

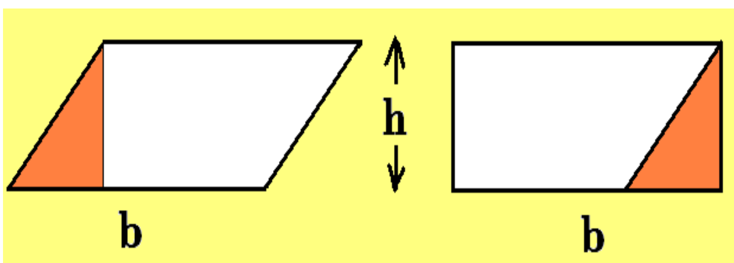
$$\begin{array}{ll} \ell_1: & x=0 \\ \ell_2: & x=b \end{array} \quad \Rightarrow \quad \overline{\text{AB}} = b = \text{change in } x\text{-values} \tag{5.37}$$

and

$$\begin{array}{llll} \ell_3: & y_0 = \dfrac{h}{x_0}x & \Rightarrow & x = \dfrac{x_0 y_0}{h} \\ \ell_4: & y_0 = \dfrac{h}{x_0}(x-b) & \Rightarrow & x = b + \dfrac{x_0 y_0}{h} \end{array} \quad \Rightarrow \quad \overline{\text{CD}} = \left(b + \frac{x_0 y_0}{h}\right) - \frac{x_0 y_0}{h} = b \tag{5.38}$$

where $\overline{\text{CD}}$ represents the change in the x-values. Comparing the equations (5.37) and (5.38) we find $\overline{\text{AB}} = \overline{\text{CD}}$ and by Cavalier's principle the area of both these figures are the same.

Note a parallelogram and rectangle with the same base and height have the same area. Area=(base)(height).



■

Example 5-13. The figures used in Cavalier's principle for determining area don't have to have the same shape as can be seen by examining the figure 5-29.

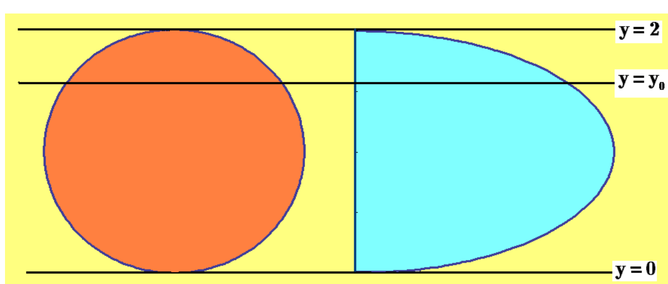


Figure 5-29.
Circle and half of ellipse.

For those of you willing to get involved in the subject you can verify that the illustration above is associated with the unit circle $(x-1)^2+(y-1)^2=1$ centered at $(1,1)$ and half of the ellipse $(y-1)^2+\frac{x^2}{4}=1$. These figures have then been translated to the positions illustrated. The area associated with both figures is π.

■

Menelaus theorem

The Menelaus'[1] theorem concerns the selection of points, one point on each side or extended side of a general triangle $\triangle ABC$ and finding a condition that these points lie on the same straight line. Remember that Ceva's theorem investigates concurrency and Menelaus theorem investigates collinearity.

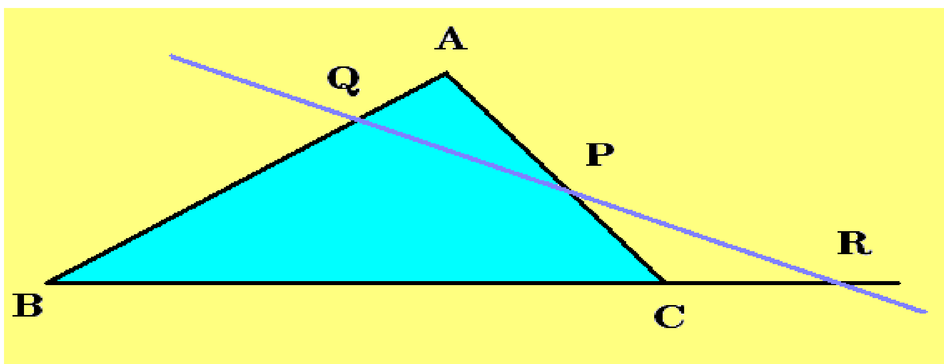


Figure 5-30.
What condition is required for points P, Q, R to lie on same line?

Consider points P, Q, R which lie on different sides or extended sides of triangle $\triangle ABC$ as illustrated in figure 5-30 where P is on side $\overline{\text{AC}}$, Q is on side $\overline{\text{AB}}$ and R is on side $\overline{\text{BC}}$. If there exists a line through the points P, Q, R which does not intersect any of the triangle vertices, then one can show the ratio condition

$$\frac{\overline{\text{AQ}}}{\overline{\text{QB}}}\cdot\frac{\overline{\text{BR}}}{\overline{\text{RC}}}\cdot\frac{\overline{\text{CP}}}{\overline{\text{PA}}}=-1 \tag{5.39}$$

or

$$\overline{\text{AQ}}\cdot\overline{\text{BR}}\cdot\overline{\text{CP}}=-\overline{\text{QB}}\cdot\overline{\text{RC}}\cdot\overline{\text{PA}} \tag{5.40}$$

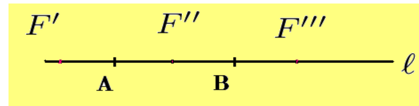
must hold.

[1] Menelaus of Alexandria (70-130)CE was a Greek mathematician and astronomer.

Note that there is no standard ordering or notation for representing the line segments in the equations (5.39) and (5.40). However there is a sign convention.

Sign convention

For a given line segment $\overline{AB}$ on a line ℓ and a point F on the line ℓ then, the ratio of line segments $\frac{\overline{AF}}{\overline{FB}}$ is defined



$$\frac{\overline{AF}}{\overline{FB}} = \begin{cases} \text{a positive value if } F \text{ is between the points } A \text{ and } B \\ \text{a negative value otherwise} \end{cases}$$

For example

$$\frac{\overline{AF'}}{\overline{F'B}} < 0, \qquad \frac{\overline{AF''}}{\overline{F''B}} > 0, \qquad \frac{\overline{AF'''}}{\overline{F'''B}} < 0$$

The equation (5.39) can be thought of as consisting of a product of line segment ratios resulting from the sequential movement of a point from a triangle vertex to a point of intersection followed by the movement of the point from the point of intersection to the next triangle vertex. When this is performed in a cyclic fashion the following ratios result.

(i) Move from vertex A to point of intersection F followed by motion from F to next vertex B giving the first ratio.

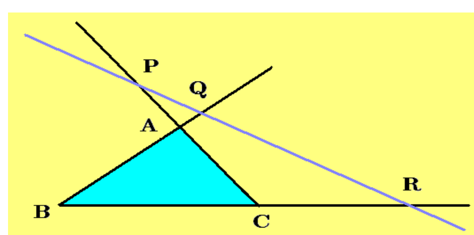
(ii) Move from vertex B to point of intersection E followed by motion from E to next vertex C giving the middle ratio.

(iii) Move from vertex C to point of intersection D followed by motion from D to next vertex A giving the last ratio.

When these ratios are multiplied the product is -1. Note for the line illustrated in figure 5-30 the first ratio is negative and the other two ratios are positive.

Proof of the Menelaus theorem:

There is a case where the line is outside the triangle and the sides of the triangle $\triangle ABC$ must be extended.



In this case all three of the ratios are negative

$$\frac{\overline{AQ}}{\overline{QB}} < 0, \qquad \frac{\overline{BR}}{\overline{RC}} < 0, \qquad \frac{\overline{CP}}{\overline{PA}} < 0$$

Consider now the case where the transversal cross two sides of triangle $\triangle ABC$ as in figure 5-30 where

$$\frac{\overline{\text{AQ}}}{\overline{\text{QB}}} > 0, \qquad \frac{\overline{\text{BR}}}{\overline{\text{RC}}} < 0, \qquad \frac{\overline{\text{CP}}}{\overline{\text{PA}}} > 0$$

In order to avoid keeping track of signs we will deal with absolute values and put the correct sign on when we finish.

Construct in figure 5-30 the perpendiculars from the vertices A, B, C to the line and call these perpendiculars a, b, c respectively.

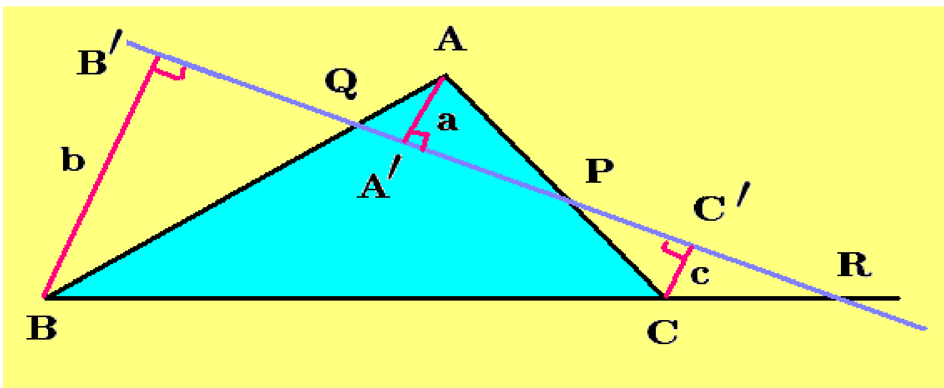


Figure 5-31.
Perpendiculars from vertices to transversal line.

Using similar right triangles from the figure 5-31 one finds the ratios

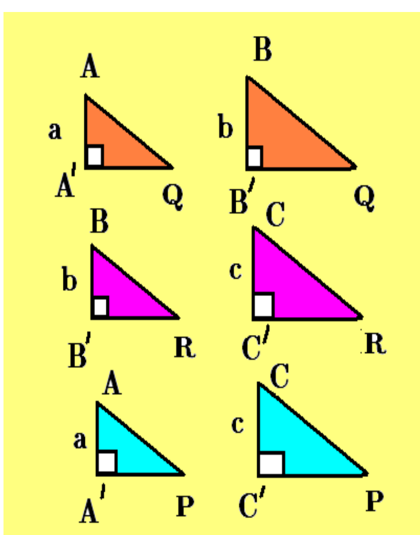


Figure 5-32.
Similar triangles (not to scale)

$$\left|\frac{\overline{\text{AQ}}}{\overline{\text{QB}}}\right| = \left|\frac{a}{b}\right|$$

$$\left|\frac{\overline{\text{BR}}}{\overline{\text{RC}}}\right| = \left|\frac{b}{c}\right|$$

$$\left|\frac{\overline{\text{CP}}}{\overline{\text{PA}}}\right| = \left|\frac{c}{a}\right|$$

The product of these ratios gives

$$\left|\frac{\overline{AQ}}{\overline{QB}}\right| \cdot \left|\frac{\overline{BR}}{\overline{RC}}\right| \cdot \left|\frac{\overline{CP}}{\overline{PA}}\right| = \left|\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}\right| = 1$$

Using the appropriate signs for the above ratios gives one the Menelaus theorem.

The converse of Menelaus theorem is that if one selects points P, Q, R on the triangle sides or extended sides $\overline{AC}, \overline{AB}, \overline{BC}$ such that

$$\frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BR}}{\overline{RC}} \cdot \frac{\overline{CP}}{\overline{PA}} = -1 \tag{5.41}$$

then the points P, Q, R are collinear.

Proof:

Assume that equation (5.41) holds and show the points P, Q, R are on the same line. We know two points defines a line, so let ℓ be the line through the points Q and R. The line ℓ and the triangle side $\overline{AC}$ intersect in a single point which we label as D'. The point D' is on the line ℓ so Menelaus' theorem must hold so that

$$\frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BR}}{\overline{RC}} \cdot \frac{\overline{CD'}}{\overline{D'A}} = -1 \tag{5.42}$$

However, our assumption is that the equation (5.41) holds true. Therefore, one can equate equations (5.41) and (5.42) to obtain

$$\frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BR}}{\overline{RC}} \cdot \frac{\overline{CD'}}{\overline{D'A}} = \frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BR}}{\overline{RC}} \cdot \frac{\overline{CP}}{\overline{PA}} \quad \Rightarrow \quad \frac{\overline{CD'}}{\overline{D'A}} = \frac{\overline{CP}}{\overline{PA}} \tag{5.43}$$

The equation (5.43) can also be expressed in the form

$$\frac{\overline{CD'}}{\overline{D'A}} + 1 = \frac{\overline{CP}}{\overline{PA}} + 1 \quad \text{or} \quad \frac{\overline{CD'} + \overline{D'A}}{\overline{D'A}} = \frac{\overline{CP} + \overline{PA}}{\overline{PA}} \quad \text{or} \quad \frac{\overline{CA}}{\overline{D'A}} = \frac{\overline{CA}}{\overline{PA}}$$

which implies that $\overline{D'A} = \overline{PA}$ or $D' = P$. That is, there is only one point that can cut the segment $\overline{AC}$ to produce the required ratio. Consequently, the points P, Q, R are collinear.

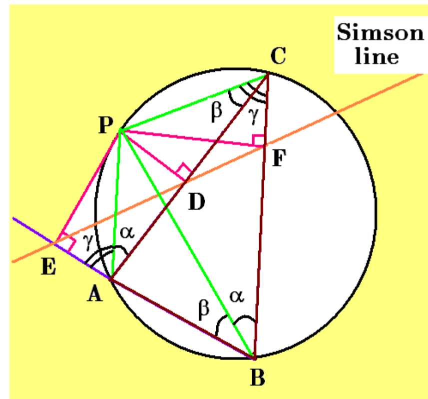
The Simson line (It should be the William Wallace line.)

Simon's theorem[2] (should be the Wallace theorem) concerns a triangle circumscribed by a circle. One can select any point P on the circumference of the circumcircle and from point P construct three perpendicular lines, one to each side

[2] Robert Simson(1687-1768) a famous English geometer was given credit for a theorem published by William Wallace (1768-1843). This mistake has been perpetrated throughout the mathematical literature on geometry for well over a hundred years. I think it is about time that this mistake be corrected.

or extended side of the triangle (red lines in figure). The three points of intersection of the perpendicular lines with the sides of the triangle will be collinear. The line through these points of intersection has since the 18th century been called the Simson line.

Proof:



Using the figure on the right we have constructed the lines from point P to each of the vertices of the triangle $\triangle ABC$ and have constructed the Simson line through the points where the perpendiculars meet the sides of the triangle. Observe that the angles

$$\angle PAD = \angle PBC = \alpha$$

since both angles subtend the same arc $\overset{\frown}{\mathrm{PC}}$.

It can be verified that the triangles $\triangle APD$ and $\triangle PBF$ are right triangles which are similar ($\triangle APD \sim \triangle PBF$). Therefore, one can write the proportion

$$\frac{\overline{\mathrm{BF}}}{\overline{\mathrm{AD}}} = \frac{\overline{\mathrm{PF}}}{\overline{\mathrm{PD}}} \tag{5.44}$$

representing the similarity associated with the triangle sides.

Note also that the angles

$$\angle PBA = \angle PBA = \beta$$

since both angles subtend the same arc $\overset{\frown}{\mathrm{PA}}$. Therefore the triangles $\triangle PDC$ and $\triangle PEB$ are right triangles and similar ($\triangle PDC \sim \triangle PEB$) and so one can form the ratio

$$\frac{\overline{\mathrm{CD}}}{\overline{\mathrm{EB}}} = \frac{\overline{\mathrm{PD}}}{\overline{\mathrm{PE}}} \tag{5.45}$$

associated with the sides of these similar triangles.

Making use of the property that the opposite interior angles of a quadrilateral inside a circle must equal π, then from quadrilateral PABC

$$\angle PAB + \angle PCB = \pi$$

and since angle $\angle BAE$ is a straight angle

$$\angle PAB + \angle PAE = \pi$$

which implies

$$\angle PCB = \angle PAE = \gamma$$

Therefore, the right triangles $\triangle PEA$ and $\triangle PFC$ are similar $(\triangle PEA \sim \triangle PFC)$ so that one can write the following proportion for the sides of these triangles

$$\frac{\overline{\text{AE}}}{\overline{\text{FC}}} = \frac{\overline{\text{PE}}}{\overline{\text{PF}}} \tag{5.46}$$

Multiply the left and right-hand sides of the equations (5.44), (5.45), and (5.46) to show

$$\frac{\overline{\text{BF}}}{\overline{\text{AD}}} \cdot \frac{\overline{\text{CD}}}{\overline{\text{EB}}} \cdot \frac{\overline{\text{AE}}}{\overline{\text{FC}}} = \frac{\overline{\text{PF}}}{\overline{\text{PD}}} \cdot \frac{\overline{\text{PD}}}{\overline{\text{PE}}} \cdot \frac{\overline{\text{PE}}}{\overline{\text{PF}}} = -1$$

or

$$\overline{\text{AE}} \cdot \overline{\text{BF}} \cdot \overline{\text{CD}} = -\overline{\text{EB}} \cdot \overline{\text{FC}} \cdot \overline{\text{DA}}$$

which is Menelaus' result needed to show the points D, E, F are collinear. Make note of the fact that the point E in the figure is outside the segment $\overline{\text{AB}}$ which produces the minus sign.

Euler line

Any triangle which is not equilateral has the property that the points defined by the orthocenter, centroid and circumcenter are collinear. These points all lie on a line called the Euler line which is named after its discoverer Leonhard Euler (1707-1783) a famous Swiss mathematician. Whenever the triangle becomes equilateral, then the orthocenter, centroid and circumcenter coalesce into a single point.

Recall a triangle $\triangle ABC$ with vertices A, B, C having coordinates $(x_1, y_1), (x_2, y_2),\ (x_3, y_3)$ respectively, has the centroid

$$x_G = \frac{1}{3}(x_1 + x_2 + x_3) \qquad y_G = \frac{1}{3}(y_1 + y_2 + y_3) \tag{5.47}$$

orthocenter

$$\begin{aligned} x_H =& \ \frac{\lambda(y_1 - y_3) + \mu(y_3 - y_2) + \nu(y_2 - y_1)}{y_2(x_1 - x_3) + y_1(x_3 - x_2) + y_3(x_2 - x_1)} \\ y_H =& -\frac{\lambda(x_1 - x_3) + \mu(x_3 - x_2) + \nu(x_2 - x_1)}{y_2(x_1 - x_3) + y_1(x_3 - x_2) + y_3(x_2 - x_1)} \end{aligned} \quad \text{where} \quad \begin{aligned} \lambda =& x_3x_1 + y_3y_1 \\ \mu =& x_2x_3 + y_2y_3 \\ \nu =& x_1x_2 + y_1y_2 \end{aligned} \tag{5.48}$$

and circumcenter given by

$$
\begin{aligned}
x_O =& \frac{(x_1^2+y_1^2)(y_3-y_2)+(x_2^2+y_2^2)(y_1-y_3)+(x_3^2+y_3^2)(y_2-y_1)}{2\,(x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1))}\\
y_O =& -\frac{(x_1^2+y_1^2)(x_3-x_2)+(x_2^2+y_2^2)(x_1-x_3)+(x_3^2+y_3^2)(x_2-x_1)}{2\,(x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1))}
\end{aligned}
\tag{5.49}
$$

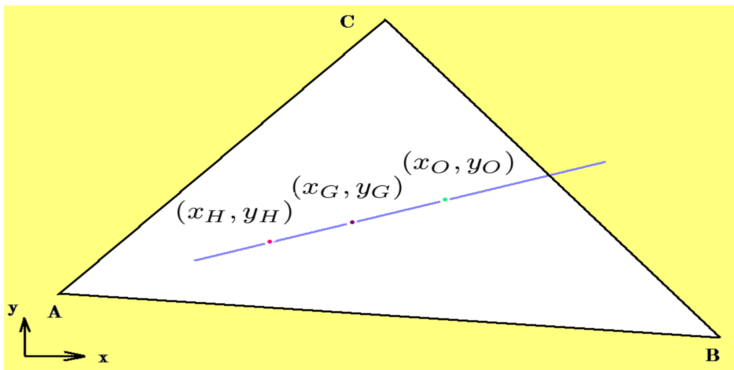


Figure 5-33.
The Euler line

The line through the orthocenter and circumcenter is found using the point-slope formula. One finds the equation for this line is

$$
y-y_H=\left(\frac{y_O-y_H}{x_O-x_H}\right)(x-x_H) \tag{5.50}
$$

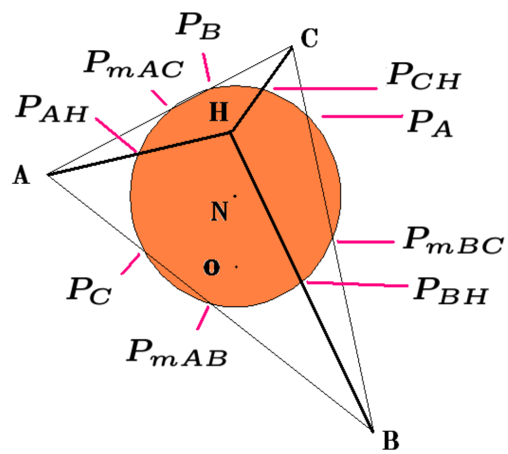
To demonstrate the point (x_G, y_G) is on this line, substitute into the equation (5.50) the values $y=y_G$ and $x=x_G$ from equation (5.47) and show

$$
y_G-y_H-\left(\frac{y_O-y_H}{x_O-x_H}\right)(x_G-x_H)=0
$$

is satisfied. This is messy algebra which can be done on a computer.

The nine point circle

Consider a general triangle with altitudes given by the line segments $\overline{AP_A}$, $\overline{BP_B}$, $\overline{CP_C}$ where the points P_A, P_B, P_C denote the points where the altitudes intersect the side opposite the vertex angles. These altitudes meet at a point of concurrency called the orthocenter and denote by H in the accompanying sketch. Examine the line segments $\overline{\text{AH}}, \overline{\text{BH}}, \overline{\text{CH}}$ which denote the line segments from each vertex to the orthocenter. Define the points P_{AH}, P_{BH}, P_{CH} as the midpoints of these line segments. Also label the midpoints of the triangle sides as P_{mAC}, P_{mBC}, P_{mAB}.



Three points determine a circle. The circle which passes through the points P_A, P_B, P_C at the base of the altitudes also passes through the points P_{AH}, P_{BH}, P_{CH} and P_{mAC}, P_{mBC}, P_{mAB}. This circle is known as the nine point circle. One can construct the line segment $\overline{\text{HO}}$ connecting the orthocenter and circumcenter, then the midpoint of this line segment is the center of the nine point circle. This center lies on the Euler line of the triangle.

Cevians

Recall that through the vertices of every triangle one can construct a cevian. These cevians may or may not meet at a point of concurrency. Note that altitudes, medians and angle bisectors are all special cases of cevians.

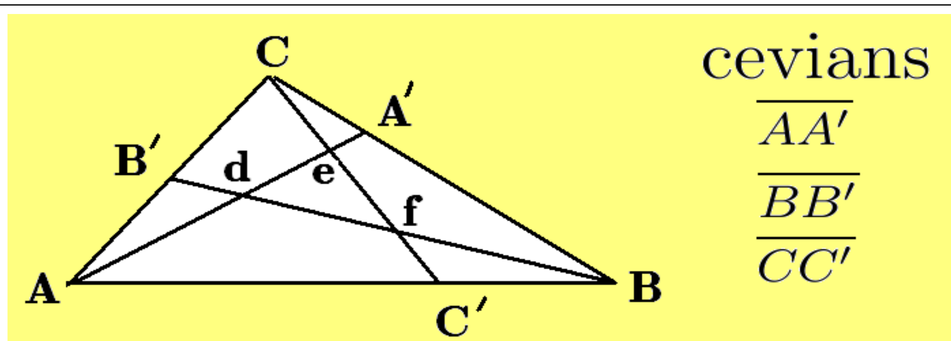
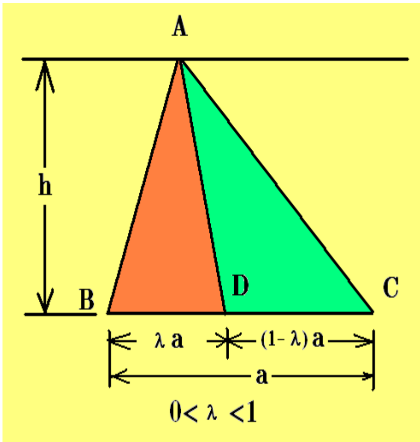


Figure 5-34. Cevians $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$

Ratio of areas theorem



The cevian of a triangle from vertex A divides the triangle into two smaller triangles. It also divides the side opposite vertex A into two parts λa and $(1-\lambda)a$. If $[ADB]$ = area of $\triangle ADB$, $[ADC]$ = area of $\triangle ADC$, and $[ABC]$ = area of $\triangle ABC$, then the ratio of areas theorem states

$$\frac{[ADB]}{[ADC]} = \frac{\overline{\text{BD}}}{\overline{\text{DC}}} = \frac{\lambda}{1-\lambda} \text{ and } \frac{[ABC]}{[ADC]} = \frac{\overline{\text{BC}}}{\overline{\text{DC}}} = \frac{1}{1-\lambda}$$

Proof: Construct a line through the vertex A which is parallel to the triangle base and let h denote the height of the triangle. One can then write

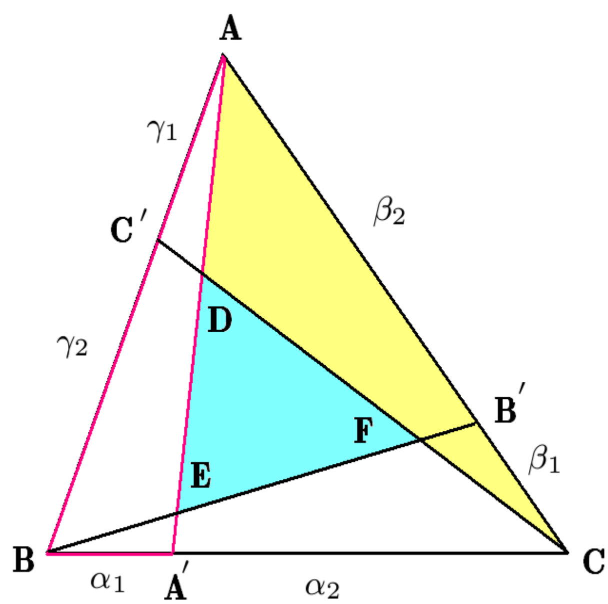
$$\frac{[ADB]}{[ADC]} = \frac{\frac{1}{2}(\overline{\text{BD}})h}{\frac{1}{2}(\overline{\text{DC}})h} = \frac{\lambda}{1-\lambda} \text{ and } \frac{[ABC]}{[ADC]} = \frac{\frac{1}{2}(\overline{\text{BC}})\,h}{\frac{1}{2}(\overline{\text{DC}})\,h} = \frac{\overline{\text{BC}}}{\overline{\text{DC}}} = \frac{1}{1-\lambda}$$

since the area of a triangle is one-half its base times height.

■

Routh's theorem

Consider a general triangle $\triangle ABC$ with area $[ABC]$. If one constructs three general cevians which intersect to form a triangle $\triangle DEF$, then the area $[DEF]$ becomes a fraction of the area $[ABC]$. Routh's theorem determines a general formula for this fraction.



Using the Menelaus theorem, let line $C'C$ intersect triangle $\triangle ABA'$ at the points C', D so that one can write

$$\frac{AC'}{C'B} \cdot \frac{BC}{CA'} \cdot \frac{A'D}{DA} = 1 \qquad (5.51)$$

which implies

$$\frac{A'D}{DA} = \frac{CA'}{BC} \cdot \frac{C'B}{AC'} = \frac{\alpha_2}{\alpha_1+\alpha_2} \cdot \frac{\gamma_2}{\gamma_1} \qquad (5.52)$$

In triangle $\triangle AA'C$ there is the sub-triangle $[ADC]$ and the ratio of the areas of these triangles is given by

$$\frac{[ADC]}{[AA'C]} = \frac{DA}{AA'} \tag{5.53}$$

Similarly, in triangle $\triangle ABC$ there is the sub-triangle $AA'C$ with ratio of areas given by

$$\frac{[AA'C]}{[ABC]} = \frac{A'C}{BC} \tag{5.54}$$

Combining the equations (5.54) and (5.53) one can write

$$[ADC] = \frac{DA}{AA'} \cdot \frac{A'C}{BC}\,[ABC] \tag{5.55}$$

Observe that

$$\frac{DA}{AA'} = \frac{DA}{A'D + DA} = \frac{1}{\frac{A'D}{DA} + 1} \tag{5.56}$$

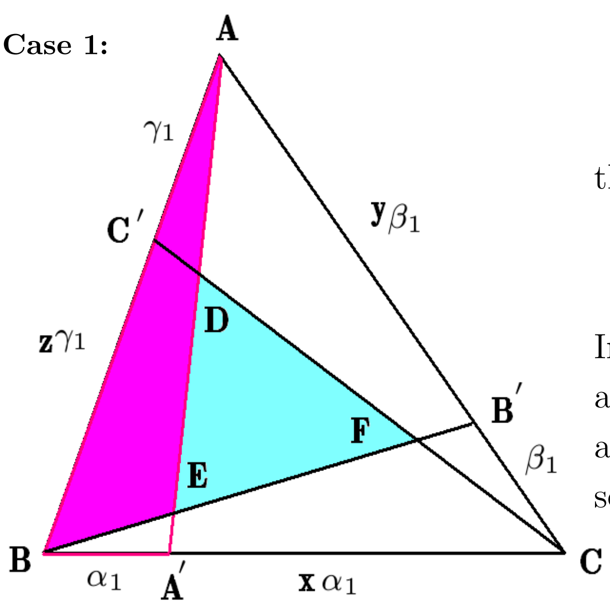
so that equation (5.55) can be expressed in the form

$$[ADC] = \frac{1}{\left[\frac{\alpha_2\gamma_2}{\gamma_1(\alpha_1+\alpha_2)} + 1\right]} \cdot \frac{\alpha_2}{(\alpha_1+\alpha_2)}\,[ABC] = \frac{\gamma_1\alpha_2}{\alpha_2\gamma_2 + \gamma_1(\alpha_1+\alpha_2)}\,[ABC] \tag{5.57}$$

which demonstrates that the area $[ADC]$ is a fraction of the area of $[ABC]$. This fraction can be expressed

$$f = \frac{\gamma_1\alpha_2}{\alpha_2\gamma_2 + \gamma_1(\alpha_1+\alpha_2)} \tag{5.58}$$

Case 1:



Let $\alpha_2 = x\alpha_1,\ \beta_2 = y\beta_1,\ \gamma_2 = z\gamma_1$ and verify the equation (5.57) becomes

$$[ADC] = \frac{x}{xz + x + 1}\,[ABC] \tag{5.59}$$

In a similar manner one can show that the areas [BEA] and [CFB] are fractions of the area [ABC] and these areas can be represented

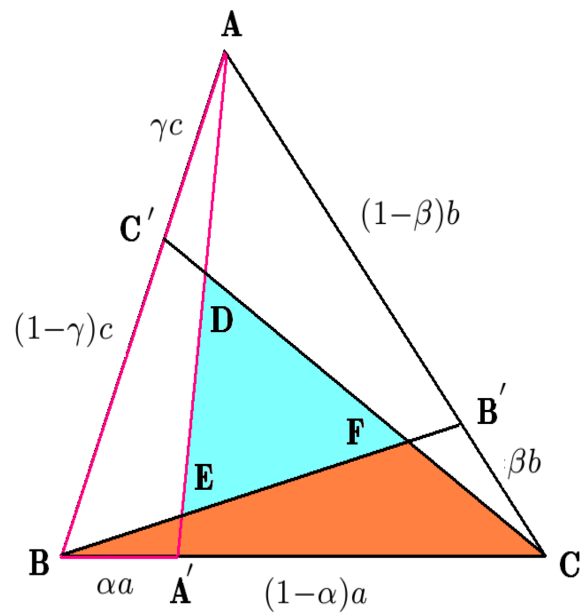
$$[BEA] = \frac{y}{xy + y + 1}\,[ABC] \tag{5.60}$$

$$[CFB] = \frac{z}{yz + z + 1}\,[ABC] \tag{5.61}$$

Note the cyclic changing of the symbols xyz can be used to obtain the other area equations. Knowing the area of the triangles $\triangle BEA$, $\triangle CFB$ and $\triangle ADC$ one can express the area of triangle $\triangle DEF$ as

$$[DEF] = [ABC] - [ADC] - [BEA] - [CFB] \tag{5.62}$$

Case 2:



Let

$$\alpha_1 = \alpha a, \quad \beta_1 = \beta b, \quad \gamma_1 = \gamma c$$
$$\alpha_2 = (1-\alpha)a, \quad \beta_2 = (1-\beta)b, \quad \gamma_2 = (1-\gamma)c$$

where a, b, c are the triangle sides and then verify the equations (5.59), (5.60), (5.61) become

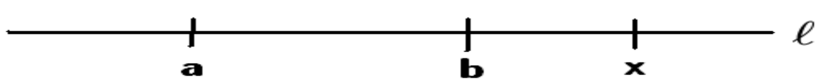
$$[ADC] = \frac{\gamma(1-\alpha)}{\alpha(\gamma-1)+1}[ABC] \tag{5.63}$$
$$[BEA] = \frac{\alpha(1-\beta)}{\beta(\alpha-1)+1}[ABC] \tag{5.64}$$
$$[CFB] = \frac{\beta(1-\gamma)}{\gamma(\beta-1)+1}[ABC] \tag{5.65}$$

with the area of triangle $\triangle DEF$ again given by equation (5.62).

Harmonic division

Given two points a and b on a line ℓ one can select a third point x on line ℓ and define the ratio

$$r(x) = r(x; a, b) = \frac{x-a}{x-b} \tag{5.66}$$



Note that

(i) If $a < x < b$, then $r(x)$ is negative.

(ii) If $x < a$ or $x > b$, then $r(x)$ is positive.

(iii) In the special case $x = \frac{a+b}{2}$ is the midpoint of $\overline{ab}$, then $r\left(\frac{a+b}{2}\right) = -1$

(iv) If there exists points c and d such that

$$\frac{r(c; a, b)}{r(d; a, b)} = -1$$

then the points a, b are said to be **divided harmonically by the points c and d.** Harmonic ratios occur in many areas of geometry and music.

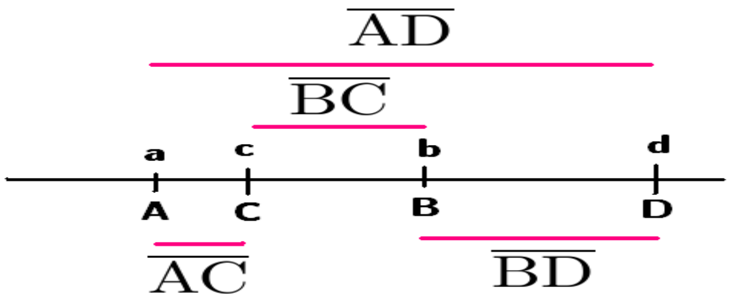


Figure 5-35. Line segments used in harmonic ratio

In terms of positive line segments, the condition for a harmonic ratio is

$$r(c;a,b) = \frac{c-a}{c-b} = \frac{\overline{AC}}{-\overline{BC}}, \qquad r(d) = \frac{d-a}{d-b} = \frac{\overline{AD}}{\overline{BD}} \tag{5.67}$$

and the condition

$$\frac{r(c)}{r(d)} = -1 \text{ becomes } \quad \frac{r(c)}{r(d)} = \frac{\frac{-\overline{AC}}{\overline{BC}}}{\frac{\overline{AD}}{\overline{BD}}} = -\frac{\overline{AC}}{\overline{BC}} \cdot \frac{\overline{BD}}{\overline{AD}} = -1 \tag{5.68}$$

Thus, if $\frac{r(c)}{r(d)} = -1$, it is required that

$$\frac{\overline{AC}}{\overline{BC}} = \frac{\overline{AD}}{\overline{BD}} \quad \text{or} \quad \overline{AC} \cdot \overline{BD} = \overline{AD} \cdot \overline{BC} \tag{5.69}$$

Reciprocal property

If the points a, b are fixed and $\frac{r(c;a,b)}{r(d;a,b)} = -1$, then the points c, d are said to divide the segment $\overline{\text{ab}}$ harmonically. If the points c, d are fixed and $\frac{r(a;c,d)}{r(b;c,d)} = -1$, then the points a, b are said to divide the segment $\overline{\text{cd}}$ harmonically. This reciprocal relationship exists between the points a, b, c, d.

Example 5-14.

Given a triangle $\triangle PAB$ with side $\overline{AP}$ extended to illustrate the interior angle P_i and exterior angle $P_e = \pi - P_i$. Construct the angle bisectors of both the interior angle P_i and exterior angle P_e. These are the lines $\overline{PC}$ and $\overline{PD}$ illustrated in the figure 5-36, where points C and D are the intersection points of the angle bisectors with the side $\overline{AB}$ (extended if necessary).

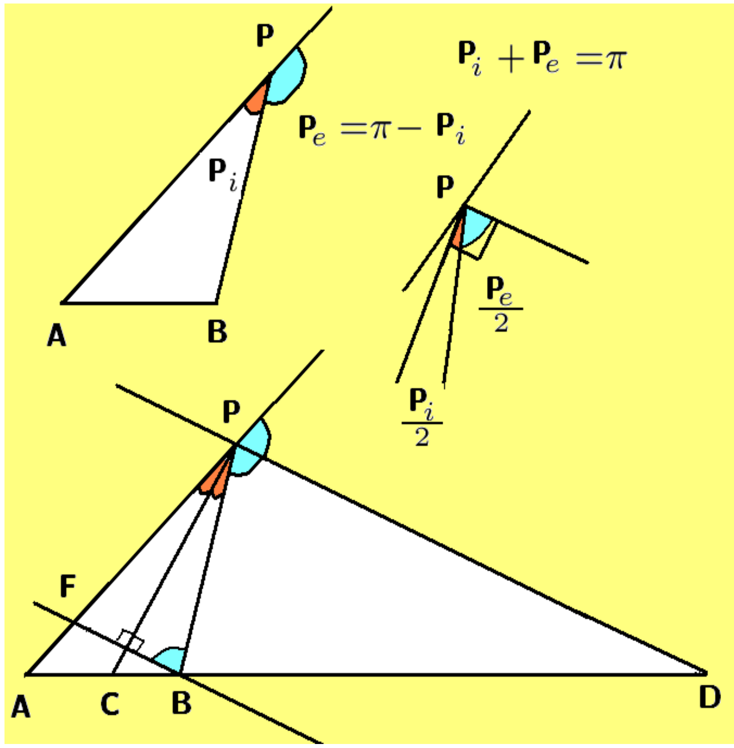


Figure 5-36. Harmonic ratio occurs

(i) Show the line $\overline{\text{PC}}$ is perpendicular to the line $\overline{\text{PD}}$. The proof follows from the fact that $P_i + P_e = \pi$ so that $\frac{P_i}{2} + \frac{P_e}{2} = \frac{\pi}{2}$.

(ii) Show

$$\frac{\overline{\text{PA}}}{\overline{\text{PB}}} = \frac{\overline{\text{AD}}}{\overline{\text{BD}}} \tag{5.70}$$

This can be demonstrated by constructing a line through point B which is parallel to line $\overline{\text{PD}}$ and intersecting side $\overline{\text{PA}}$ at the point F as illustrated. It follows that triangle $\triangle FBP$ is isosceles so that $\overline{\text{PB}} = \overline{\text{PF}}$.

In triangle $\triangle ADP$ one can use the Thales intercept theorem to show

$$\frac{\overline{\text{PA}}}{\overline{\text{PF}}} = \frac{\overline{\text{AD}}}{\overline{\text{BD}}} \tag{5.71}$$

Using the result that $\overline{\text{PB}} = \overline{\text{PF}}$ the equation (5.71) becomes

$$\frac{\overline{\text{PA}}}{\overline{\text{PF}}} = \frac{\overline{\text{AD}}}{\overline{\text{BD}}} = \frac{\overline{\text{PA}}}{\overline{\text{PB}}} \tag{5.72}$$

which was to be demonstrated

(iii) Show

$$\frac{\overline{AC}}{\overline{BC}} = \frac{\overline{AD}}{\overline{BD}} \tag{5.73}$$

Using the angle bisector theorem in triangle $\triangle ABP$ one finds

$$\frac{\overline{PA}}{\overline{PB}} = \frac{\overline{AC}}{\overline{BC}} \tag{5.74}$$

The equation (5.74) combined with equation (5.72) completes the proof

The equation (5.73) shows that the side $\overline{AB}$ has been divided harmonically. See the equations (5.69).

■

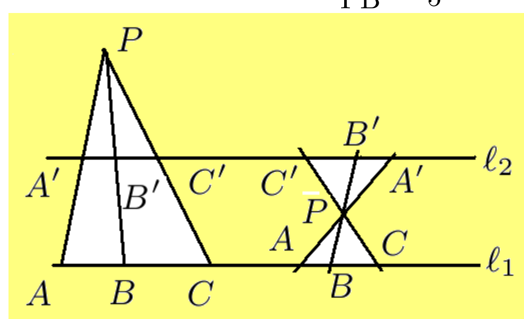
Exercises

► **5-1.** Determine the point P which divides the given line $\overline{AB}$ into the ratio specified.

$$(a)\quad A:(0,0),\ B:(100,0),\quad \frac{\overline{AP}}{\overline{PB}} = \frac{3}{5} \qquad (c)\quad A:(0,0),\ B:(81,81),\quad \frac{\overline{AP}}{\overline{PB}} = \frac{2}{7}$$

$$(b)\quad A:(0,0),\ B:(0,50),\quad \frac{\overline{AP}}{\overline{PB}} = \frac{2}{5} \qquad (d)\quad A:\ (0,2),\ B:\ (100,6),\quad \frac{\overline{AP}}{\overline{PB}} = \frac{3}{5}$$

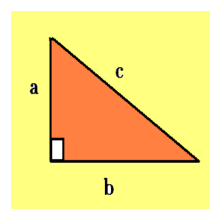
► **5-2.**



Show that three or more straight lines through a point P which also intersect parallel lines, then there is produced proportional parallel line segments. For the illustration given show

$$\frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{B'C'}}{\overline{BC}}$$

► **5-3.**



Find the centroid of the given triangle if

$$(a)\quad a=3,\ b=4 \qquad (b)\quad a=8,\ b=15 \qquad (c)\quad a=9,\ b=40$$

► **5-4.** Given the parallel lines $y=0$ and $y=10$ along with the lines

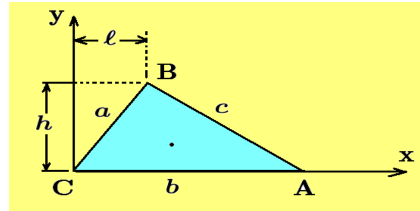
$$\ell_1:\ y=3x, \qquad \ell_2:\ y=-4(x-8)$$

(a) Find the point P where the two lines meet.
(b) Find where the line ℓ_1 intersects the parallel lines.
(c) Find where the line ℓ_2 intersects the parallel lines.
(d) Sketch the above lines and points and verify Thales intercept theorem.

▶ **5-5.**

For the triangle illustrated, show the centroid is located at

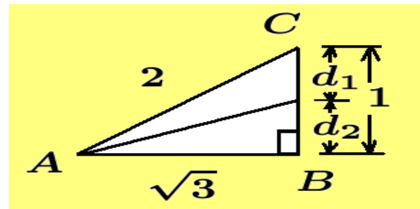
$$\bar{x} = \frac{b+\ell}{3} \quad \text{and} \quad \bar{y} = \frac{h}{3}$$



▶ **5-6.**

Give the right triangle $\triangle ABC$ with sides

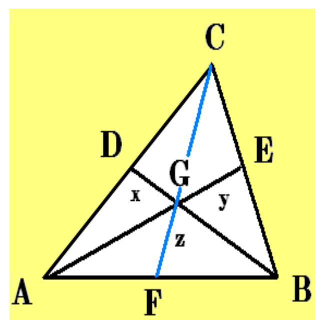
$$\overline{\text{AB}} = \sqrt{3}, \quad \overline{\text{AC}} = 2, \quad \overline{\text{BC}} = 1$$



The bisector of the vertex angle A divides the side $\overline{\text{BC}}$ into two parts of lengths d_1 and d_2. Find the lengths d_1 and d_2 and the length of the angle bisector.

▶ **5-7.**

Given the triangle $\triangle ABC$ with $\overline{\text{DG}} = x$, $\overline{\text{EG}} = y$, $\overline{\text{GF}} = z$ and lines $\overline{\text{AE}}, \overline{\text{BD}}, \overline{\text{CF}}$ medians. The label G denoting the centroid of the triangle.



(a) Find $\overline{\text{GB}}$ in terms of x

(b) Find $\overline{\text{GA}}$ in terms of y

(c) Find $\overline{\text{GC}}$ in terms of z

(d) Show the line segment $\overline{\text{DE}}$ is parallel to $\overline{\text{AB}}$

(e) What is the ratio of distance of G from a vertex to distance of median from same vertex.

▶ **5-8.** Apply a cyclic rotation of symbols to the equations (5.18) and (5.21) to find the length of the cevians $\overline{BB'}$ and $\overline{CC'}$ which are angle bisectors associated with the vertex angles B and C of triangle $\triangle ABC$.

▶ **5-9.**

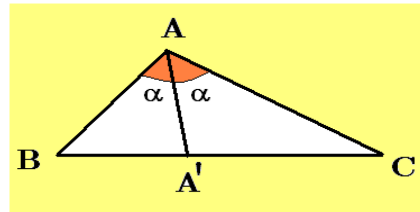
Examine figure 5-18 and prove that the points P, Q, R are collinear.

▶ **5-10.** Find the incenter of an equilateral triangle with sides of length s and height $\frac{\sqrt{3}s}{2}$.

▶ **5-11.** Find the incenter of an isosceles triangle with equal sides of length 100, base equal to $100\sqrt{3}$ and height equal to 50.

▶ **5-12.**

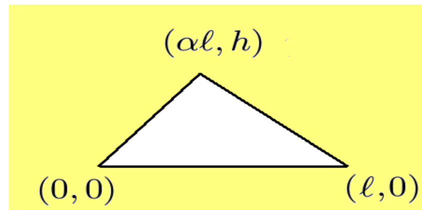
Given triangle $\triangle ABC$ with cevian $\overline{AA'}$ representing the angle bisector of the vertex angle A. Use the angle bisector theorem and Stewarts theorem to show



$$(\overline{AA'})^2 = (\overline{\text{AB}})(\overline{\text{AC}}) - (\overline{BA'})(\overline{A'C})$$

▶ **5-13.**

Find the incenter of the triangle $\triangle ABC$ illustrated where α is a parameter restricted to the values $0 < \alpha < 1$.



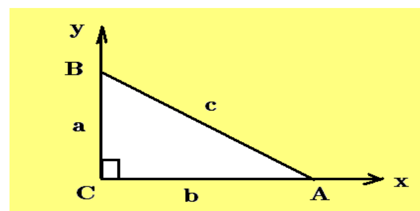
▶ **5-14.** Find the orthocenter, centroid and circumcenter associated with an equilateral triangle having a side of length s.

▶ **5-15.**

For the right triangle illustrated.

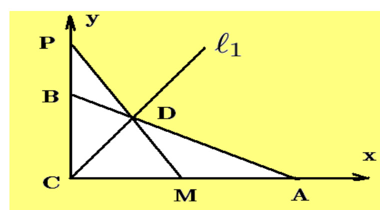
(a) Find the radius r_0 of the circumscribed circle.

(b) Find the radius r_I of the inscribed circle.



▶ **5-16.**

Given the right triangle $\triangle ABC$ with line ℓ_1 dividing the hypotenuse such that $\overline{\text{DB}} = 4\sqrt{\frac{2}{3}}$ and $\overline{\text{DA}} = 20\sqrt{\frac{2}{3}}$. Point B has the coordinates $(0, 8)$, point A has the coordinates $(8\sqrt{5}, 0)$, point M has the coordinates $(4\sqrt{5}, 0)$ and P has the coordinates $(0, y)$. Use Menelaus's theorem to find y such that the points P, D, M are collinear.



▶ **5-17.**

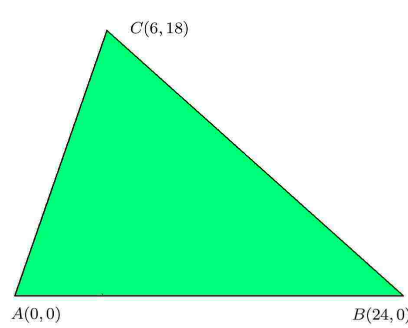
Given the triangle $\triangle ABC$ illustrated.

(a) Find the orthocenter.

(b) Find the circumcenter.

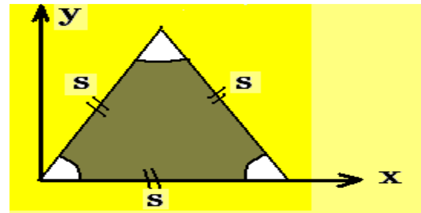
(c) Find the centroid.

(d) Find the Euler line.



▶ **5-18.**

For the given equilateral triangle $\triangle ABC$ find the point where the orthocenter, centroid and circumcenter coalesce.



▶ **5-19.** Given a triangle $\triangle ABC$ inside the circle $x^2 + y^2 = 1$. The vertices of the triangle have the coordinates

$$A(-1, 0), \quad B(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \quad C(0, 1)$$

Select point $P(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ on the circle and

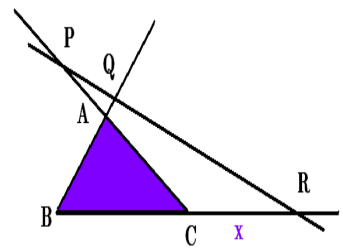
(a) find the points associated with the Simson line (William Wallace line).

(b) find the equation of the Simson line (William Wallace line).

▶ **5-20.**

Use the Menelaus theorem to find the point R if points P, Q, R are to be collinear. Using the distances

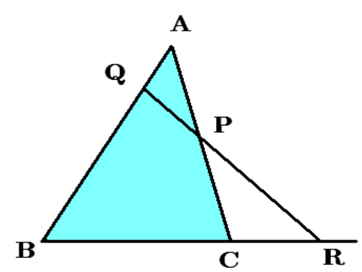
$$\overline{AQ} = \frac{10}{3} \qquad \overline{BR} = 8 + x \qquad \overline{CP} = 2\sqrt{41}$$
$$\overline{QB} = \frac{25}{3} \qquad \overline{RC} = -x \qquad \overline{PA} = \sqrt{41}$$



▶ **5-21.**

Use the Menelaus theorem to determine the point P so that the points P, Q, R are collinear. Use the following distances for the triangle given.

$$\overline{AQ} = 2 \qquad \overline{RC} = -2$$
$$\overline{QB} = 8 \qquad \overline{CP} = x$$
$$\overline{BR} = 10 \qquad \overline{PA} = 9 - x$$



▶ **5-22.**

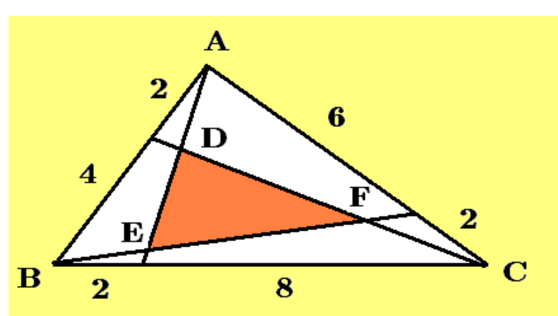
The triangle illustrated has the sides

$$\overline{AB} = 6, \quad \overline{BC} = 10, \quad \overline{CA} = 8$$

(a) Find the area of triangle $\triangle ABC$

(b) Find the area of triangle $\triangle DEF$

(c) Find the ratio $\dfrac{[DEF]}{[ABC]}$



Geometry

Chapter 6

Introduction to Mathematical Fundamentals

Generalization and abstraction

By observing mathematical operations certain concepts lead to abstractions and generalizations. For example $2 + 3 = 5$ is a special case of $x + y = z$ for the values $x = 2$, $y = 3$ and $z = 5$. The equation $x + y = z$ is an abstraction of the concept of addition. It represents all possible sums of two quantities to produce a third quantity. In mathematics, "Everything is a special case of something more general." Many mathematicians search for the more general concepts in order to classify and correlate mathematical concepts and ideas into some kind of an organized structure.

The study of triangles, squares, pentagons, hexagons,..., n-gons illustrates the search for generalized concepts concerning all polygons. This is a way of collecting facts about polygons and organizing these facts to enlarge our data base of knowledge.

Organization of knowledge

The building of a foundation of knowledge is composed of definitions, axioms and postulates and then the organization and formalization of the knowledge to obtain an understanding of the basic material in order to generate by deductive arguments new ideas and concepts represented by theorems, corollaries and lemmas. These facts can be generalized and organized into various categories and subcategories. This is what Euclid started some 2300 years ago and is still done today in mathematics. The theorems, corollaries and lemmas introduce new ideas and illustrate how the new ideas logically follow from using the fundamentals and previously develop old facts which have already been verified. The proofs of the theorems are usually given by the presentation of a sequence of steps which produce the conclusion stated by the theorem. Euclid and his followers were the first to organize how mathematics was to be approached in a logical way and to develop the concept of a proof.

The wonderful thing about theorems and proofs is that they are timeless. What Euclid proved 2300 years ago is still valid today. Theorems build on our data base of knowledge and the advancement of the sciences.

Euclid's introduction of a proof has been modified in the modern era of computers. Computer calculations, simulations, modeling using Monte Carlo techniques and statistics along with numerical methods are some of the more modern acceptable proofs describing what is happening in various fields of science. Mathematics is now used in all areas of physics, engineering, science and business.

Logic symbols

Let letters of the alphabet $A, B, \ldots, P, Q, \ldots$ denote sentences or statements that are either true or false, but not both. Composite statements or compound statements consist of sub-statements along with different types of connectives. The truth or falseness property of a compound statement depends upon the truth of the sub-statements and how the sub-statements are connected.

Negation $(\sim A)$

Given the statement

A: It is raining outside.

The negation of the statement A is written $(\sim A)$ which is read as ≫ not A ≪ and is constructed by inserting the word ≫ not ≪ into the statement A.

$\sim A$: It is not raining outside.

If A is true, then $\sim A$ is false and if A is false, then $\sim A$ is true. A truth table for $\sim A$ is the following table where T represents true and F represents false.

Truth Table

A	$\sim A$	$\sim(\sim A)$
T	F	T
F	T	F

Note that $\sim(\sim A) = A$. That is, the negation of any statement is always the opposite truth value of the original statement.

Conjunction $(A \wedge B)$

Whenever two statements A and B are combined using the word ≫ and ≪, then a compound statement is formed. The compound statement $(A \wedge B)$, read as ≫ A and B ≪, is called the conjunction of the original statements A and B.

A: John has a cell phone.

B: Mary has a cell phone.

$A \wedge B$: John has a cell phone and Mary has a cell phone.

or John and Mary have cell phones.

The truth table for the above compound statement $A \wedge B$ is as follows

Truth Table

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

The T and F values in a truth table represent the truth values for A and B in a compound sentence. Note that $A \wedge B$ is true, only when both A and B are true. The first line of the truth table is read ≫ If A is true and B is true, then $A \wedge B$ is true.≪

Disjunction $(A \vee B)$

Whenever two statements A and B are combined using the word ≫ or ≪ to form a compound statement, then the new statement is called a disjunction of the original two statements and written as $A \vee B$ and read as ≫ A or B ≪.

Be careful, the word ≫ or ≪ can be used in different ways. For example, $A \vee B$ could mean

(i) A or B or both (at least one of the two alternatives occurs)

(ii) A or B, but not both (only one of the two alternatives can occur)

The use of the word ≫ or ≪ is to be used in the sense of "and/or" as given in (i) above.

A truth table for the compound statement $A \vee B$ is as follows

Truth Table

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Here $A \vee B$ is false only when both A and B are false.

Implication

In mathematics one can come across statements or expressions such as:
(i) if-then statement. If (hypothesis A is true), then (conclusion B is true)
(ii) A implies B
(iii) B is a necessary condition for A
(iv) A is a sufficient condition for B
(v) B follows from A

Each of the above statements all mean that the truth of statement B follows from the truth of statement A. In these statements A is called the premise or hypothesis and B is called the conclusion or result. Each of the above statements can be written symbolically by using one of the notations $A \Rightarrow B$ or $A \rightarrow B$, which is read using one of the interpretations given by (i) through (v) above. For example, $A \Rightarrow B$ could be interpreted as $\gg$ If A, then B $\ll$.

Converse

When two parts of a sentence are interchanged or turned about, one part is called the **converse** of the other part. In mathematics a necessary and sufficient condition for the completeness of a definition is that its converse is also true. For example the definition, " The centroid of a triangle is defined as the point of intersection of the medians of the triangle. " has the converse statement, "The point of intersection of the medians of a triangle defines the centroid of a triangle. " If one has a theorem $A \Rightarrow B$ (If A, then B), then its converse is $B \Rightarrow A$ (If B, then A). The **converse** of an if-then statement is obtained by interchanging the hypothesis (A) and the conclusion (B). Note that the converse of a true statement may or may not be true. For example, the statement, "If a quadrilateral is a square, then all the sides must be equal" has the converse, "If all the sides of a quadrilateral are equal, then the quadrilateral must be a square", is a false statement because the quadrilateral could be a rhombus. The **law of detachment** states that if the statement, "If hypothesis, then conclusion" (if A, then B) is accepted as being true, then when you know condition A holds, a valid logical conclusion is B.

The **law of syllogism** states that if $A \Rightarrow B$ and $B \Rightarrow C$ are true statements, then the statement $A \Rightarrow C$ is also a true statement. If you have a hypothesis H and conclusion C, so that $H \Rightarrow C$, (H implies C), then the **contrapositive** of this statement is obtained by negating both the hypothesis and conclusion and then interchanging the results. For example,

$$H \Rightarrow C \quad \text{has the contrapositive statement} \quad \sim C \Rightarrow \sim H$$

Remember that

(i) If the original statement is true, then the contrapositive statement is also true.

(ii) If the original statement is false, then the contrapositive statement is also false.

Example 6-1.

Consider the statements {A: The computer is working.} and {B: I'll solve your problem.} and the following if-then combinations.

Original $A \Rightarrow B$

$$\text{If } \underbrace{\text{the computer is working}}_{A}\text{, then } \underbrace{\text{I'll solve your problem.}}_{B}$$

Converse $B \Rightarrow A$

$$\text{If } \underbrace{\text{I solve your problem}}_{B}\text{, then } \underbrace{\text{the computer is working.}}_{A}$$

Inverse $\sim A \Rightarrow \sim B$

$$\text{If } \underbrace{\text{the computer is not working}}_{\sim A}\text{, then } \underbrace{\text{I will not solve your problem.}}_{\sim B}$$

Contrapositive $\sim B \Rightarrow \sim A$

$$\text{If } \underbrace{\text{I don't solve your problem}}_{\sim B}\text{, then } \underbrace{\text{the computer is not working.}}_{\sim A}$$

■

Example 6-2.

The following statements are all true.

Original $(A \Rightarrow B)$ If $\triangle ABC$ and $\triangle DEF$ are similar, then their corresponding sides are proportional.

Converse $(B \Rightarrow A)$ If the corresponding sides of $\triangle ABC$ and $\triangle DEF$ are proportional, then the triangles are similar.

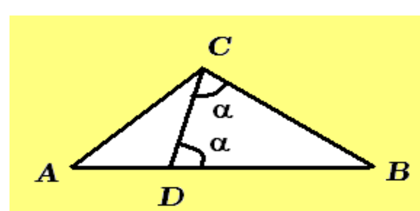
Inverse $(\sim A \Rightarrow \sim B)$ If $\triangle ABC$ and $\triangle DEF$ are not similar, then their corresponding sides are not proportional.

Contrapositive $(\sim B \Rightarrow \sim A)$ If the corresponding sides of $\triangle ABC$ and $\triangle DEF$ are not proportional, then the triangles are not similar.

■

Example 6-3.

If two sides of a triangle are unequal, then the angles opposite these sides are also unequal with the larger angle opposite the larger side.
(Euclid, Book 1, Propositions 18,19)

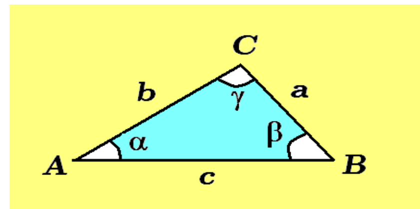


Proof: Given triangle $\triangle ABC$ with larger side $\overline{\text{AB}}$. Measure the distance $\overline{\text{BC}}$ and mark off the distance $\overline{\text{DB}} = \overline{\text{BC}}$ on side $\overline{\text{AB}}$. We know the angles opposite equal sides are equal. Therefore, $\angle DCB = \angle CDB = \alpha$. Hence one can conclude

Statement	Reason
$\angle ACB > \alpha$	The whole is greater than any of its parts.
$\alpha > \angle DAC$	Exterior angle greater than any single opposite interior angle.
$\angle ACB > \angle DAC$	law of syllogism

■

Example 6-4.



If two angles of triangle $\triangle ABC$ are unequal, the sides opposite these angles are also unequal with the greater side opposite the greater angle.
(Euclid, Book 1, Propositions 18,19)

Proof: Assume angle β is the greater than angle α. One can then state that one of the following conditions must hold.

$$b < a, \quad b = a, \quad \text{or} \quad b > a$$

Statement	Reason
$\beta > \alpha$	Assumption
$a = b$ cannot be true	If true, then $\alpha = \beta$ contradicts original assumption.
$b < a$ cannot be true	If true, then $\beta < \alpha$ contradicts original assumption.
$b > a$ must be true	Only remaining case

This type of proof is known as, **a proof by exclusion.** As Sherlock Holmes would say, "When you have ruled out all the other possibilities, whatever remains must be the truth."

■

Biconditional

If the connective between statements of facts A and B is expressed in any of the forms

(a) A is true if and only if B is true.
(b) A is equivalent to statement B.
(c) A is a necessary and sufficient condition for B.

then this is referred to as a biconditional connective between statements.

Some examples of biconditional compound statements are:

(i) A quadratic equation having rational coefficients can be factored if and only if its discriminant is a perfect square.

(ii) Two polynomials are equal if and only if their difference produces the zero polynomial.

(iii) A necessary and sufficient condition for two planes

$$Ax + By + Cz + D = 0 \quad \text{and} \quad \alpha x + \beta y + \gamma z + \delta = 0$$

to be parallel is for the coefficients to satisfy $\dfrac{A}{\alpha} = \dfrac{B}{\beta} = \dfrac{C}{\gamma}$

Sometimes biconditional statements are used in definitions.

(i) A triangle is isosceles if and only if the triangle has two equal sides.

(ii) The absolute value of x is $|\, x \,|$, by definition $|\, x \,| = 0$ if and only if $x = 0$.

(iii) If $a > 0$, then $|\, x \,| < a$ if and only if $-a < x < a$.

Biconditionals are expressed mathematically using the symbol ($\iff$). The expression $A \iff B$, is interpreted using one of the statements (a),(b) or (c) above. For example, one could read $A \iff B$ as $\gg$ A is true if and only if B is true. $\ll$ Here the connective between statements A and B is $\gg$ if and only if $\ll$ or one of the above statements (b) or (c) above. Sometimes the expression "if and only if" is expressed using the shorthand notation "iff".

The following is a truth table for a biconditional if and only if statement.

Truth Table

A	B	$A \iff B$
T	T	T
T	F	F
F	T	F
F	F	T

A biconditional (fact 1 $\iff$ fact 2) is true when the truth or falseness of both facts are the same. Either both facts are true or both facts are false.

Example 6-5. Show that for a,b,c,d integers different from zero, then the equality $\frac{a}{b} = \frac{c}{d}$ is true if and only if $ad = bc$.

Solution

Assume that $\frac{a}{b} = \frac{c}{d}$ is a true statement. Then multiply both sides of this relation by the non zero product bd. One then obtains the result that equals multiplied by equals the results are equal. This gives

$$\frac{a}{b}\,bd = \frac{c}{d}\,bd \quad \text{which simplifies to} \quad ad = bc$$

Conversely, assume that $ad = bc$ is true and multiply both sides by the non zero identity $\frac{1}{bd} = \frac{1}{bd}$. Equals multiplied by equals gives the result

$$\frac{ad}{bd} = \frac{bc}{bd} \quad \text{which simplifies to} \quad \frac{a}{b} = \frac{c}{d}$$

Hence one can write

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

■

Indirect poofs

Indirect proofs, sometimes referred to as proof by exclusion, consists of assuming a contradiction to the conclusion of a proposition and then showing that this assumption leads to a contradiction. For example, to prove a statement such as $A = B$, one can make the assumption $A \neq B$. If such is the case, then either $A > B$ or $A < B$. If one can then show that both the statements $A < B$ and $A > B$ are false, then the only case left is $A = B$ which contradicts the original assumption. This type of proof is known as *reductio ad absurdum.*

Inductive and Deductive reasoning

Inductive reasoning is starting with a conjecture, hypothesis or a statement of general behavior arrived at because of

(i) pattern recognition

(ii) observation of data changes

(iii) belief obtained from experience

One can use written out statements, numbers, diagrams, tables and patterns associated with a given problem to try and discover a way of obtaining a solution. One tries to formulate a general conjecture or hypothesis concerning the solution to the problem and then from this starting point **prove** to everyones satisfaction, if possible, that your conjecture is true. Note that to disprove a conjecture one need only find one counterexample to show the conjecture is false.

Example 6-6. The hypothesis that $f(n) = \frac{1}{6}\left(18 - 14n + 9n^2 - n^3\right)$ produces a prime number for integers $n = 1, 2, 3, \ldots$ can be shown to be false by substituting in the integer $n = 5$ as a counterexample. One can show that

$$\begin{array}{ll} n = 1,\ f(1) = 2 & n = 4,\ f(4) = 7, \\ n = 2,\ f(2) = 3, & n = 5,\ f(5) = 8 \\ n = 3,\ f(3) = 5, & n = 6,\ f(6) = 7 \end{array}$$

Remember it doesn't matter how many times you show something to be true—that is not a proof. It only takes one time to show something is false to have a counterexample.

■

The ancient Greeks asked the question, "What are correct reasoning and incorrect reasoning techniques used in solving problems?" In proving something is true
(i) There must not be any ambiguity in the statement of a problem or in the presentation of its solution. Hence the need for definitions, axioms, postulates and understanding of basic knowledge associated with a given problem.
(ii) One must show a step-by-step methodology in arriving at a solution.
(iii) Principles of logic should be applied correctly.
(iv) There can be no flaws, inconsistencies or contradictions in any of the assumptions made to solve a problem or to prove a theorem.

Deductive reasoning

Deductive reasoning uses a premise, axiom, postulate, theorems or facts which are accepted as true and then using these facts as a starting point begin a step-by-step process marching toward some conclusion. As you proceed to prove your conclusion verify the truth of each step of your reasoning process. Deductive reasoning is starting with a premise and then taking logical steps toward a conclusion. Deductive reasoning uses the law of syllogism to arrive at a conclusion. That is, if $A \to B$, $B \to C$, $C \to D$ is true, then one can conclude from these facts that $A \to D$ is true. The two column (Statements | Reasons) proofs used in geometry is an example of this law of syllogism.

Note that sometimes in solving problems there may occur a mixture of inductive and deductive reasoning techniques. New ideas in mathematics different from axioms or postulates must be proven using precise statements. The two column (Statments

| Reasons) approach is one way to introduce the rigor of a mathematical proof. All of modern mathematics require proofs when new concepts, ideas or methods are introduced.

Occam's Razor

Occam's razor is a principle of logic which states- Any proof should use the least set of postulates and rules of logic. This principle was set forth by William of Occam (1288-1348) as a way to minimize internal contradictions in a proof and to make clear where ones starting points for ideas originate.

Mathematical induction

The method of mathematical induction is **a deductive method of reasoning** used to prove propositions or formulas which **depend upon positive integers** n being used to determine the solution. Mathematical induction requires three conditions to be satisfied.

(i) The proposition or formula can be verified to be true for small values of n. Usually one shows the proposition or formula holds for the values $n = 1, n = 2$, and sometimes $n = 3$ just to verify the results hold for small values of n.

(ii) One next extends the range of n for which the formula or proposition holds. One tries to prove that **if the formula holds for** n **equal to a positive integer** $k > 2$**, then it must also hold for the next integer** $k + 1$**.**

(iii) Make use of the results from part (i) and (ii) to prove **the truth of the case** $n = k$ **implies the truth of the case** $(k + 1)$**, then the proposition or formula will hold for all values of** $n = 1, 2, 3, \ldots$**.**

The method of mathematical induction is somewhat like having an infinite set of dominos standing on end. If you knock down the first domino and the second domino falls, does this mean all the dominos will fall? If you assume the kth domino falls and **prove that this implies the** $(k + 1)$**st domino falls**, then you can substitute $k = 1, 2, 3, \ldots, n, \ldots$ into the formula for dominos falling, the idea being that if the kth domino falling guarantees that the $(k + 1)$st domino will also fall, then this implies that all the dominos must fall.

Example 6-7. Let S_n denote the sum of the geometrical series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}, \quad n \text{ a positive integer, } a \text{ is constant} \tag{6.1}$$

where r is called the common ratio associated with consecutive terms.

Prove by mathematical induction that the sum of this series is given by the formula

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad \text{for } r \neq 1 \tag{6.2}$$

and $n = 1, 2, 3, \ldots$ is an integer.

Solution

In the case $n = 1$ equation (6.1) becomes $S_1 = a$ and equation (6.2) becomes $S_1 = a\frac{1-r}{1-r} = a$ and so equation (6.2) is true for this case.

In the case $n = 2$ equation (6.1) becomes $S_2 = a + ar$ and equation (6.2) becomes

$$S_2 = a\frac{1-r^2}{1-r} = a\frac{(1-r)(1+r)}{1-r} = a(1+r) = a + ar$$

and so equation (6.2) also holds for the next integer 2.

Assume equation (6.2) holds for $n = k$ and write

$$S_k = \underbrace{a + ar + ar^2 + \cdots ar^{k-1}}_{k\ terms} = \frac{a(1-r^k)}{1-r} \tag{6.3}$$

Using the assumption that equation (6.3) is true, show that this guarantees that the formula

$$S_{k+1} = \underbrace{a + ar + ar^2 + \cdots + ar^{k-1} + ar^k}_{(k+1)\ terms} = \frac{a(1-r^{k+1})}{1-r} \tag{6.4}$$

must also hold true.

To prove that this is indeed the case, write

$$\begin{aligned} S_{k+1} = S_k + ar^k =& \frac{a(1-r^k)}{1-r} + ar^k \quad \text{add fractions} \\ =& \frac{a - ar^k + ar^k - ar^{k+1}}{1-r} \\ =& \frac{a - ar^{k+1}}{1-r} = \frac{a(1-r^{k+1})}{1-r} \end{aligned} \tag{6.5}$$

Here we have added equals to equals to obtain, after simplification, equal results. Hence the truth of the $n = k$ case implies the truth of the $n = k+1$ case. This implies the given equation is true for all values of the integer n.

■

Example 6-8. Show that the sum $S_n = 1 + 2 + 3 + \cdots + (n-1) + n$ of the first n integers is given by $S_n = \dfrac{n(n+1)}{2}$

Solution

To verify that the formula

$$S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = \frac{n(n+1)}{2} \tag{6.6}$$

is true for all values of n, one must first show the validity of the formula for small values of n.

For $n = 1$ equation (6.6) reduces to $S_1 = 1 = \dfrac{1 \cdot 2}{2} = 1$

For $n = 2$ equation (6.6) reduces to $S_2 = 1 + 2 = \dfrac{2 \cdot 3}{2} = 3$

and so the equation (6.6) is valid for the small values of $n = 1$ and $n = 2$.

Next assume the given equation holds for some positive integer $n = k$ so that

$$S_k = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \tag{6.7}$$

Finally, prove that the assumption of truth for $n = k$ implies the truth of the formula for $n = k + 1$. Note that equals added to equals produces equals so that

$$\begin{aligned} S_{k+1} = S_k + (k+1) =& \frac{k(k+1)}{2} + (k+1) \\ =&(k+1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2} \end{aligned} \tag{6.8}$$

which is the formula (6.6) with n replaced by $k + 1$. Hence, the truth of the kth statement implies the truth of the $(k+1)st$ statement which implies the given equation is true for all values of n.

■

The summation formula in equation (6.6) has been a known formula for over 2000 years. It can be quickly obtained by writing the equation (6.6) followed by the same equation with the integers written in reverse order. Observe in the right-hand columns that the numbers adds up to $(n+1)$ and there are n of these numbers. This gives

$$\begin{aligned} S_n =& \; 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ S_n =& \; n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ \hline 2S_n =& \; \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)}_{n \; terms} \end{aligned} \tag{6.9}$$

$$2S_n = n(n+1)$$

or $$S_n = \frac{n(n+1)}{2}$$

An addition of the sums as written in equation (6.9) together with some algebra produces the result indicated by equation (6.6).

The summation sign $\sum$

The mathematical symbol $\sum$ (Greek letter sigma) is used to denote a **summation of terms**. If $f = f(x)$ is a function whose domain contains all the integers and m is an integer, then the notation

$$\sum_{j=1}^{m} f(j) = f(1) + f(2) + f(3) + \cdots + f(m) \tag{6.10}$$

is used to denote the summation of the terms $f(j)$ as j varies from the 1 to m. Here $j = 1$ is called the **starting index** for the sum and the m above the sigma sign is used to denote the **ending index** for the sum. The quantity j is called the **dummy summation index** because **the letter j does not occur in the answer** and j can be replaced by some other index if one desires to do so. The sum is understood to be zero if the ending index is less than the starting index. If the ending index is ∞, the series is called an infinite series.

The result for the geometric series in example 6-7 can be expressed

$$\sum_{j=1}^{n} ar^{j-1} = \frac{a(1-r^n)}{1-r} \tag{6.11}$$

The summation of the integers formula from example 6-6 can be written as

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6.12}$$

The sigma summation convention can save you a lot of writing.

Example 6-9. Calculate the sum

$$\sum_{m=100}^{1000} m = 100 + 101 + 102 + \cdots + 1000 \tag{6.13}$$

Solution

We know the sum $\sum_{m=1}^{99} m = \frac{99(100)}{2}$ from equation (6.12) with $n = 99$ and we know the sum $\sum_{k=1}^{1000} k = \frac{1000(1001)}{2}$ again from equation (6.12) with $n = 1000$. The difference gives us

$$\sum_{j=100}^{1000} j = \sum_{j=1}^{1000} j - \sum_{j=1}^{99} j = \frac{1000(1001)}{2} - \frac{99(100)}{2} = 495\,550$$

This result can be generalized by writing

$$\sum_{j=m}^{n} j = \sum_{j=1}^{n} j - \sum_{j=1}^{m-1} j \tag{6.14}$$

where m and n can be any integers you want, provided $m < n$.

■

Example 6-10. Find the sum of the squares of the first n integers

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \sum_{j=1}^{n} j^2$$

Solution

Here we do not have a formula for the sum and so we cannot use the method of mathematical induction to prove the formula. This is a chance to do some inductive reasoning. Writing out the squares of the integers

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \; \ldots \; (n-2) \quad (n-1) \quad n$$

one obtains

$$0^2 \quad 1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2 \; \ldots \; (n-2)^2 \quad (n-1)^2 \quad n^2 \tag{6.15}$$

Examine what we have and make note that the difference between two consecutive squares is given by

$$n^2 - (n-1)^2 = n^2 - [n^2 - 2n + 1] = 2n - 1$$

which is **an odd number for all integers n.** Subtract a lower square from the next higher square in equation (6.15) and show there results the odd numbered differences listed below

$$\begin{array}{cccccccccc} 0^2 & 1^2 & 2^2 & 3^2 & 4^2 & 5^2 & \ldots & (n-2)^2 & (n-1)^2 & n^2 \\ & 1 & 3 & 5 & 7 & 9 & \ldots & & (2n-3) & (2n-1) \end{array}$$

The above differences shows that **the square of any integer** can be written as **the sum of consecutive odd integers** or

$$\sum_{k=1}^{n}(2k-1)=n^2$$

One can construct a table of such sums.

$$\begin{aligned}
1^2 =&1\\
2^2 =&1+3\\
3^2 =&1+3+5\\
4^2 =&1+3+5+7\\
5^2 =&1+3+5+7+9\\
\vdots \qquad & \qquad \vdots\\
\sum_{j=1}^{n-1}(2j-1)=(n-1)^2 =&1+3+5+7+\cdots+(2n-3)\\
\sum_{j=1}^{n}(2j-1)=n^2 =&1+3+5+7+\cdots+(2n-3)+(2n-1)
\end{aligned} \tag{6.16}$$

Examine the sum of the terms that are to the left of the equal signs in the equations (6.16). This is the sum of integers squared that we desire. On the right hand side of the equal signs is a triangular array of numbers. This triangle array of numbers represents the answer that we desire. If we can sum these numbers we have our answer.

Here we have used pattern recognition to come up with an answer. If we can write the answer for specific values of n and then try to obtain a more general form of the answer for any integer n, then we can use mathematical induction to prove the more general result. Playing around with the triangular array of numbers one might discover the following more general method of obtaining the desired summation.

The triangular array of numbers from equation (6.16) can be written in three different ways, as illustrated by the following listings.

$$\begin{array}{lllllll}
1 \\
1 & 3 \\
1 & 3 & 5 \\
1 & 3 & 5 & 7 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
1 & 3 & 5 & 7 & \cdots & (2n-3) \\
1 & 3 & 5 & 7 & \cdots & (2n-3) & (2n-1)
\end{array}$$

$$\begin{array}{ccccccc}
1 & & & & & & \\
3 & 1 & & & & & \\
5 & 3 & 1 & & & & \\
7 & 5 & 3 & 1 & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
(2n-3) & (2n-5) & \cdots & 5 & 3 & 1 & \\
(2n-1) & (2n-3) & \cdots & 7 & 5 & 3 & 1
\end{array}$$

$$\begin{array}{ccccccc}
(2n-1) & & & & & & \\
(2n-3) & (2n-3) & & & & & \\
(2n-5) & (2n-5) & (2n-5) & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
5 & 5 & 5 & \cdots & 5 & & \\
3 & 3 & 3 & \cdots & 3 & 3 & \\
1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{array}$$

The sum of the numbers in each array represents the sum $S = \sum_{j=1}^{n} j^2$ that we are trying to calculate. Add the numbers from each array which **come from the same position in each of the triangular arrays listed above** and find that in all of the **sums from the same positions** is given by $(2n+1)$. We can then write the sums from all three arrays as

$$3S = (2n+1) times (\text{ number of positions in the triangular array})$$

The number of positions in each array is

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$$

Therefore,

$$3S = \frac{(2n+1)n(n+1)}{2} \qquad \text{or} \qquad S = \frac{n}{6}(n+1)(2n+1)$$

so that

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \sum_{j=1}^{n} j^2 = \frac{n}{6}(n+1)(2n+1) \tag{6.17}$$

Lets see if we can prove this formula is true for all values of n by using mathematical induction. Let $n = 1$ in equation (6.17) and find $1^2 = \frac{1}{6}(2)(3) = 1$ and so the formula is true for $n = 1$. Substitute $n = 2$ into equation (6.17) and find

$1^2 + 2^2 = \frac{2}{6}(3)(5) = 5$ and so the formula is also true for $n = 2$. Assume the formula is true for $n = k$. Let

$$S_k = 1^2 + 2^2 + \cdot + k^2 = \frac{k}{6}(k+1)(2k+1) \tag{6.18}$$

denote the sum of integers squared through $n = k$ and show by adding $(k+1)^2$ to both sides of equation (6.18) one obtains

$$\begin{aligned} S_{k+1} = S_k + (k+1)^2 =& \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ =&(k+1)\left[\frac{k}{6}(2k+1) + (k+1)\right] \\ =&(k+1)\left[\frac{2k^2+7k+6}{6}\right] \\ =&\frac{(k+1)}{6}(k+2)(2k+3) \end{aligned}$$

representing the sum of the squares through $k+1$. Here the truth of the kth formula implies the truth of the $(k+1)$st formula. Therefore, equation (6.17) is valid for all values of n.

■

Example 6-11.

Let S denote the sum of the cubes of the first n integers, written

$$S = \sum_{m=1}^{n} m^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 \qquad \text{Find S}$$

Solution

Examine the telescoping series

$$\sum_{m=1}^{n}[m^4 - (m-1)^4] = [1^4 - 0^4] + [2^4 - 1^4] + [3^4 - 2^4] + [4^4 - 3^4] + \cdots + [n^4 - (n-1)^4]$$

and note that for every integer $j < n$, when a term j^4 occurs it is followed by a term $-j^4$ and so these terms always add to zero. This is why the series is called a telescoping series. All the terms in the telescoping series add to zero except one term. Study this series and show

$$\sum_{m=1}^{n}[m^4 - (m-1)^4] = n^4$$

Now we can do some deductive reasoning using just plain ordinary algebra. We will make use of known facts to combine the facts and solve the problem.

Use some algebra and show

$$\begin{aligned} m^4-(m-1)^4 =& m^4-[m^4-4m^3+6m^2-4m+1] \\ =& 4m^3-6m^2+4m-1 \end{aligned} \tag{6.19}$$

Now sum each term in equation (6.19) from $m=1$ to $m=n$ and show

$$\sum_{m=1}^{n}[m^4-(m-1)^4]=n^4=4\sum_{m=1}^{n}m^3-6\sum_{m=1}^{n}m^2+4\sum_{m=1}^{n}m-\sum_{m=1}^{n}1 \tag{6.20}$$

Examine the last three summations in equation (6.20) and show

$$\begin{aligned} \sum_{m=1}^{n}1 =& \underbrace{1+1+1+\cdots+1}_{\text{n ones}}=n \\ \sum_{m=1}^{n}m =& 1+2+3+\cdots+n=\frac{n(n+1)}{2} \qquad \text{from equation (6.12)} \\ \sum_{m=1}^{n}m^2 =& 1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6} \qquad \text{from equation (6.17)} \end{aligned} \tag{6.21}$$

Substitute the value of these sums into the equation (6.20) and verify that

$$n^4=4\sum_{m=1}^{n}m^3-n(n+1)(2n+1)+2n(n+1)-n$$

which simplifies to

$$\begin{aligned} 4\sum_{m=1}^{n}m^3 =& \, n^4+n(n+1)(2n+1)-2n(n+1)+n \\ =& \, n\left[n^3+2n^2+n+2n+1-2n-2+1\right] \\ =& \, n[n^3+2n^2+n] \\ =& \, n^2[n^2+2n+1] \\ =& \, n^2(n+1)^2 \end{aligned}$$

or

$$S=\sum_{m=1}^{n}m^3=1^3+2^3+3^3+4^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2=(1+2+3+\cdots+n)^2 \tag{6.22}$$

Examining the sum of cubes

Examine the sums

$$\begin{aligned} 1^3 =& 1 \\ 1^3+2^3 =& 9 \\ 1^3+2^3+3^3 =& 36 \\ 1^3+2^3+3^3+4^3 =& 100 \\ 1^3+2^3+3^3+4^3+5^3 =& 225 \\ 1^3+2^3+3^3+4^3+5^3+6^3 =& 441 \end{aligned}$$

and observe that the sums are all square numbers. This is a further indication that the formula we derived is correct.

One can double check our results, in case we made an algebra mistake, by using mathematical induction. Examine the equation (6.22) for $n = 1$ and $n = 2$ by showing

$$\text{for } n = 1, \quad 1^3 = (\frac{1(2)}{2})^2 = 1$$
$$\text{and for } n = 2, \quad 1^3 + 2^3 = \left(\frac{2(3)}{2}\right)^2 = 3^2 = 9$$

are both true statements.

Assume equation (6.22) is true for a positive integer $n = k$ so that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2 \tag{6.23}$$

then add $(k+1)^3$ to both sides of equation (6.23) and obtain the equation

$$1^3 + 2^3 + 3^3 + \cdot + k^3 + (k+1)^3 = (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 \tag{6.24}$$

to be verified as having the same form as equation (6.22) with n replaced by $k+1$. Let S_{k+1} denote the sum $(1 + 2 + 3 + \cdots + k + (k+1))$ and write equation (6.24) in the form

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdot + k^3 + (k+1)^3 =& (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 \\ =& \left(1 + 2 + 3 + \cdots + k + \underbrace{(k+1) - (k+1)}_{add\ and\ subtract}\right)^2 + (k+1)^3 \\ =& (S_{k+1} - (k+1))^2 + (k+1)^3 \\ =& S_{k+1}^2 - 2S_{k+1}(k+1) + (k+1)^2 + (k+1)^3 \end{aligned}$$

Replace S_{k+1} by its equivalent $\left[\frac{(k+1)(k+2)}{2}\right]$ to find

$$\begin{aligned}1^3+2^3+3^3+\cdot+k^3+(k+1)^3 =&S_{k+1}^2 - 2\left[\frac{(k+1)(k+2)}{2}\right](k+1)+(k+1)^2+(k+1)^3\\ =&S_{k+1}^2-(k+1)\underbrace{(k+1+1)}_{1+1=2}(k+1)+(k+1)^2+(k+1)^3\\ =&S_{k+1}^2-(k+1)^3-(k+1)^2+(k+1)^2+(k+1)^3\end{aligned}$$

which simplifies to equation (6.23) with k replaced by $k+1$ everywhere. Hence the truth of the kth formula implies the truth of the $(k+1)$st formula.

■

Example 6-12. (Sum of cubes)

Examine the representations

$$\begin{aligned}1^3 =&1\\ 2^3 =&3+5\\ 3^3 =&7+9+11\\ 4^3 =&13+15+17+19\\ 5^3 =&21+23+25+27+29\end{aligned} \tag{6.25}$$

Do you recognize any patterns? It appears that n^3 would have n-terms, but what is the beginning term for n^3? Examine the following patterns.

The numbers being cubed are

1 2 3 4 5 ... n

and the starting values on the right-hand sums are

1 3 7 13 21 31 ... ?

Taking difference between consecutive starting values gives

2 4 6 8 10 ... ?

and taking differences of the differences one finds

2 2 2 2 2 ... 2

Mathematicians who examine patterns have discovered that when differences of differences are all constant, then the original pattern must have the form

$$f(n) = c_0 + c_1(n-1) + c_2(n-1)(n-2) \tag{6.26}$$

where c_0, c_1 and c_2 are constants to be determined. We can find the values for these constants by substituting the values $n = 1, 2$ and 3 into the formula (6.26). Here $f(n)$ represents the first term of the summations on the right-hand side of the equations (6.25) and so we require

$$\begin{aligned}
\text{For } n=1 \quad f(1) =& c_0 = 1 \\
\text{For } n=2 \quad f(2) =& c_0 + c_1(2-1) = 3 \qquad \Rightarrow c_1 = 2 \\
\text{For } n=3 \quad f(3) =& c_0 + c_1(3-1) + c_2(3-1)(3-2) = 7 \qquad \Rightarrow c_2 = 1 \\
\text{giving the equation} \quad f(n) =& 1 + 2(n-1) + (n-1)(n-1) = n^2 - n + 1
\end{aligned}$$

We now can generalize the listing (6.25) by making the following observations

$$\begin{aligned}
1^3 =& 1 \\
2^3 =& 3 + 5 \\
3^3 =& 7 + 9 + 11 \\
4^3 =& 13 + 15 + 17 + 19 \\
5^3 =& 21 + 23 + 25 + 27 + 29 \\
\vdots \quad & \quad \vdots \\
n^3 =& \underbrace{(n^2 - n + 1) + (n^2 - n + 3) + (n^2 - n + 5) + \cdots + (n^2 - n + (2n-1))}_{n-terms}
\end{aligned} \tag{6.27}$$

n	$\frac{n(n+1)}{2}$
1	1
2	3
3	6
4	10
5	15
6	21

A summation of the results given in the equations (6.27) can be expressed using our Σ summation convention. On the right-hand side of the equations (6.27) there is a summation of odd numbered terms. Odd numbers are represented $2i - 1$ for $i = 1, 2, 3, \ldots, m$ where the last odd integer is $2m-1$ which we want to equal $n^2-n+2n-1$. This requires that we select m to satisfy

$$2m - 1 = n^2 - n + 2n - 1 \;\Rightarrow\; m = \frac{n(n+1)}{2} \tag{6.28}$$

A summation of the terms on the left-hand side of the equal signs from the equations (6.27) followed by a summation of terms on the right-hand side of the equal signs gives the result

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{\frac{n(n+1)}{2}} (2i - 1) \tag{6.29}$$

where the right hand side is a summation of odd numbers $2i-1$ with index i ranging from 1 to $\frac{n(n+1)}{2}$. Suppose we add and then subtract all the missing even numbers to equation (6.29). This will change the equation (6.29) into a form where previous summation formulas can be used to obtain our answer. For example,

$$\text{add} \quad 2+4+6+\cdots+(n^2+n)$$

to change the right hand side into the form

$$\sum_{m=1}^{n^2+n} m = 1+2+3+\cdots+(n^2+n) \tag{6.30}$$

Here we have added the even numbers to change the summation (6.29) into a form where we know the answer. In order not to change equation (6.29) after adding the even numbers we must now subtract what we have just added. That is

$$\text{subtract} \qquad 2\left[1+2+3+\cdots+\frac{n^2+n)}{2}\right] \tag{6.31}$$

We can now use the first summation formula from the equations (6.23) to sum the equations (6.30) and (6.31). The equation (6.29) can now be written as

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{\frac{n(n+1)}{2}} (2i-1) = \sum_{i=1}^{n^2+n} i - 2\sum_{i=1}^{\frac{n(n+1)}{2}} i \tag{6.32}$$

The first sum is

$$\sum_{i=1}^{n^2+n} i = \frac{(n^2+n)(n^2+n+1)}{2} = \frac{2n^2(n+1)^2}{4} + \frac{n(n+1)}{2} \tag{6.33}$$

and the second sum is

$$2\sum_{i=1}^{\frac{n(n+1)}{2}} i = 2\left[\frac{\left(\frac{n(n+1)}{2}\right)\left(\frac{n(n+1)}{2}+1\right)}{2}\right] = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)}{2} \tag{6.34}$$

Subtracting the equation (6.34) from equation (6.33) gives the final result

$$S = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{6.35}$$

or

$$1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} \tag{6.36}$$

■

All of the summations

$$\begin{aligned}\sum_{i=1}^{n} 1 =& 1+1+1+\cdots+1 = n\\ \sum_{i=1}^{n} i =& 1+2+3+\cdots+n = \frac{n(n+1)}{2}\\ \sum_{i=1}^{n} i^2 =& 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}\\ \sum_{i=1}^{n} i^3 =& 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2 = (1+2+3+\cdots+n)^2\end{aligned} \tag{6.37}$$

were known to the famous mathematician, physicist, engineer and astronomer Archimedes of Syracuse (287-212) CE. Archimedes used these formulas to make many discoveries about various shapes in geometry. We will also use these results to make our own discoveries.

In particular, given a plane or solid figure one can cut the figure into many small slices to form elements of area or volume. The summation formulas (6.37) can then be used to sum these small elements to calculate the area or volume.

Example 6-13.

Find the area of a right triangle by slicing it up into small pieces.

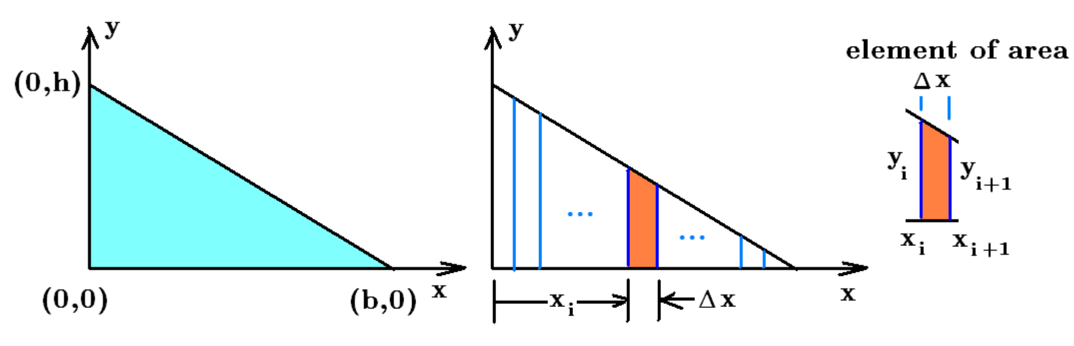


Figure 6-1. Right triangle sliced into n-parts.

Solution

Given the right triangle illustrated in the figure 6-1. The equation of the hypotenuse is given by the point-slope formula $y - y_0 = m(x - x_0)$ where

$$m = \frac{\text{change in y values}}{\text{change in x values}} = -\frac{h}{b} \text{ and } (x_0, y_0) = (b, 0) \Rightarrow \; y = -\frac{h}{b}(x - b)$$

Slice the triangle into n-parts. The distance between each slice being $\Delta x = \dfrac{b}{n}$.

$$\begin{aligned}
\text{Start at } x =&0 \text{ with } y = h\\
\text{The first slice is at } x_1 =&\frac{b}{n} \quad \text{with height } y_1 = -\frac{h}{b}(x_1 - b)\\
\text{The second slice is at } x_2 =&2\,\frac{b}{n} \quad \text{with height } y_2 = -\frac{h}{b}(x_2 - b)\\
\vdots \qquad\qquad & \qquad \vdots\\
\text{The } i\text{th slice is at } x_i =&i\,\frac{b}{n} \quad \text{with height } y_i = -\frac{h}{b}(x_i - b)\\
\text{The } (i+1)\text{st slice is at } x_{i+1} =&(i+1)\,\frac{b}{n} \quad \text{with height } y_{i+1} = -\frac{h}{b}(x_{i+1} - b)\\
\vdots \qquad\qquad & \qquad \vdots\\
\text{The last slice is at } x_{n-1} =&(n-1)\,\frac{b}{n} \quad \text{with height } y_{n-1} = -\frac{h}{b}(x_{n-1} - b)
\end{aligned}$$

The slice at $x = x_i$ produces an element of area in the shape of a trapezoid. The area of this trapezoid is the average of the bases times the height or

$$\begin{aligned}
\text{area of } i\text{th trapezoid} = A_{ti} =&\frac{1}{2}(y_i + y_{i+1})\,\Delta x\\
=&\frac{1}{2}\left[-\frac{h}{b}(i\frac{b}{n} - b) - \frac{h}{b}((i+1)\frac{b}{n} - b)\right]\frac{b}{n}\\
=&\frac{1}{2}\left[(2n-1)\frac{bh}{n^2} - \frac{2bh}{n^2}\,i\right]
\end{aligned}$$

Perform a summation on all the elements of area and note that we have created a situation where the equations (6.37) can be employed to evaluate the sums. Here we have

$$\text{Sum of area elements} = \sum_{i=0}^{n-1}\frac{bh}{n} - \sum_{i=0}^{n-1}\frac{1}{2}\frac{bh}{n^2} - \sum_{i=0}^{n-1}\frac{bh}{n^2}i \tag{6.38}$$

Here only the i terms are being summed with n some fixed integer, so that equation (6.38) simplifies to

$$\text{Sum of area elements} = \frac{bh}{n}\sum_{i=0}^{n-1}1 - \frac{1}{2}\frac{bh}{n^2}\sum_{i=0}^{n-1}1 - \frac{bh}{n^2}\sum_{i=0}^{n-1}i \tag{6.39}$$

From the equations (6.37) select the summations

$$\sum_{i=1}^{n}1 = n \qquad \text{and} \qquad \sum_{i=1}^{n}i = \frac{n(n+1)}{2}$$

and modify them to read

$$\sum_{i=0}^{n-1} 1 = n \quad \text{and} \quad \sum_{i=0}^{n-1} i = \frac{(n-1)n}{2}$$

Now substitute these results into the equation (6.39) and show there results

$$\text{Sum of area elements} = bh - \frac{bh}{2n} - \frac{bh}{2} + \frac{bh}{2n} = \frac{1}{2}bh \tag{6.40}$$

and consequently the equation (6.40) reduces to the result that the area of the triangle is one-half the base times the height.

This is an example of what can be done in future chapters to calculate areas and even volumes of complicated figures.

■

Introduction to determinants

A two by two (2x2) array of elements having the form

$$M_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{6.41}$$

is called a two by two (2x2) square matrix. A three by three (3x3) array of elements having the form

$$M_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{6.42}$$

is called a three by three (3x3) square matrix. In this introduction we will only consider square matrices M_{nxn} for $n = 2$ and $n = 3$ and leave square matrices M_{nxn} associated with $n > 3$ for another time. Associated with every **square matrix** is a single scalar quantity called the **determinant of the square array**. Matrix arrays which are not square do not have determinants.

The determinant of the two by two (2x2) array given by equation (6.41) can be calculated by the shortcut method[1]

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{6.43}$$

which represents the product of the numbers along the main diagonal minus the product of the numbers along the secondary diagonal as illustrated in the figure 5-2.

---

[1] The determinant of a M_{nxn} matrix is a single number calculated by considering all possible products having n-factors obtained by selecting one and only one number from each row and each column. There are $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ such products. Each product is assigned a plus or minus sign which depend upon how the numbers $123\ldots n$ are arranged. A summation of all these products, with the correct sign, produces the single number called the determinant.

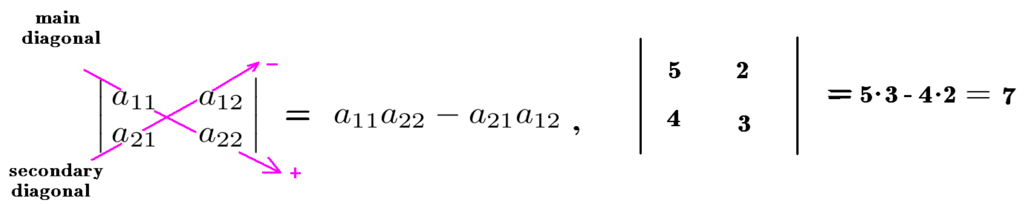


Figure 6-2. Determinant of two by two square array.

One method for calculating the determinant of the three by three (3x3) array given by equation (6.42) is the method using an expansion involving the elements in the first row. This method is called expansion by cofactors and is defined

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{6.44}$$

Here one starts with the first row of elements. Take the first element a_{11} and then cross out all the numbers in the first row and first column which leaves a (2×2) array. Next select the middle element from first row and change its sign giving $-a_{12}$. Now cross out all the elements in the first row and second column leaving another (2×2) array. Repeating this process, select the last element in the first row and then cross out the elements in the first row and last column. This gives the final (2×2) array. The determinants of the (2×2) arrays are then multiplied by there corresponding elements from the first row followed by a summation of these products as illustrated in the equation (6.44).

For example,

$$\begin{vmatrix} 2 & 8 & 3 \\ 5 & 6 & 4 \\ 8 & 7 & 9 \end{vmatrix} = 2 \begin{vmatrix} 6 & 4 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 5 & 4 \\ 8 & 9 \end{vmatrix} + 3 \begin{vmatrix} 5 & 6 \\ 8 & 7 \end{vmatrix} = (2)(26) - (8)(13) + (3)(-13) = -91$$

Properties of determinants

1)

One can interchange the rows and columns of a determinant without changing the value of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

2)

If two rows or columns of a determinant are changed, then the value of the determinant will change sign.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$$

3)

Multiplying all the elements of a row or column of a determinant by a constant k produces a new determinant with a value equal to k times the value of the original determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ k \cdot g & k \cdot h & k \cdot i \end{vmatrix} = k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

This means that if a row or column has a common factor k, then you can factor out the common factor.

4)

If all the elements in any row or column of a determinant are zero, then the value of the determinant will be zero.

5)

You can multiply the numbers in any row (or column) of a determinant by a constant value k and then add the results to some different row (or different column) without changing the value of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d+ka & e+kb & f+kc \\ g & h & i \end{vmatrix}$$

6)

If the elements of a row or column of a determinant can be written as the sum of two quantities, then the determinant can be written as the sum of two determinants

$$\begin{vmatrix} a+3x & b & c \\ d+x & e & f \\ g+2x & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 3x & b & c \\ x & e & f \\ 2x & h & i \end{vmatrix}$$

7)

If two rows or columns of a determinant are the same, then the value of the determinant is zero.

Solution of system of equations by determinants

Cramer's[2] rule for the solution of 2x2 and 3x3 systems of equations has the following representation.

The solution to the system of two equations in two unknowns x and y

$$\begin{aligned} a_1x + b_1y =& c_1 \\ a_2x + b_2y =& c_2 \end{aligned}$$

is given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad \text{and} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

The solution of the system of three equations in three unknowns x, y and z

$$\begin{aligned} a_1x + b_1y + c_1z =& d_1 \\ a_2x + b_2y + c_2z =& d_2 \\ a_3x + b_3y + c_3z =& d_3 \end{aligned}$$

is given by

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

The determinant in the denominator is called the determinant of the coefficients associated with the given system of equations. Observe the determinants in the numerators have a pattern to be recognized. That is, to solve for x, the first column of the determinant of the coefficients is replaced by the right-hand side values and to solve for y the second column of the determinant of the coefficients is replaced by the right-hand side values. Similarly, when solving for z the third column is replaced by the right-hand side values. The pattern continues for higher ordered systems of equations.

---

[2] Gabriel Cramer (1704-1752) a Swiss mathematician.

Calculating an estimate for π

Archimedes (287-212)BCE, one of the worlds greatest minds in mathematics and science determined an estimate for the value of π by examining the perimeters of both inscribed and circumscribed polygons associate with a circle.

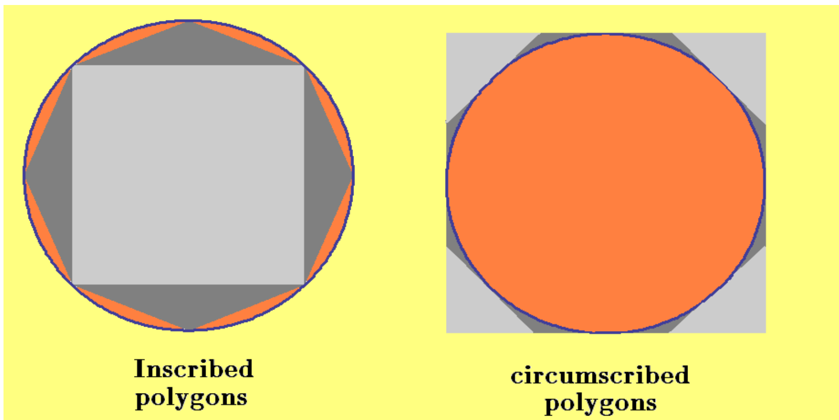


Figure 6-3.
Inscribed and circumscribed polygons for circle with radius $1/2$.

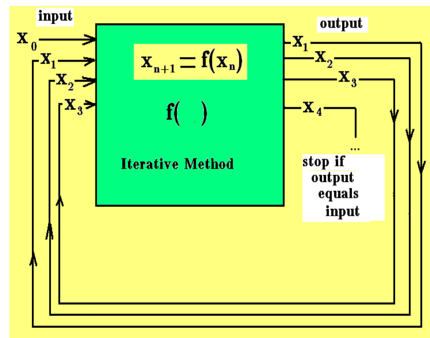
The following estimate for π is a demonstration of Archimedes method which is modified slightly. Consider a circle with diameter 1 or radius 1/2 as illustrated in the figure 6-3 and let p_n denote the perimeter of an inscribed regular polygon with n-sides and let P_n denote the perimeter of a circumscribed regular polygon with n-sides. The circle with radius 1/2 has a circumference of π and so one can write the inequality

$$p_n < \pi < P_n \tag{6.45}$$

Let us examine the inequality (6.45) as n increases in size.

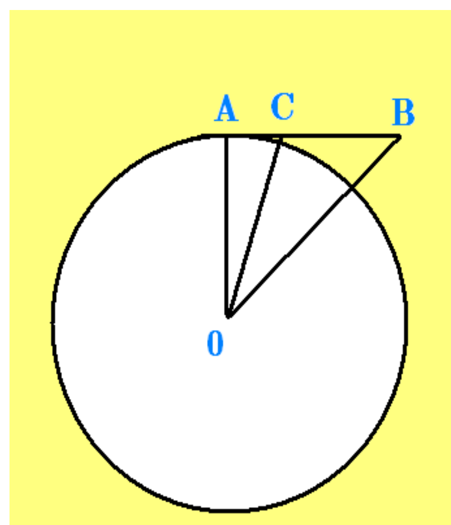
If you start with two squares, one inscribed within the circle and one circumscribed outside the circle and ask the question, **What happens when the number of the polygon sides are doubled?** Let x_n denote the length of one side associated with a regular polygon having n sides and let x_{2n} denote the length of one side associated with a regular polygon having $2n$ sides. There are two cases to consider. The first case you start with a circumscribed square and start the doubling process keeping track of the resulting regular polygon perimeters. The second case you start with an inscribed square and start the doubling process keeping track of the resulting regular polygon perimeters. In each of the two cases one can find a relation between the size of the sides x_n and x_{2n} and one can solve for x_{2n} in terms of x_n. The resulting formula can then be used in an iterative way to find the value of π.

An iterative procedure consists of having a starting value substituted into an equation to produce a $result_1$, the process is repeated **by substituting** $result_1$ **into the same equation** to produce $result_2$. This process of taking an old result and substituting it back into an equation to get a new result is called an iterative or cyclic way of repeating what you have just done.



As an example of an iterative process is to put the number 8 into your hand held calculator and then continue to hit the square root button until you obtain the number 1. When you do this, you are using the iterative equation $x_{n+1} = \sqrt{x_n}$, with $x_0 = 8$ the starting value, to solve the equation $x^2 = x$. This iterative procedure produces the positive root $x = 1$.

Circumscribed polygons



Let $\frac{x_n}{2} = \overline{\text{AB}}$ denote half the length of one side associated with a polygon with n-sides which circumscribes a circle with radius $1/2$. The perimeter of the polygon with n-sides is then given by $p_n = nx_n$. Similarly, let $\frac{x_{2n}}{2} = \overline{\text{AC}}$ denote the half length of one side associated with a polygon with $2n$-sides which circumscribes the same circle. The perimeter of this second polygon is $P_{2n} = 2nx_{2n}$. We wish to find a relation between the side length x_{2n} in terms of the side length x_n as the circumscribed polygon doubles in the number of sides.

The accompanying figure shows what happens when the number of sides of the circumscribed polygon is doubled. Observe that the line segment $\overline{0\text{C}}$ divides triangle $\triangle 0AB$ into two parts where one can employ Thales theorem to obtain the ratios

$$\frac{\overline{\text{AC}}}{\overline{\text{A0}}} = \frac{\overline{\text{CB}}}{\overline{\text{B0}}} \tag{6.46}$$

where

$$\begin{aligned}\overline{\mathrm{CB}} &= \frac{x_n}{2} - \frac{x_{2n}}{2}\\ \overline{\mathrm{A0}} &= \frac{1}{2}\\ \overline{\mathrm{AC}} &= \frac{x_{2n}}{2}\\ \overline{\mathrm{AB}} &= \frac{x_n}{2}\end{aligned} \tag{6.47}$$

and $\overline{\mathrm{B0}}^2 = \overline{\mathrm{AB}}^2 + \overline{\mathrm{A0}}^2$ by the Pythagorean theorem. Therefore,

$$\overline{\mathrm{B0}} = \sqrt{\overline{\mathrm{AB}}^2 + \overline{\mathrm{A0}}^2} \quad \Rightarrow \quad \overline{\mathrm{B0}} = \sqrt{\left(\frac{x_n}{2}\right)^2 + \frac{1}{4}} \tag{6.48}$$

Substitute the results from the equations (6.47) and (6.48) into the equation (6.46) and then simplify the results to verify that

$$x_{2n} = \frac{x_n}{1 + \sqrt{1 + x_n^2}} \tag{6.49}$$

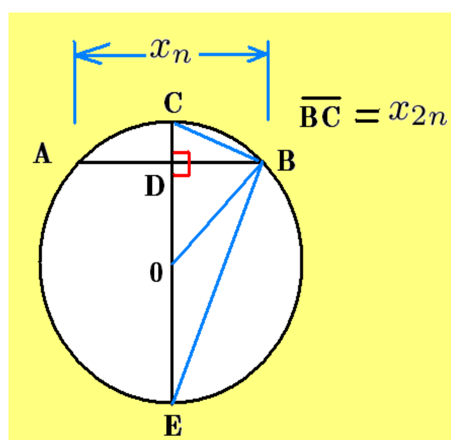
The equation (6.49) tells you that if you start with a square with 4 sides where you know each side of the circumscribing square has a length $x_4 = 1$, then the next circumscribed polygon, which has double the number of sides, has the length of each side given by

$$x_8 = \frac{x_4}{1 + \sqrt{1 + x_4^2}} = \frac{1}{1 + \sqrt{1 + 1}} = 0.414214$$

and the perimeter is $P_8 = 8(0.401214) = 3.313708$. The value for x_8 can now be used to calculate x_{16} and P_{16} and this process can be repeated using the equation (6.49) in an iterative fashion.

Inscribed polygons

Consider the same circle as above with unit diameter as illustrated. Examining the inscribe polygons we want to find a formula expressing side x_{2n} in terms of side x_n. The accompanying figure illustrates the changes when the number of sides of the inscribed polygon doubles. By Thales theorem the triangle $\triangle EBC$ is a right triangle and so it is similar to the right triangle $\triangle CDB$ ($\triangle EBC \sim \triangle CDB$), as illustrated in the accompanying figure. The sides of these triangles are proportional so that



$$\frac{x_{2n}}{\overline{\mathrm{CD}}} = \frac{\overline{\mathrm{CE}}}{x_{2n}} \quad \Rightarrow \quad x_{2n}^2 = \overline{\mathrm{CD}} \cdot \overline{\mathrm{CE}} \tag{6.50}$$

The diameter of the circle is $1 = \overline{\text{CE}}$ so that equation (6.50) simplifies to

$$x_{2n}^2 = \overline{\text{CD}} = \overline{\text{C0}} - \overline{\text{D0}} \tag{6.51}$$

The radius $\overline{\text{C0}} = \frac{1}{2}$ and from the Pythagorean theorem applied to triangle $\triangle 0DB$ one finds

$$\overline{\text{D0}}^2 + \overline{\text{DB}}^2 = \overline{\text{0B}}^2 \Rightarrow \ \overline{\text{D0}} = \sqrt{\frac{1}{4} - \left(\frac{x_n}{2}\right)^2} \tag{6.52}$$

since $\overline{\text{DB}} = \frac{x_n}{2}$ and $\overline{\text{0B}} = \frac{1}{2}$. Substituting the results from the equations (6.52) into the equation (6.51) and simplifying the result one finds

$$x_{2n} = \sqrt{\frac{1 - \sqrt{1 - x_n^2}}{2}} \tag{6.53}$$

The equation (6.53) tells you that if you start with an inscribed square with 4 sides, where you know the size x_4, then the equation (6.53) gives the size x_8 of each of the sides of the next regular inscribed polygon having double the number of sides. This is followed by the calculation of the new inscribed perimeter p_8. For example, to start the process, the side of the inscribed square is x_4 which by the Pythagorean theorem must satisfy $x_4^2 + x_4^2 = 1$ or $x_4 = \frac{1}{\sqrt{2}}$ since the diameter of the circle is the hypotenuse having the value 1. The equation (6.53) then gives

$$x_8 = \sqrt{\frac{1 - \sqrt{1 - x_4^2}}{2}} = \sqrt{\frac{1 - \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}}{2}}$$

and the perimeter of the polygon with this side is $p_8 = 8x_8 = 3.06147$. The value for x_8 is now used to substitute into the equation (6.53) to find x_{16} and p_{16}, then one can continue this iterative process until you exceed the accuracy of your calculator. The results from the equations (6.50) and (6.53) are summarized by the table of values on the next page. This table gives a comparison of sides and perimeters associated with various inscribed and circumscribed polygons. One finds the inscribed and circumscribed polygons approach the perimeter length π which is the circumference of the circle. If the circles in the above figures had a radius r, then the circumference would approach $2\pi r$.

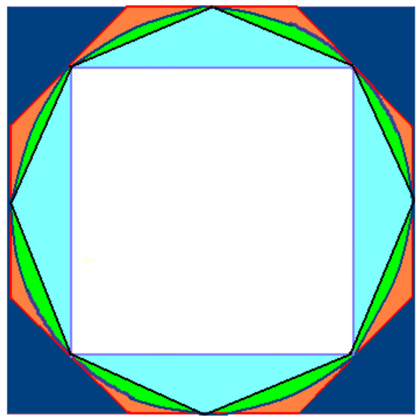
Recall the area of a regular n-sided polygon is given by

$$\text{Area of polygon} = \frac{1}{2}(p_n)(a)$$

where p_n is the perimeter and a is the apothem. In the limit as n increases in size the apothem approaches the radius r of the circle and the perimeter p_n approaches $2\pi r$. Therefore, in the limit as n becomes very large, the polygons approach a circle with area

$$\text{Area of circle} = \frac{1}{2}(2\pi r)(r) = \pi r^2$$

The following table gives the results of using equations (6.49) and (6.53) in an iterative fashion to calculate lower and upper bounds for the estimate of π.



number of sides	Inscribed Polygon Side	Inscribed Polygon Perimeter p_n	Circumscribed Polygon Side	Circumscribed Polygon Perimeter P_n
4	$x_4 = \frac{1}{\sqrt{2}}$ $x_{2n} = \sqrt{\frac{1-\sqrt{1-x_n^2}}{2}}$	$\frac{4}{\sqrt{2}}$	$x_4 = 1$ $x_{2n} = \frac{x_n}{1+\sqrt{1+x_n^2}}$	4
8	0.382683432	3.06146746	0.414213562	3.31370850
16	0.198912367	3.18259788	0.198912367	3.18259788
32	0.0980171403	3.13654849	0.0984914034	3.15172491
64	0.0490676743	3.14033116	0.0491268498	3.14411839
128	0.0245412285	3.14127725	0.0245486221	3.14222363
256	0.0122715383	3.1415138	0.0122724624	3.14175037
512	0.00613588465	3.14157294	0.00613600016	3.14163208
1024	0.00306795676	3.14158773	0.0030679712	3.14160251
2048	0.00153398019	3.14159142	0.00153398199	3.14159512
4096	0.000766990319	3.14159235	0.000766990544	3.14159327
8192	0.000383495188	3.14159258	0.000383495216	3.14159281
16384	0.000191747597	3.14159263	0.000191747601	3.14159269

The value of π to 10 decimal places is

$$3.1415926535$$

Note that the top line in the above table gives you the sides of the inscribed and circumscribed squares which are used as the starting values for using the equations (6.49) and (6.53) in an iterative fashion which is associated with doubling the sides of the inscribed and circumscribed polygons. The left side of the above table lists inscribed polygons, the length of one side and the perimeter length associated with the polygon of n-sides. The right side of the table lists circumscribed polygons, the length of one side and the perimeter length associated with the polygon of n-sides. As the number of sides increases the inscribed perimeter approaches π from below and the circumscribed perimeter approaches π from above. This method sandwiches π between an upper and lower values getting closer and closer to the actual value.

Exercises

▶ **6-1.** If two triangles are similar, then their corresponding sides are proportional, is of the form $A \Rightarrow B$.

(a) Write the converse $B \Rightarrow A$

(b) Write the opposite $\sim A \Rightarrow \sim B$

(c) Write the contrapositive $\sim B \Rightarrow \sim A$

▶ **6-2.** If two sides of a triangle are congruent, then the angles opposite these sides are also congruent, is of the form $A \Rightarrow B$

(a) Write the converse $B \Rightarrow A$

(b) Write the opposite $\sim A \Rightarrow \sim B$

(c) Write the contrapositive $\sim B \Rightarrow \sim A$

▶ **6-3.** Fill in the following truth table.

Truth Table

A	B	$\sim A$	$\sim B$	$\sim A \vee \sim B$	$\sim A \wedge \sim B$
T	T				
T	F				
F	T				
F	F				

▶ **6-4.** Write the given sums using the $\sum$ summation sign and then prove the statements using mathematical induction.

(a) $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

(b) $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + (2n - 1) \cdot (2n) = \frac{1}{3}n(n + 1)(4n - 1)$

(c) $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$

▶ **6-5.** Use mathematical induction to prove the following statements.

(a) Prove the number of diagonals in a regular n-gon is $\frac{1}{2}n(n - 3)$

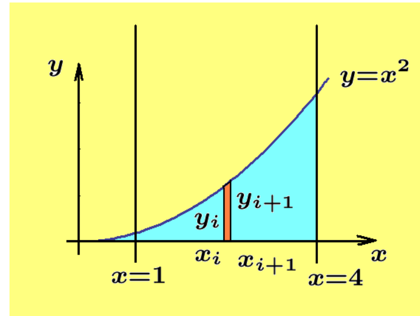
(b) Prove the interior angles of a regular n-gon sum to $(n - 2)\pi$ radians.

► **6-6.**

(a) Use slicing to find an estimate for the area bounded by the x-axis, the lines $x = 1$ and $x = 4$ and the parabola $y = x^2$.

(b) Show that as n increases without bound the area is 21 square units.

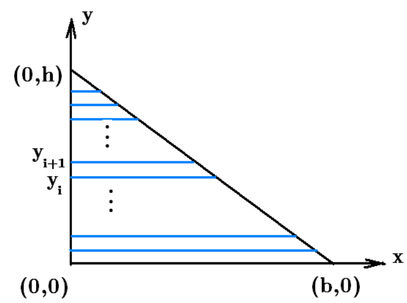
Hint: Let $\Delta x = \frac{3}{n}$ be width of each trapezoid.



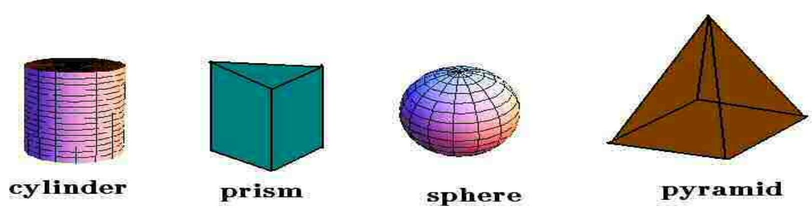
► **6-7.**

Use horizontal slicing of the given right triangle and summation to find the area.

Hint: Elements of area are trapezoids with $\Delta y = \frac{h}{n}$.



► **6-8.** What kind of volume element would result if the figures below were sliced horizontally by parallel planes?



► **6-9.** Evaluate the following determinants

$$(a)\quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 10 & 5 \\ 3 & -2 \end{vmatrix}$$

► **6-10.** Evaluate the following determinants

$$(a)\quad \begin{vmatrix} 1 & 3 & 5 \\ -2 & 3 & 0 \\ -1 & 3 & 6 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 3 & -1 & 2 \\ 0 & 0 & 4 \\ 1 & 1 & 3 \end{vmatrix}$$

► **6-11.** Prove that the value of the determinant is unchanged if the rows and columns are interchanged.

(a) Prove $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

(b) Prove $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

▶ **6-12.** Solve the 2×2 system of equations using determinants

$$(a)\quad \begin{aligned} 3x - 4y &= -5 \\ 2x + 3y &= 93 \end{aligned} \qquad (b)\quad \begin{aligned} 5x + 4y &= 59 \\ 7x - 5y &= 19 \end{aligned} \qquad (c)\quad \begin{aligned} 2x - 3y &= -6 \\ 3x + 2y &= 17 \end{aligned}$$

▶ **6-13.** Solve the 3×3 system of equations using determinants

$$(a)\quad \begin{aligned} 2x + 3y - z &= 5 \\ 5x - 3y + 2z &= 5 \\ 2x - 2y + 3z &= 8 \end{aligned} \qquad (b)\quad \begin{aligned} x + y - z &= 7 \\ 2x - 3y + 2z &= 3 \\ 3x - 2y + 3z &= 12 \end{aligned} \qquad (c)\quad \begin{aligned} 6x - y + 2z &= 25 \\ 4x + 2y - 3z &= 16 \\ x - 3y + 3z &= 1 \end{aligned}$$

▶ **6-14.** Given the iterative equation $x_{n+1} = x_n - \frac{(x_n^2 - 10)}{2x_n}$, $n = 0, 1, 2, 3, \ldots$ with the starting value $x_0 = 3$. Find the iterations x_1, x_2, x_3, x_4, x_5.

▶ **6-15.** Find the following sums

$$(a)\quad \sum_{i=1}^{100} i \qquad (b)\quad \sum_{j=1}^{100} j^2 \qquad (c)\quad \sum_{k=1}^{100} k^3$$

▶ **6-16.** Find the following sums

$$(a)\quad \sum_{i=10}^{100} i \qquad (b)\quad \sum_{j=10}^{100} j^2 \qquad (c)\quad \sum_{k=10}^{100} k^3$$

▶ **6-17.** Find the sum of the given geometric series

$$(a)\quad \sum_{i=1}^{10} 2(3)^{i-1} \qquad (b)\quad \sum_{j=1}^{5} \left(\frac{1}{2}\right) 2^{j-1} \qquad (c)\quad \sum_{k=1}^{5} 3(4)^{k-1}$$

▶ **6-18.** An arithmetic series has the form

$$S_n = \underbrace{a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + (a_1 + (n-1)d)}_{n \ terms}$$

where a_1 is the starting term, d is the common difference between terms and the last term is $a_n = a_1 + (n-1)d$. The arithmetic sum can also be expressed in the form $\sum_{j=1}^{n} [a_1 + (j-1)d]$.

(a) Reverse the sum and add to original sum to show $2S_n = n(a_1 + a_n)$

(b) Prove by mathematical induction $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

▶ **6-19.** Show that $\sum_{j=k}^{m} f(j) = \sum_{i=0}^{m-k} f(k+i)$ This is called shifting the summation index

▶ **6-20.** Find the sum of the arithmetic series

$$(a)\quad \sum_{j=0}^{5}[3+5j] \qquad (b)\quad \sum_{j=0}^{6}[4+3j] \qquad (c)\quad \sum_{k=2}^{14}[3+3k]$$

▶ **6-21.** Find a formula to calculate the summation $\sum_{k=m}^{n}[a_0+kd]$

▶ **6-22.** Find the following sums

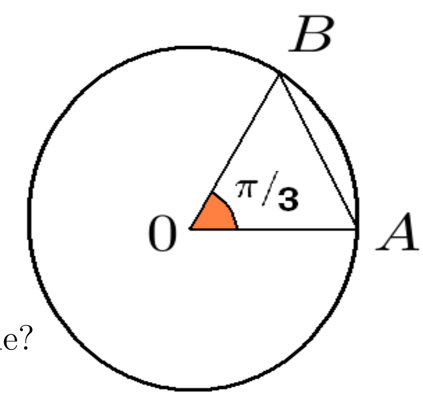
$$(a)\quad \sum_{m=1}^{(n-1)} m \qquad (b)\quad \sum_{m=1}^{(n-1)} m^2 \qquad (c)\quad \sum_{m=1}^{(n-1)} m^3$$

▶ **6-23.** Use multiplication to show

$$1+x+x^2+\cdots+x^{(n-2)}+x^{(n-1)} = \sum_{m=0}^{(n-1)} x^m = \frac{x^n-1}{x-1}$$

▶ **6-24.**

Given a circle with radius $r = \overline{0A}$. Rotate the radius $\overline{0A}$ through a 60° angle to end in position $\overline{0B}$ as illustrated.



(a) Find the length of the chord $\overline{AB}$.
(b) Find the area of triangle $\triangle 0AB$.
(c) How many triangles $\triangle 0AB$ will fit inside of the circle?

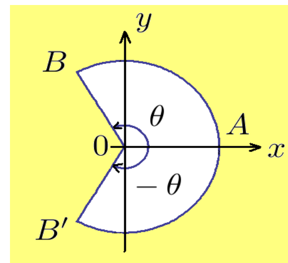
<u>*Historical note*</u>

George Cantor, a German mathematician who developed set theory, noted that the Babylonians were aware that the chord $\overline{AB}$ fits inside the circle six times and each chord subtends an arc corresponding to 60 degrees. This is possibly the start of the sexagesimal system.

Geometry
Chapter 7
Trigonometry I

Trigonometry

The word trigonometry comes from the Greek language and means 'measure of triangles'. Plane trigonometry deals with triangles in a plane and spherical trigonometry deals with spherical triangles on a sphere.



In plane trigonometry if a line $\overline{0A}$ is drawn through the origin of a coordinate system and then rotated counterclockwise about the origin to the position $\overline{0B}$, then the positive angle $\angle A0B = \theta$ is said to have been generated. If $\overline{0B}'$ is the reflection of $\overline{0B}$ about the x-axis, then when the line $\overline{0A}$ is rotated in a clockwise direction to the position $\overline{0B}'$, a negative angle $-\theta$ is said to be generated. The situation is illustrated in the accompanying figure.

By definition if two angles add to 90°, then they are called complementary angles and if two angles add to 180°, then the angles are called supplementary. Angles are sometimes measured in degrees $(^\circ)$, minutes $(')$, and seconds $('')$ where there are 360° in a circle, 60 minutes in one degree and 60 seconds in one minute. Another unit of measurement for the angle is the radian. An angle of one radian subtends an arc length on the circumference of a circle which has the same length as the radius of the circle. The circumference of a circle is given by the formula $c = 2\pi r$, where r is the radius of the circle. One finds that the conversion between radians and degrees is given by

$$\begin{array}{llll} (a) & 2\pi\,\text{radians} = 360^\circ & (c) & 1\,\text{radian} = \dfrac{180^\circ}{\pi} \\ (b) & \pi\,\text{radians} = 180^\circ & (d) & \dfrac{\pi}{180}\,\text{radians} = 1^\circ \end{array} \tag{7.1}$$

Thus, to convert 30° to radians, just multiply both sides of equation (7.1) (d) by 30 to get the result that $30^\circ = \pi/6\,\text{radians}$.

An angle is classified as acute if it is less than 90° and called obtuse if it lies between 90° and 180°. An angle θ is called a right angle when it has the value of 90° or $\frac{\pi}{2}$ radians.

In scientific computing one always uses the radian measure in all calculations. Also note that most hand-held calculators have a switch for converting from one angular measure to another and so owners of such calculators must learn to set them appropriately before doing any calculations.

Functions

Whenever there is a one-to-one correspondence, where to each value of x there is one and only one value of y, then y is said to be a function of x, written $y = f(x)$. Sometimes y is referred to as a single-valued function of x to emphasize the one-to-one correspondence between x and y.

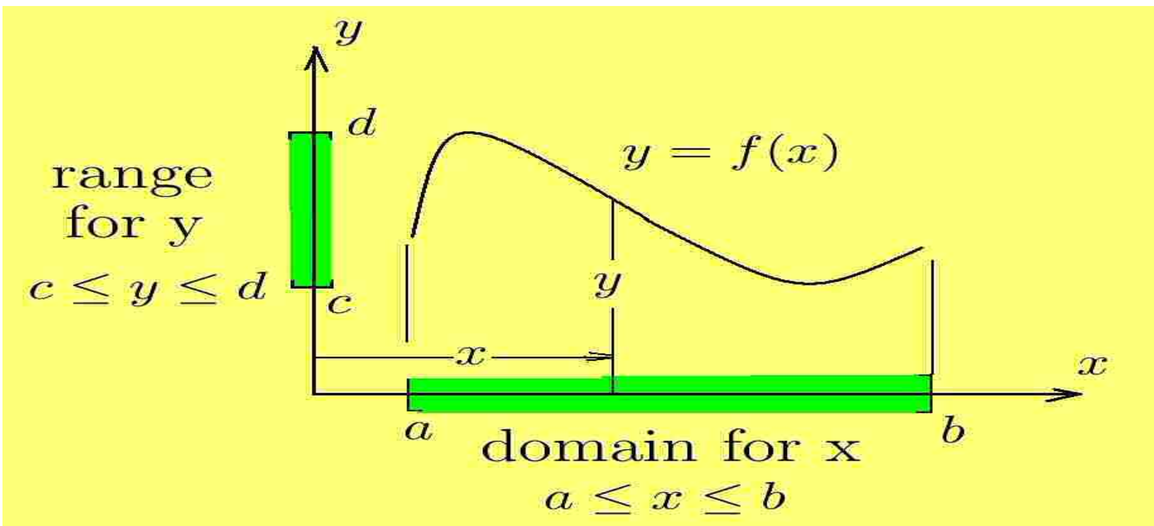


Figure 7-1.
Graphic display for representation of a function.

The variable x is called the independent variable and when the values assigned to x are restricted to some set of values, then these values are said to be the **domain of the function.** Usually values assigned to x are restricted so that the values for y are real numbers. For example, the function $y = f(x) = \sqrt{x-3}$ must be restricted to the domain where $x - 3 > 0$ or $x > 3$ in order for y to have real values. If $y = f(x)$ is a single-valued function of x, then the values for y depends upon the values selected for x and so y is called the dependent variable. The values assigned to y, as x varies through its values from the domain of the function, is called the **range of the function.**

The straight line with slope m which passes through the point (x_0, y_0) is given by

$$y = f(x) = y_0 + m(x - x_0)$$

is an example of a function. Some other examples of functions are

$$\left.\begin{aligned} y = f(x) &= x \\ y = g(x) &= x^2 \\ y = h(x) &= x^3 \\ &\vdots \\ y = y(x) &= x^n \end{aligned}\right\} \quad \text{polynomial functions}$$

$$\begin{aligned} y = f_1(x) &= \sqrt{x} && \text{square root function} \\ y = f_2(x) &= 2^x && \text{exponential function} \\ y = f_3(x) &= |\,x\,| && \text{absolute value function} \\ y = f_4(x) &= \log(x) && \text{logarithmic function} \end{aligned}$$

The graph of a function

A pictorial representation of a function $y = f(x)$ can be constructed by creating a table of (x, y) values

x	$y = f(x)$
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

where the x-values $\{x_0, x_1, x_2, \ldots, x_n\}$ are numbers selected from the domain of the function and the y-values

$$\{y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)\}$$

are calculated from the given function $f(x)$. One can then plot all the (x, y) points from the table and then join neighboring points using straight line segments. If only a few points are plotted the resulting curve will not look very smooth. The curve will look smoother as the number of points plotted increases. The figure 7-2 illustrates some smooth curves associated with selected functions.

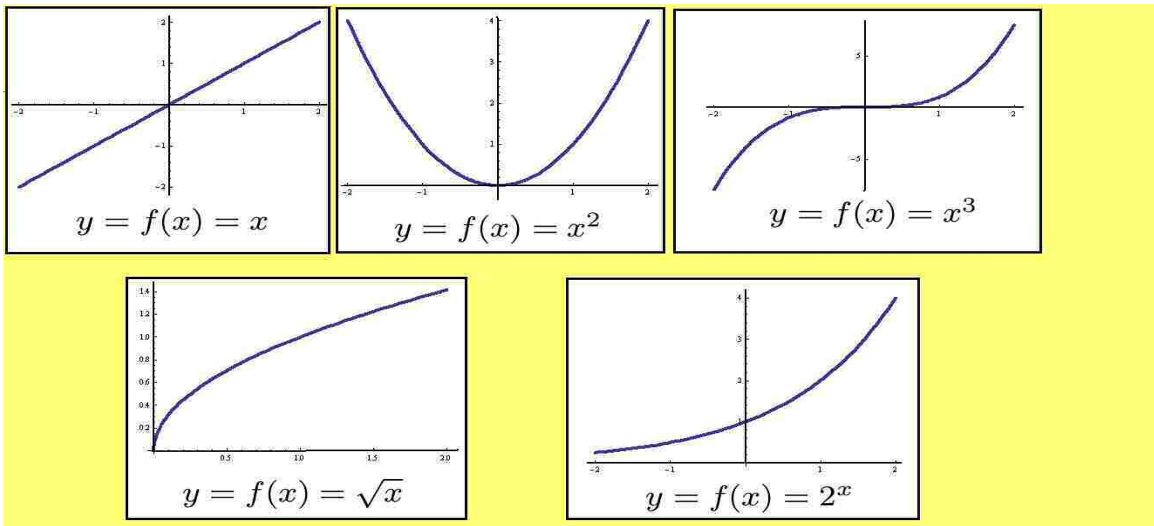


Figure 7-2.
Graphs of selected functions

Vertical line test

Remember that in order to have a function it is necessary that for each value of x there can be only one value for y. The function must be single-valued. This requirement[1] can be tested by constructing various vertical lines. No vertical line can intersect the graph of the function more than once.

Trigonometric functions

The first recorded trigonometric tables comes from the Greeks around 150 BCE. However, it is inferred that trigonometry was used in sailing, astronomy and construction going back thousands of years before this time, but only limited records are available.

The **ratio of sides of a right triangle are used to define the six trigonometric functions associated with one of the acute angles of the right triangle.** These definitions can then be extended to apply to positive and negative angles associated with a point moving on a unit circle.

[1] If more than one value for y exists, then the function is called a **multi-valued function** and steps must be taken to try and convert it into a singe-valued function. This is usually done by breaking the multi-valued function up into different parts called branches where in each branch the function is single-valued.

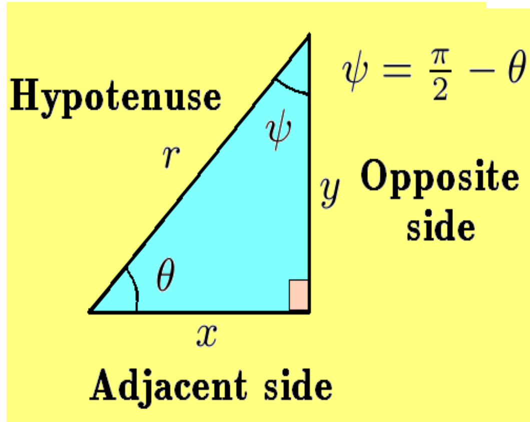
The six trigonometric functions associated with the angle θ of a right triangle are

sine **tangent** **secant**

cosine **cotangent** **cosecant**

which are abbreviated respectively as

sin, tan, sec, cos, cot , and csc



and are defined[2]

$$
\begin{array}{lll}
\sin\,\theta = \dfrac{y}{r} = \dfrac{\text{opposite side}}{\text{hypotenuse}} & \cos\,\theta = \dfrac{x}{r} = \dfrac{\text{adjacent side}}{\text{hypotenuse}} & \tan\,\theta = \dfrac{y}{x} = \dfrac{\text{opposite side}}{\text{adjacent side}} \\
\csc\theta = \dfrac{r}{y} = \dfrac{\text{hypotenuse}}{\text{opposite side}} & \sec\theta = \dfrac{r}{x} = \dfrac{\text{hypotenuse}}{\text{adjacent side}} & \cot\theta = \dfrac{x}{y} = \dfrac{\text{adjacent side}}{\text{opposite side}}
\end{array}
\tag{7.2}
$$

provided none of the denominators are zero.

Observe that the following functions are reciprocals

$$\sin\theta \quad \text{and} \quad \csc\theta, \qquad \cos\theta \quad \text{and} \quad \sec\theta, \qquad \tan\theta \quad \text{and} \quad \cot\theta \tag{7.3}$$

and satisfy the relations

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta} \tag{7.4}$$

Also note that $\cos\theta = \sin\psi = \sin(\frac{\pi}{2} - \theta)$, where cosine represents the sine of the complementary angle $\psi = \frac{\pi}{2} - \theta$. Similarly, $\cot\theta = \tan\psi = \tan(\frac{\pi}{2} - \theta)$, where cotangent represents the tangent of the complementary angle. Also $\csc\theta = \sec\psi = \sec(\frac{\pi}{2} - \theta)$, where cosecant represents the secant of the complementary angle.

The figure 7-3 gives a graphical representation of the above trigonometric functions in terms of **distances** associated with **the unit circle.** Note that in figure 7-3 the unit circle has $0D = 0G = 0A = 1$ and

[2] A mnemonic devise to remember the sine, cosine and tangent definitions is the expression "**S**ome **O**ld **H**orse **C**ame **A** **H**opping **T**owards **O**ur **A**lley " $(S = \frac{O}{H},\ C = \frac{A}{H},\ T = \frac{O}{A})$

$$
\begin{aligned}
\text{In triangle } \triangle 0FD \quad \sin\theta = \frac{DF}{0D} &\Rightarrow \sin\theta = DF\\
\text{In triangle } \triangle 0FD \quad \cos\theta = \frac{0F}{0D} &\Rightarrow \cos\theta = 0F\\
\text{In triangle } \triangle 0GC \quad \tan\theta = \frac{CG}{0G} &\Rightarrow \tan\theta = CG\\
\text{In triangle } \triangle 0AB \quad \cot\theta = \frac{AB}{0A} &\Rightarrow \cot\theta = AB\\
\text{In triangle } \triangle 0GC \quad \sec\theta = \frac{0C}{0G} &\Rightarrow \sec\theta = 0C\\
\text{In triangle } \triangle 0BA \quad \csc\theta = \frac{0B}{0A} &\Rightarrow \csc\theta = 0B
\end{aligned}
$$

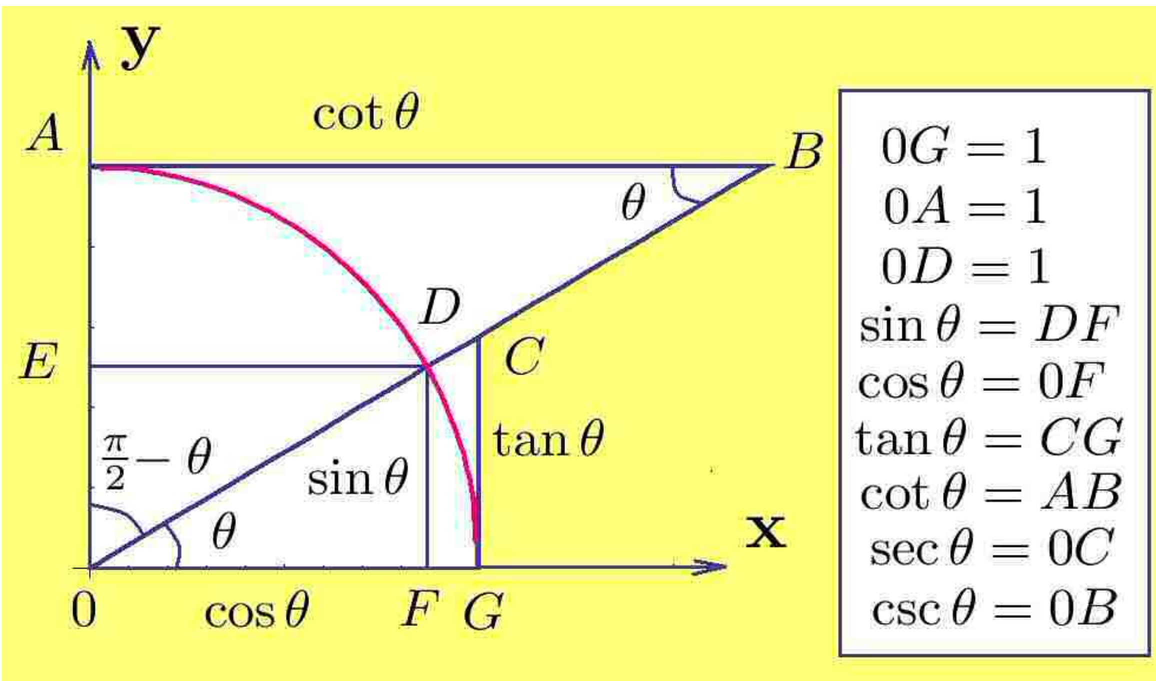


Figure 7-3.

Trigonometric functions in terms of distances associated with the unit circle.

Note that as the angle θ, in radians, varies from 0 to 2π, one finds

$$
\begin{aligned}
-1 \le \sin\theta \le 1\\
-1 \le \cos\theta \le 1\\
-\infty < \tan\theta < +\infty\\
-\infty < \cot\theta < +\infty
\end{aligned}
$$

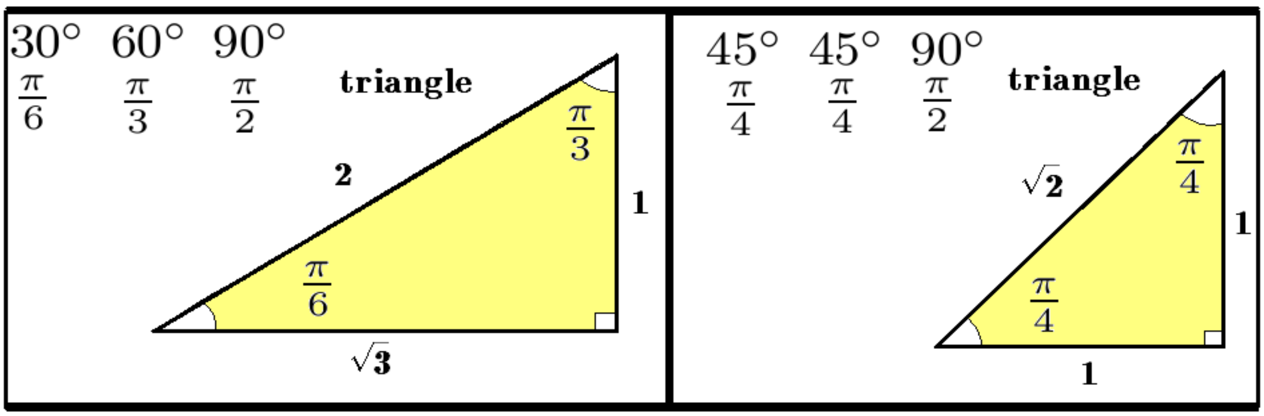
Also observe that for n an integer one can write

$$\sin(\theta \pm 2n\pi) = \sin\theta, \qquad \tan(\theta \pm n\pi) = \tan\theta$$
$$\cos(\theta \pm 2n\pi) = \cos\theta, \qquad \cot(\theta \pm n\pi) = \cot\theta$$

The tangent function is not defined for $\theta = \frac{\pi}{2} \pm n\pi$ and the cotangent function is not defined for $\theta = \pm n\pi$.

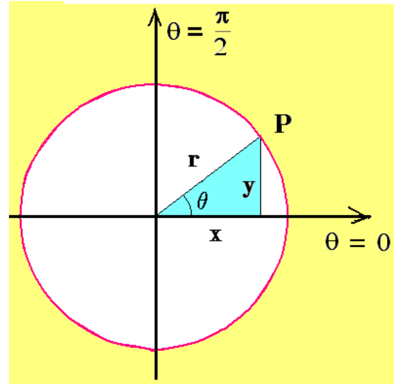
Special right triangles

Consider the special right triangles illustrated



Note that in the 45 degree triangle, the sides opposite the 45 degree angles are equal and so any convenient length can be used to represent the equal sides. If the number 1 is used, then by the Pythagorean theorem the hypotenuse has the value $\sqrt{2}$. Similarly, in the 30-60-90 degree right triangle, the side opposite the 30 degree angle is always half the hypotenuse and so by selecting the value of 2 for the hypotenuse one obtains from the Pythagorean theorem the sides $2, 1, \sqrt{3}$ as illustrated. By using the trigonometric definitions given by the equations (7.2) together with the special 30-60-90, and 45 degree right triangles, the following table of values for the trigonometric functions result.

Angle θ in degrees	Angle θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$0°$	0	0	1	0	undefined	1	undefined
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
$90°$	$\pi/2$	1	0	undefined	0	undefined	1



The values for $\theta = 0$ and $\theta = 90$ degrees are special cases which can be examined separately by examining a triangle which changes as a point P moves around the circumference of a circle having radius r. The 45 degree triangle being a special case when $x = 1$, $y = 1$ and $r = \sqrt{2}$. The 30-60-90 degree triangle is a special case when there is a circle with $x = 1$, $r = 2$ and $y = \sqrt{3}$. The trigonometric functions associated with the limiting values of $\theta = 0$ and $\theta = 90$ degrees can be obtained from these special triangles by examining the values of x and y associated with the point P for the limiting conditions $\theta = 0$ and $\theta = \pi/2$.

Cofunctions

In the definitions of the trigonometric functions certain combinations of these functions are known as **cofunctions.** For example, **the sine and cosine functions are known as cofunctions.** Thus the cofunction of the sine function is the cosine function and the cofunction of the cosine function is the sine function. Similarly, **the functions tangent and cotangent are cofunctions** and **the functions secant and cosecant are cofunctions.**

Theorem: Each trigonometric function of an acute angle θ in a right triangle is equal to the cofunction of the complementary angle $\psi = \frac{\pi}{2} - \theta$.

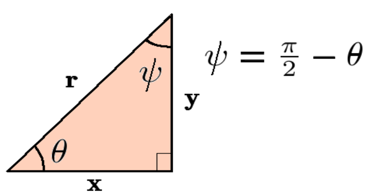


Figure 7-4.
Complementary angles related to cofunctions

The above theorem follows directly from the definitions of the trigonometric functions giving the results

$$\begin{array}{lll} \sin\theta = \cos(\frac{\pi}{2} - \theta) & \tan\theta = \cot(\frac{\pi}{2} - \theta) & \sec\theta = \csc(\frac{\pi}{2} - \theta) \\ \cos\theta = \sin(\frac{\pi}{2} - \theta) & \cot\theta = \tan(\frac{\pi}{2} - \theta) & \csc\theta = \sec(\frac{\pi}{2} - \theta) \end{array} \tag{7.5}$$

The above results are known as the **cofunction formulas.**

Trigonometric Functions Defined for Other Angles

Consider a circle with radius r constructed with respect to some xy coordinate system and let P denote a point on the circumference of the circle which moves around the circle in a counterclockwise direction. The figure 7-5 illustrates the point P lying in various quadrants of the x, y coordinate system. These quadrants are denoted by the Roman numerals I,II,III,IV.

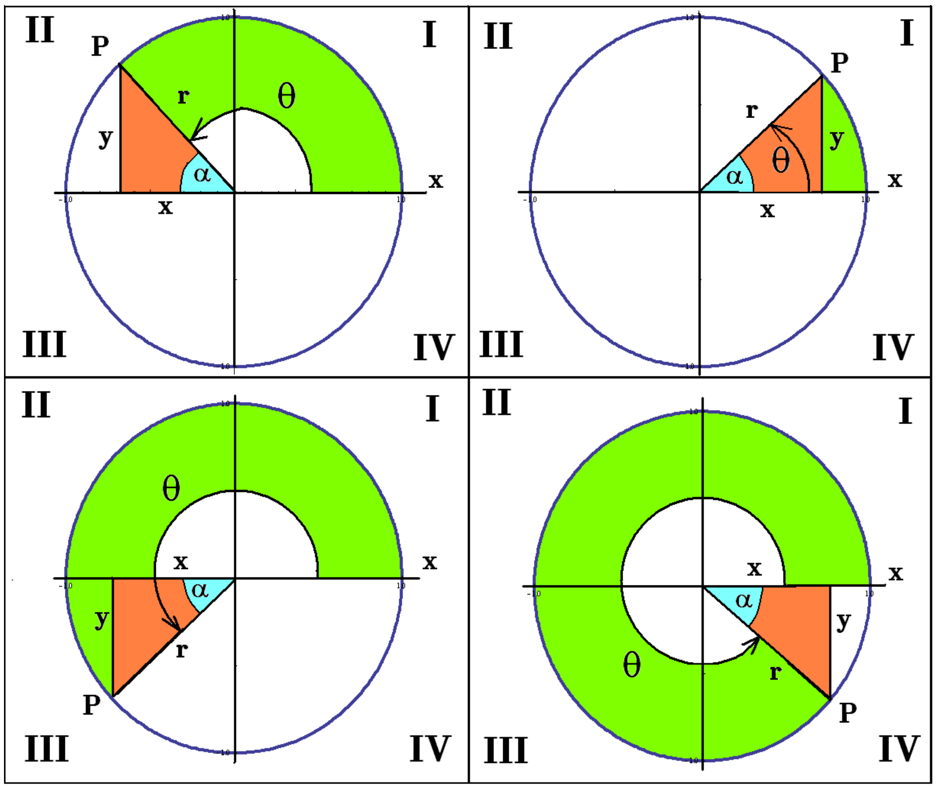


Figure 7-5.
Point P on circle of radius r where P is
in quadrant I, quadrant II, quadrant III and quadrant IV.

Let θ denote the angle swept out as P moves counterclockwise about the circle and define the six trigonometric functions of θ as follows

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}, \quad \cot\theta = \frac{x}{y}, \quad \sec\theta = \frac{r}{x}, \quad \csc\theta = \frac{r}{y} \tag{7.6}$$

Here y denotes the ordinate of the point P, x denotes the abscissa of the point P and r denotes the polar distance of the point P from the origin. These distances are illustrated in the figures 6-5. Note that these definitions imply that

$$\begin{array}{lll} \tan\theta = \dfrac{\sin\theta}{\cos\theta}, & \sec\theta = \dfrac{1}{\cos\theta}, & \\ & & \cot\theta = \dfrac{1}{\tan\theta} \\ \cot\theta = \dfrac{\cos\theta}{\sin\theta}, & \csc\theta = \dfrac{1}{\sin\theta}, & \end{array} \tag{7.7}$$

In figure 7-5 there is the angle α called **the reference angle** associated with the angle θ. The **reference angle is defined to be the positive acute angle formed from the terminal side of the angle θ and the x-axes.**

The value of any trigonometric function of the angle θ equals the same trigonometric function of the reference angle α with the appropriate sign attached.

For example, in quadrant I, $\sin\theta = \sin\alpha$, in quadrant II, $\sin\theta = \sin\alpha$, in quadrant III, $\sin\theta = -\sin\alpha$ and in quadrant IV, $\sin\theta = -\sin\alpha$.

The trigonometric function of θ equals the same trigonometric function of the reference angle α with the following signs.

In quadrant I	In quadrant II	In quadrant III	In quadrant IV	
$x > 0,\ y > 0,\ r > 0$	$x < 0,\ y > 0,\ r > 0$	$x < 0,\ y < 0,\ r > 0$	$x > 0,\ y < 0,\ r > 0$	
$\sin\theta > 0$	$\sin\theta > 0$	$\sin\theta < 0$	$\sin\theta < 0$	
$\cos\theta > 0$	$\cos\theta < 0$	$\cos\theta < 0$	$\cos\theta > 0$	
$\tan\theta > 0$	$\tan\theta < 0$	$\tan\theta > 0$	$\tan\theta < 0$	(7.8)
$\cot\theta > 0$	$\cot\theta < 0$	$\cot\theta > 0$	$\cot\theta < 0$	
$\sec\theta > 0$	$\sec\theta < 0$	$\sec\theta < 0$	$\sec\theta > 0$	
$\csc\theta > 0$	$\csc\theta > 0$	$\csc\theta < 0$	$\csc\theta < 0$	

Sign Variation of the Trigonometric Functions

In the figure 7-5 make note of the following.

(i) The six trigonometric functions of θ, as the point P moves about the circle, are sometimes referred to as circular functions.

(ii) The ray $\overline{0P}$ defines not only the angle θ but also the angles $\theta \pm 2m\pi$ where m is a positive integer or zero.

(iii) As P moves around the circle, the radial distance $r = \overline{0P}$ always remains constant, but the values of x and y will change sign as the point P moves through the different quadrants. An analysis of these sign changes produces the table of signs given in the figure 7-6.

(iv) The six trigonometric functions take on special values whenever $x = 0$ or $y = 0$. One of these special values will occur whenever θ is equal to some multiple of $\frac{\pi}{2}$.

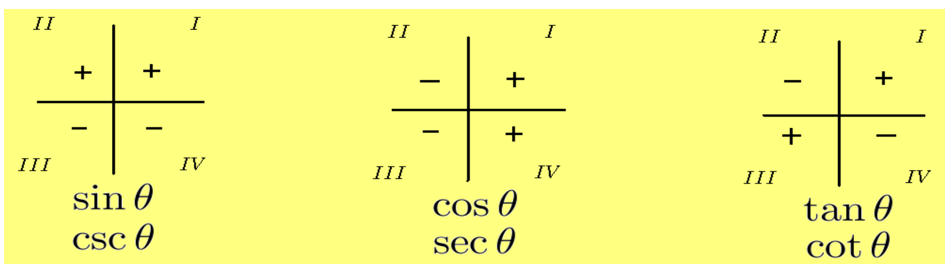
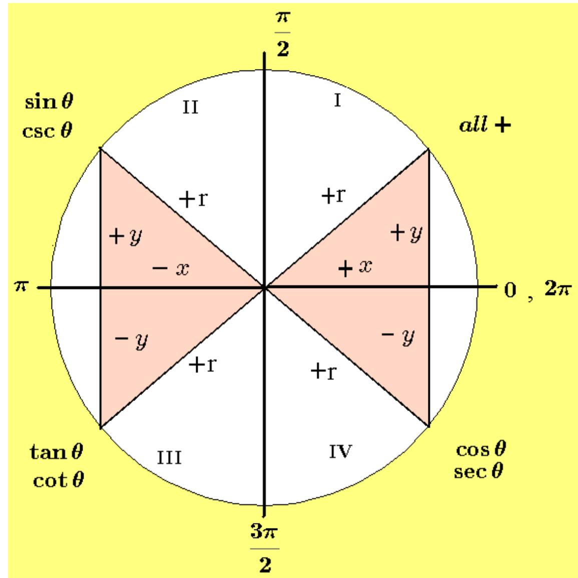


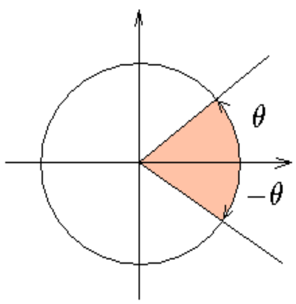
Figure 7-6.
Sign variation for the trigonometric functions by quadrant.

(v) The angle α in the figures 7-5 is **the smallest nonnegative angle between the line $\overline{0P}$ and the x-axis.** Limiting values for α are $\frac{\pi}{2}$ and 0 radians. The angle α is called **a reference angle.** If the reference angle is different from 0 or $\frac{\pi}{2}$, then it can be viewed as **an acute positive angle** in the first quadrant. Note that there is then a definite relationship between the six trigonometric functions of θ and the six trigonometric functions of the reference angle α. The six trigonometric functions of the reference angle α are all positive and so **one need only add the appropriate sign change** to obtain the six trigonometric functions of θ in the other quadrants. The appropriate sign changes are given in the figure 7-6.

(vi) The figures 7-5 and 7-6 can be combined into one figure so that by using the definition of the trigonometric functions, the correct sign of a trigonometric function can be determined corresponding to θ in any quadrant. For example, in quadrant II the functions $\sin\theta$ and $\csc\theta$ are positive. In quadrant III the functions $\tan\theta$ and $\cot\theta$ are positive and in quadrant IV, the functions $\cos\theta$ and $\sec\theta$ are positive.



(vii)



If the angle θ is a positive acute angle, then $-\theta$ lies in the fourth quadrant. The reference angle is $\alpha = \theta$ is positive and gives the **negative angle formulas**

$$\begin{array}{lll} \sin(-\theta) = -\sin\theta & \cos(-\theta) = \cos\theta & \tan(-\theta) = -\tan\theta \\ \csc(-\theta) = -\csc(\theta) & \sec(-\theta) = \sec\theta & \cot(-\theta) = -\cot\theta \end{array} \tag{7.9}$$

(viii) Any function $f(\theta)$ satisfying $f(-\theta) = f(\theta)$ is called an **even function** of θ and functions satisfying $f(-\theta) = -f(\theta)$ are called **odd functions** of θ. The above arguments show that the functions sine, tangent, cotangent and cosecant are odd functions of θ and the functions cosine and secant are even functions of θ. These results are sometimes referred to as even-odd identities. If a trigonometric function $y = f(\theta)$ is an **even function** of θ, then its graph will be **symmetric about the y-axis.** If it is an **odd function** of theta, then the graph will be **symmetric about the origin.**

Special Values–Quadrant I

Angle θ degrees	Angle θ radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0	0	0	1	0	$undefined$	1	$undefined$
90	$\pi/2$	1	0	$undefined$	0	$undefined$	1
180	π	0	-1	0	$undefined$	-1	$undefined$
270	$3\pi/2$	-1	0	$undefined$	0	$undefined$	-1
360	2π	0	1	0	$undefined$	1	$undefined$

Quadrant II

Angle θ degrees	Angle θ radians	Reference angle α	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
120	$2\pi/3$	$\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
135	$3\pi/4$	$\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150	$5\pi/6$	$\pi/6$	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2

Quadrant III

Angle θ degrees	Angle θ radians	Reference angle α	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
210	$7\pi/6$	$\pi/6$	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2
225	$5\pi/4$	$\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240	$4\pi/3$	$\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$

Quadrant IV

Angle θ degrees	Angle θ radians	Reference angle α	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
300	$5\pi/3$	$\pi/3$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$
315	$7\pi/4$	$\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330	$11\pi/6$	$\pi/6$	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2

Table 6.1 Special values for the Trigonometric Functions

The Pythagorean identities

Write the Pythagorean theorem $x^2 + y^2 = r^2$ for the right triangles in each quadrant of figure 7-5. Divide each term in the Pythagorean theorem by either x^2, y^2

or r^2 and show the Pythagorean theorem can be expressed in any of the alternative forms

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \qquad 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \qquad \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

which by the above trigonometric definitions become

$$\cos^2\theta + \sin^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta \tag{7.10}$$

These are fundamental relations between the trigonometric functions and are known as **trigonometric identities.** The above trigonometric identities are called the **Pythagorean identities.**

Other trigonometric identities

The Pythagorean identities and other identities such as

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

are all examples of trigonometric equations called trigonometric identities which are satisfied for all angles θ whenever both sides of the equations are well defined.

The trigonometric definitions and identities are often used to (i) prove additional identities (ii) change equations with trigonometric terms into a different form in order to illustrate some point or express the equation in a simpler form.

Whenever one is asked to prove a trigonometric equation is an identity it is best to start with one side of the given equation use some algebra together with making trigonometric substitutions to produce the other side of the equation. If this is possible then the given trigonometric equation is called a trigonometric identity.

Example 7-1. Prove the trigonometric identity $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}$

Solution

Performing a heuristic examination of the given equation one can see that cross multiplication produces a Pythagorean identity. We want to start with one side of the given equation and derive the other side of the equation. Hence start with say the left side and multiply both numerator and denominator by $1-\sin\theta$ and simplify the result to obtain

$$\left(\frac{1+\sin\theta}{\cos\theta}\right)\left(\frac{1-\sin\theta}{1-\sin\theta}\right) = \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)} = \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)} = \frac{\cos\theta}{1-\sin\theta}$$

where we have used the Pythagorean identity $1 - \sin^2\theta = \cos^2\theta$ to simplify the above expression to verify the given equation is a trigonometric identity.

■

Example 7-2. Prove the trigonometric identity $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha\sin^2\alpha$ is true for all angles α for which the tangent function is well defined.

Solution

Starting with the left-hand side of the given trigonometric equation write

$$\begin{aligned}\tan^2\alpha - \sin^2\alpha =& \frac{\sin^2\alpha}{\cos^2\alpha} - \sin^2\alpha = \sin^2\alpha\left(\frac{1}{\cos^2\alpha} - 1\right)\\ =& \sin^2\alpha\left(\frac{1-\cos^2\alpha}{\cos^2\alpha}\right) = \tan^2\alpha(1-\cos^2\alpha) = \tan^2\alpha\sin^2\alpha\end{aligned}$$

and so the given equation is a trigonometric identity.

■

Graphs of the Trigonometric Functions

Consider the special case where the radius r in figure 7-5 is unity. The advantage of setting $r = 1$ in the above definitions for the trigonometric functions is that in this special case the ordinate value y represents $\sin\theta$ and the abscissa value x represents $\cos\theta$. By plotting y vs θ, as the point P moves counterclockwise around the circle, one obtains the figure 7-7(a). Similarly, by rotating the unit circle through $90°$ it is possible to plot a graph of x vs θ, as the point P moves in a counterclockwise direction about the circle. The plotting of the height of the point P as a function of θ, as the point P moves in a counterclockwise direction around the circumference of the unit circle gives rise to the name circular functions for the trigonometric functions. The graphs of $\sin\theta$ and $\cos\theta$ are illustrated in the figures 6-7(a) and 6-7(b). A plot the ratio $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ vs θ for the tangent function is illustrated in the figure 7-7(c). The three functions sine, cosine and tangent are also illustrated in the figure 7-9 over the domain $0 \le x \le 4\pi$.

A continuous function $y = f(x)$, which is well defined everywhere, is said to be a **periodic function** of period p, if p is the smallest number such that $f(x+p) = f(x)$ is satisfied for all values of x. In figure 7-8 make note of the following.

(i) The trigonometric functions $\sin\theta$, $\cos\theta$, and consequently the reciprocal functions $\csc\theta$ and $\sec\theta$ are periodic functions of θ with period 2π and satisfy

$$\sin(\theta+2\pi) = \sin\theta, \quad \cos(\theta+2\pi) = \cos\theta, \quad \csc(\theta+2\pi) = \csc\theta, \quad \sec(\theta+2\pi) = \sec\theta$$

(ii) Similarly, the tangent and cotangent functions are periodic functions of θ with period π and satisfy

$$\tan(\theta+\pi)=\tan\theta, \qquad \cot(\theta+\pi)=\cot\theta$$

(iii) The function $\tan\theta$ becomes undefined at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ This is because $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and the cosine function takes on the value of zero at these values.

(iv) The sine and cosine functions oscillate between $+1$ and -1 as well as being periodic.

(v) Let f denote any trigonometric function of θ, then note that $f(\theta+2n\pi)=f(\theta)$, where n is any integer. This is because the angle $\theta \pm 2n\pi$ is coterminal[3] with the angle θ.

Graphs of trigonometric functions

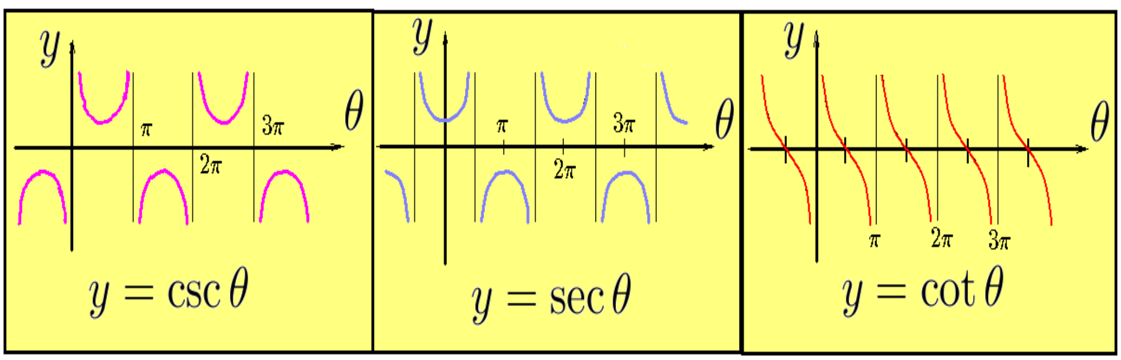


Figure 7-7.
The trigonometric functions cosecant, secant and cotangent

3 Angles which share the same initial side and terminal side.

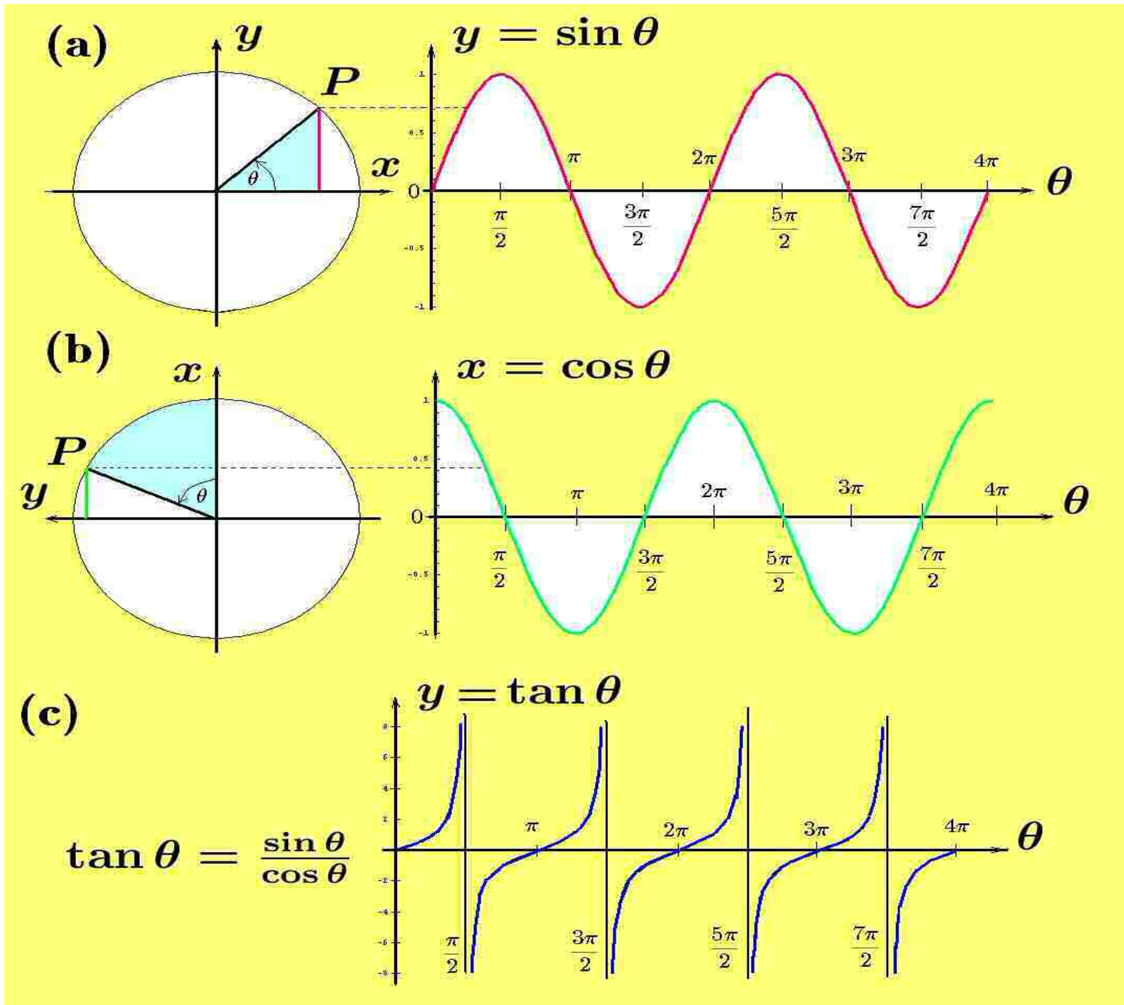


Figure 7-8.

Graphs of the trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$

Scaling

Curves of the form $y = A\sin(\beta x + \gamma)$ or $y = A\cos(\beta x + \gamma)$, with A, β, γ constants, are said to have an amplitude A and to be periodic of period $2\pi/\beta$. The graphs of these functions can be obtained by **translation of the graphs** $y = A\sin\beta x$ and $y = A\cos\beta x$ by a distance of γ/β in the negative x-direction. Another way to view these curves is to set $\theta = \beta x + \gamma$ and then ask the questions, "Where is θ equal to zero?", and "Where is θ equal to 2π?". One finds $\theta = 0$ when $x = -\gamma/\beta$ and $\theta = 2\pi$ when $x = (2\pi - \gamma)/\beta$. Subtracting these values gives the period $2\pi/\beta$ and the zero value for θ gives the translational distance $-\gamma/\beta$. These curves oscillate between $+A$ and $-A$ and are scaled versions of the basic sine and cosine curves.

Graphs for the reciprocal relations

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \text{and} \qquad \cot\theta = \frac{1}{\tan\theta}$$

are illustrated in the figures 7-7 and 7-10.

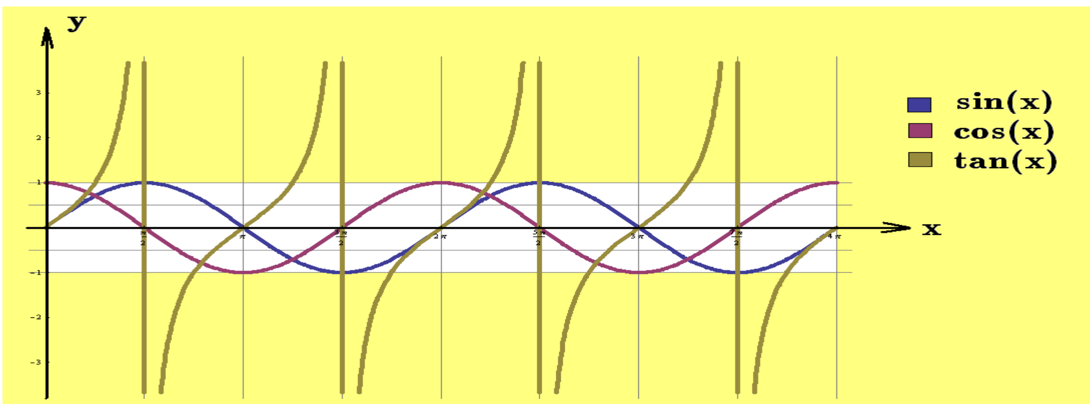


Figure 7-9. The trigonometric functions $\sin(x), \cos(x), \tan(x),\ \ 0 \le x \le 4\pi$

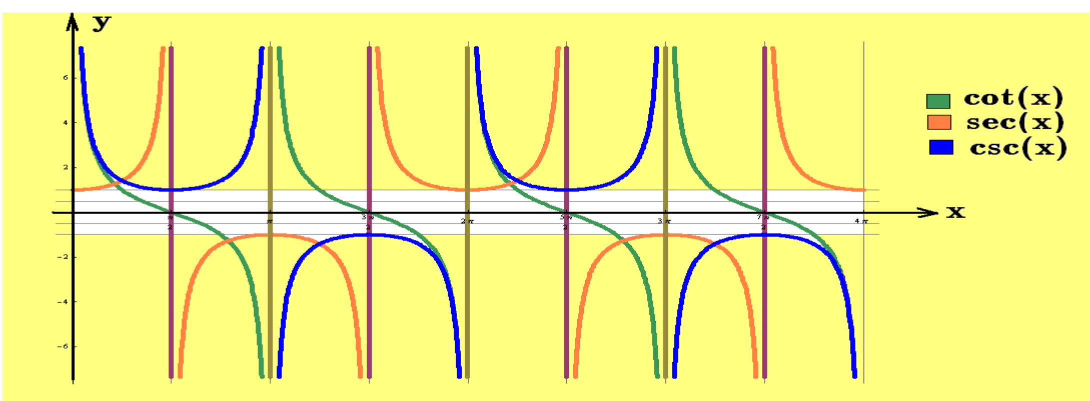


Figure 7-10. The trigonometric functions $\cot(x), \sec(x), \csc(x),\ \ 0 \le x \le 4\pi$

Trigonometric Functions of Sums and Differences

Consider **a unit circle** with the points P_0, P_1, P_2 lying on the circumference of the circle. Let the rays $\overline{0P_0}, \overline{0P_1}, \overline{0P_2}$ make positive angles of $0, \alpha$ and $\alpha+\beta$ radians respectively with respect to the x-axis, as is illustrated in the figure 7-11(a).

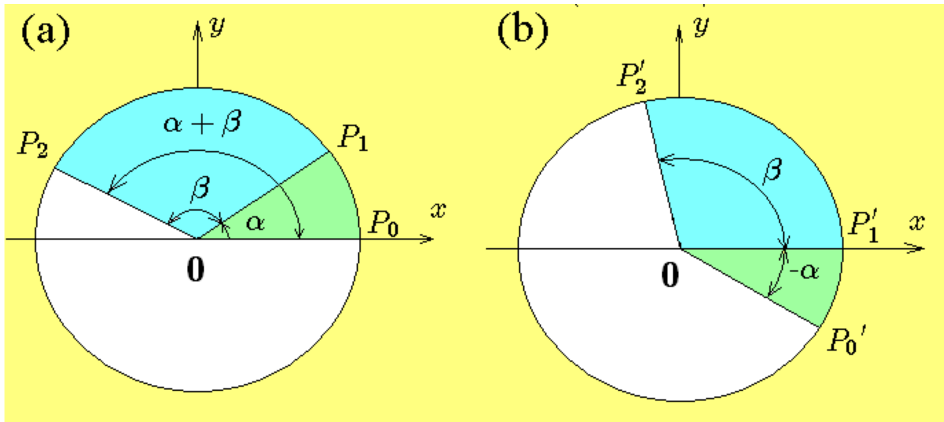


Figure 7-11.
Distance between points $\overline{P_0P_2}$ and $\overline{P_0'P_2'}$ remains invariant under rotation of circle.

The point P_0, P_1, P_2 are on **the unit circle** so that the Cartesian coordinates of these points can be written as

$$\begin{aligned} P_0: &\quad (x_0, y_0) = (1, 0), \\ P_1: &\quad (x_1, y_1) = (\cos\alpha, \sin\alpha), \\ P_2: &\quad (x_2, y_2) = (\cos(\alpha+\beta), \sin(\alpha+\beta)) \end{aligned} \tag{7.11}$$

If the points in figure 7-11(a) are rotated clockwise through an angle α, the points P_0, P_1, P_2 move to the primed positions illustrated in the figure 7-11(b). The coordinates of the primed points are given by

$$\begin{aligned} P_0': &\quad (x_0', y_0') = (\cos\alpha, -\sin\alpha), \\ P_1': &\quad (x_1', y_1') = (1, 0), \\ P_2': &\quad (x_2', y_2') = (\cos\beta, \sin\beta) \end{aligned} \tag{7.12}$$

Now the distance $\overline{P_2P_0}$ remains invariant under a rotation of axis and so this distance must be the same as the distance $\overline{P_2'P_0'}$. This requires that

$$\begin{aligned} \overline{P_2P_0} &= \overline{P_2'P_0'} \\ \sqrt{(x_2-x_0)^2+(y_2-y_0)^2} &= \sqrt{(x_2'-x_0')^2+(y_2'-y_0')^2} \\ \sqrt{(\cos(\alpha+\beta)-1)^2+(\sin(\alpha+\beta)-0)^2} &= \sqrt{(\cos\beta-\cos\alpha)^2+(\sin\beta-(-\sin\alpha))^2} \end{aligned} \tag{7.13}$$

Square both sides of equation (7.13) and expand the squared terms to obtain

$$\cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta) = \cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta + 2\sin\alpha\cos\beta + \sin^2\alpha$$

Observe the Pythagorean identities $\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta) = 1$ as well as $\cos^2\beta+\sin^2\beta = 1$ and $\cos^2\alpha + \sin^2\alpha = 1$. Use these identities to simplify the above result and obtain the addition formula

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{7.14}$$

In equation (7.14) replace the angle β by $-\beta$ to obtain

$$\cos(\alpha + -\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

and then use the even-odd identities to obtain

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{7.15}$$

Observe that a utilization of the cofunction formulas gives

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] = \cos\left[(\frac{\pi}{2} - \alpha) - \beta\right]$$

In equation (7.15) replace α by $\frac{\pi}{2} - \alpha$ to show

$$\cos(\frac{\pi}{2} - (\alpha + \beta)) = \sin(\alpha + \beta) = \cos(\frac{\pi}{2} - \alpha)\cos\beta + \sin(\frac{\pi}{2} - \alpha)\sin\beta$$

which simplifies using the cofunction formulas. This gives the addition rule

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{7.16}$$

It is an easy exercise to replace β by $-\beta$ in equation (7.16) and then use the even-odd properties to show

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{7.17}$$

Now if $\cos\alpha \neq 0$ and $\cos(\alpha + \beta) \neq 0$, then

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Divide each of the terms in the numerator and denominator by $\cos\alpha\cos\beta$ and then simplify to obtain the result

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{7.18}$$

Replacing β by $-\beta$ in equation (7.18) and simplifying the result produces the difference formula

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{7.19}$$

In a similar fashion, if $\sin\alpha\neq 0$ and $\sin(\alpha+\beta)\neq 0$, then

$$\cot(\alpha+\beta)=\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Now divide each term in the numerator and denominator by $\sin\alpha\sin\beta$ to obtain

$$\cot(\alpha+\beta)=\frac{\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta}{\sin\beta}-1}{\frac{\cos\beta}{\sin\beta}+\frac{\cos\alpha}{\sin\alpha}}$$

which simplifies to the summation formula

$$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha} \tag{7.20}$$

Replacing β by $-\beta$ in equation (7.20) and simplifying produces

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha} \tag{7.21}$$

In summary, the trigonometric functions sine, cosine and tangent obey the following sum and difference formulas

$$\begin{aligned}\sin(\alpha\pm\beta)&=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta\\ \cos(\alpha\pm\beta)&=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta\\ \tan(\alpha\pm\beta)&=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}\\ \cot(\alpha\pm\beta)&=\frac{\cot\alpha\cot\beta\mp 1}{\cot\beta\pm\cot\alpha}\end{aligned} \tag{7.22}$$

Double-angle Formulas

In the addition formulas (7.14), (7.16), (7.18), and (7.20) make the substitution $\alpha=\beta$ to obtain the double-angle formulas

$$\sin 2\beta = 2\sin\beta\cos\beta \tag{7.23}$$

$$\cos 2\beta = \cos^2\beta-\sin^2\beta \tag{7.24}$$

$$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} \tag{7.25}$$

$$\cot 2\beta = \frac{\cot^2\beta-1}{2\cot\beta} \tag{7.26}$$

Using the Pythagorean identity $\cos^2\beta + \sin^2\beta = 1$ the equation (7.24) can be expressed in one of the alternative forms

$$\cos 2\beta = 2\cos^2\beta - 1 \qquad \text{or} \qquad \cos 2\beta = 1 - 2\sin^2\beta \tag{7.27}$$

Multiple angle formulas

Formulas for

$$\begin{array}{ccc} \sin(3\alpha) & \sin(4\alpha) & \sin(5\alpha) \\ \cos(3\alpha) & \cos(4\alpha) & \cos(5\alpha) \\ \tan(3\alpha) & \tan(4\alpha) & \tan(5\alpha) \end{array}$$

can be generated using the previous summation formulas and double angle formulas. If one substitutes $\alpha = A$ and $\beta = 2A$ in the equation (7.16), then one obtains

$$\sin(\alpha+\beta) = \sin(A+2A) = \sin(3A) = \sin(A)\cos(2A) + \cos(A)\sin(2A)$$

and then substitute for $\sin(2A)$ and $\cos(2A)$ from equations (7.23) and (7.27) to produce the equation

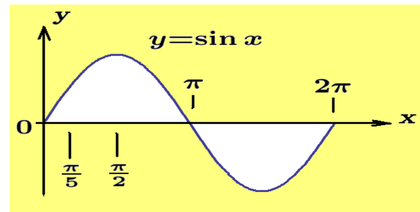
$$\begin{aligned} \sin(3A) =& \sin(A)\left[1 - 2\sin^2(A)\right] + \cos(A)\left[2\sin(A)\cos(A)\right] \\ \sin(3A) =& \sin(A) - 2\sin^3(A) + 2\sin(A)\cos^2(A) \\ \sin(3A) =& \sin(A) - 2\sin^2(A) + 2\sin(A)\left[1 - \sin^2(A)\right] \\ \sin(3A) =& 3\sin(A) - 4\sin^3(A) \end{aligned} \tag{7.28}$$

In a similar fashion one can derive the following multiple angle formulas

$$\begin{aligned} \sin 3\alpha =& 3\sin\alpha - 4\sin^3\alpha \\ \cos 3\alpha =& -3\cos\alpha + 4\cos^3\alpha \\ \tan 3\alpha =& \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} \\ \sin 4\alpha =& 8\cos^3\alpha\sin\alpha - 4\cos\alpha\sin\alpha \\ \cos 4\alpha =& 8\cos^4\alpha - 8\cos^2\alpha + 1 \\ \tan 4\alpha =& \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha} \\ \sin 5\alpha =& 5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha \\ \cos 5\alpha =& \cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha \\ \tan 5\alpha =& \frac{\tan 4\alpha + \tan\alpha}{1 - \tan 4\alpha\tan\alpha} \end{aligned} \tag{7.29}$$

Using the Pythagorean identity $\sin^2\alpha+\cos^2\alpha=1$, the above formulas **can be altered to take on forms which differ** from those presented above. These multiple angle formulas can be used to calculate exact trigonometric values for many special angles.

Example 7-3. Find the exact value for $\sin\frac{\pi}{5}$.



Solution:

Use the multiple angle formula

$$\sin 5\alpha = 5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha$$
$$\sin 5\alpha = 5(1-\sin^2\alpha)(1-\sin^2\alpha)\sin\alpha - 10(1-\sin^2\alpha)\sin^3\alpha + \sin^5\alpha$$

Let $x=\sin\alpha$ and show the above equation becomes

$$\sin 5\alpha = 16x^5 - 20x^3 + 5x \tag{7.30}$$

If $\alpha=\frac{\pi}{5}$, then $\sin 5\alpha=\sin\pi=0$ and equation (7.30) becomes the algebraic equation

$$16x^4 - 20x^2 + 5 = 0 \tag{7.31}$$

The equation (7.31) is a quadratic equation in x^2 with solution

$$x^2 = \frac{5}{8} \pm \frac{\sqrt{5}}{8} \tag{7.32}$$

We know $\frac{\pi}{5}<\frac{\pi}{4}$ which implies $\sin\frac{\pi}{5}<\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ so that we want to select the minus sine in equation (7.32). One finds

$$x=\sin\frac{\pi}{5}=\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}=\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}} \tag{7.33}$$

■

Half-angle Formulas

In the equations (7.27) replace the angle β by $\alpha/2$ to obtain the half-angle formulas

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \tag{7.34}$$
$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \tag{7.35}$$

Taking the ratio of the above results gives

$$\tan^2\frac{\alpha}{2} = \frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} = \frac{1-\cos\alpha}{1+\cos\alpha} \quad \text{provided } \boldsymbol{\cos\alpha \neq -1} \tag{7.36}$$

The equations (7.34), (7.35), and (7.36) can be expressed in the alternative forms

$$\begin{aligned} \boldsymbol{\sin\frac{\alpha}{2}} &= \boldsymbol{\pm\sqrt{\frac{1-\cos\alpha}{2}}} \\ \boldsymbol{\cos\frac{\alpha}{2}} &= \boldsymbol{\pm\sqrt{\frac{1+\cos\alpha}{2}}} \\ \boldsymbol{\tan\frac{\alpha}{2}} &= \boldsymbol{\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}} \end{aligned} \tag{\textbf{7.37}}$$

The correct algebraic sign (+ or −) is determined by the quadrant the angle $\alpha/2$ lies in. Also one finds

$$\cot\frac{\alpha}{2} = \frac{1}{\tan\frac{\alpha}{2}} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

There are alternative forms for representing $\tan\frac{\alpha}{2}$ and $\cot\frac{\alpha}{2}$ which are derived by employing the half angle formulas (7.34) and (7.35) as follows.

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\sin^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{1-\cos\alpha}{\sin\alpha} \tag{7.38}$$

or

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} = \frac{\sin\alpha}{1+\cos\alpha} \tag{7.39}$$

In a similar fashion one can show

$$\cot\frac{\alpha}{2} = \frac{\sin\alpha}{1-\cos\alpha} \tag{7.40}$$

$$\cot\frac{\alpha}{2} = \frac{1+\cos\alpha}{\sin\alpha} \tag{7.41}$$

Product, Sum and Difference Formula

Products of the sine and cosine functions can be replaced by certain sums or differences and conversely, sums and differences of sine and cosine functions can be replaced by certain products. This is accomplished by adding the equations (7.16) and (7.17) to obtain the sum formula

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta \tag{7.42}$$

and subtracting the equations (7.16) and (7.17) gives

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\sin\beta\cos\alpha \tag{7.43}$$

Similarly, by adding the equations (7.14) and (7.15) one obtains the result

$$\cos(\alpha-\beta)+\cos(\alpha+\beta)=2\cos\alpha\cos\beta \tag{7.44}$$

Subtracting these equations gives

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta \tag{7.45}$$

In summary, the **product formulas** for the trigonometric functions are

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta) \tag{7.46}$$

$$2\sin\beta\cos\alpha=\sin(\alpha+\beta)-\sin(\alpha-\beta) \tag{7.47}$$

$$2\cos\alpha\cos\beta=\cos(\alpha-\beta)+\cos(\alpha+\beta) \tag{7.48}$$

$$2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta) \tag{7.49}$$

The equations (7.42), (7.43), (7.44), and (7.45) allow one to express certain sine-cosine products as sums or differences. Make the following substitutions into the equations (7.42), (7.43), (7.44), and (7.45) ,

$$\alpha+\beta=A \qquad \text{and} \qquad \alpha-\beta=B \tag{7.50}$$

Adding and subtracting the equations (7.50) implies that

$$\alpha=\frac{A+B}{2}, \qquad \text{and} \qquad \beta=\frac{A-B}{2} \tag{7.51}$$

The equations (7.42), (7.43), (7.44), and (7.45) then become the **summation formulas**

$$\boldsymbol{\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} \tag{7.52}$$

$$\boldsymbol{\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} \tag{7.53}$$

$$\boldsymbol{\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} \tag{7.54}$$

$$\boldsymbol{\cos B-\cos A=2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} \tag{7.55}$$

Many of the above trigonometric identities can be used in geometry to change the way results are expressed.

Inverse Functions

Two functions $f(x)$ and $g(x)$ are said to be inverse functions of one another if $f(x)$ and $g(x)$ have the properties that

$$g(f(x)) = x \qquad \text{and} \qquad f(g(x)) = x \tag{7.56}$$

If $g(x)$ is an inverse function of $f(x)$, the notation f^{-1}, read "f-inverse", is used to denote the function g. That is, an inverse function of $f(x)$ is denoted $f^{-1}(x)$ and has the properties

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x \tag{7.57}$$

Given a function $y = f(x)$, then by interchanging the symbols x and y there results $x = f(y)$. This is an equation which defines the inverse function. If the equation $x = f(y)$ can be solved for y in terms of x, to obtain a single valued function, then this function is called the inverse function of $f(x)$. There then results the equivalent statements

$$x = f(y) \qquad \Longleftrightarrow \qquad y = f^{-1}(x) \tag{7.58}$$

The process of interchanging x and y in the representation $y = f(x)$ to obtain $x = f(y)$ implies that geometrically the graphs of f and f^{-1} are mirror images of each other about the line $y = x$. In order that the inverse function be single valued it is necessary that there are no vertical lines, $x = constant$, which intersect the graph $y = f(x)$ more than once. An example of a function and its inverse is given in the figure 7-12.

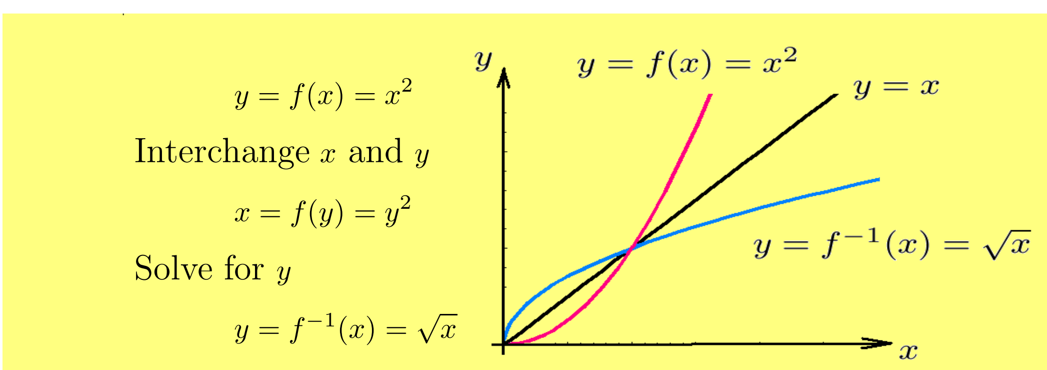


Figure 7-12. A function and its inverse.

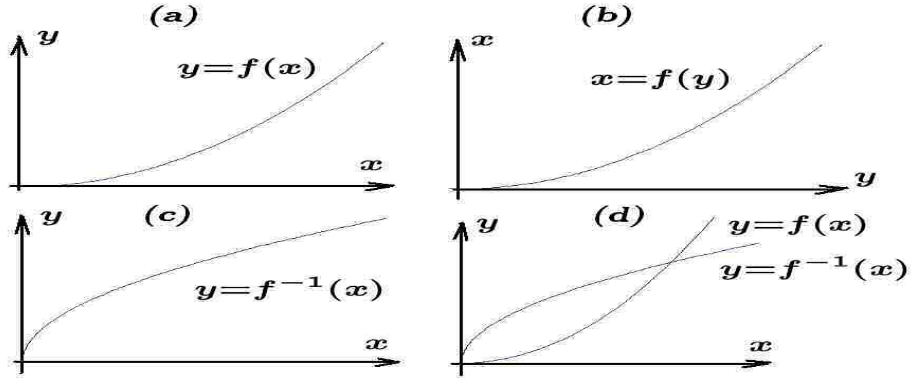


Figure 7-13. A function and its inverse.

A graphical interpretation of the inverse function is given in the figure 7-13. Consider the graph of a function $y = f(x)$ as illustrated in the figure 7-13(a). On this graph interchange x and y **everywhere** to obtain the figure 7-13(b). Flip the figure (b) so that the x-axis points to the right and obtain the figure 7-13(c). Now place the figure 7-13(c) on top of the figure 7-13(a) to obtain the last figure (d).

Now do this to all of the graphs associated with the trigonometric functions to obtain graphs of the inverse trigonometric functions. Make note of the fact that these inverse functions have angular measure.

Inverse Trigonometric Functions

The notation $y = \sin^{-1} x$ is read as, "y is the inverse sine of x". An alternative notation is to write $y = \arcsin x$. These notations define the angle whose sine is x Thus,

$$y = \arcsin x = \sin^{-1} x \quad \Rightarrow \quad \sin y = x$$

Here y is the angle whose sine is x. In a similar fashion one can write

$y = \sin^{-1} x \quad \Rightarrow \quad \sin y = x$ $\quad y$ is the angle whose sine is x

$y = \cos^{-1} x \quad \Rightarrow \quad \cos y = x$ $\quad y$ is the angle whose cosine is x

$y = \tan^{-1} x \quad \Rightarrow \quad \tan y = x$ $\quad y$ is the angle whose tangent is x

$y = \cot^{-1} x \quad \Rightarrow \quad \cot y = x$ $\quad y$ is the angle whose cotangent is x

$y = \sec^{-1} x \quad \Rightarrow \quad \sec y = x$ $\quad y$ is the angle whose secant is x

$y = \csc^{-1} x \quad \Rightarrow \quad \csc y = x$ $\quad y$ is the angle whose cosecant is x

One way to make the inverse trigonometric functions single-valued is given in the following table.

Inverse Trigonometric Functions			
Function	Alternate notation	Definition	Interval for single-valuedness
$\arcsin x$	$\sin^{-1} x$	$\sin^{-1} x = y$ if and only if $x = \sin y$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\arccos x$	$\cos^{-1} x$	$\cos^{-1} x = y$ if and only if $x = \cos y$	$0 \le y \le \pi$
$\arctan x$	$\tan^{-1} x$	$\tan^{-1} x = y$ if and only if $x = \tan y$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\operatorname{arccot} x$	$\cot^{-1} x$	$\cot^{-1} x = y$ if and only if $x = \cot y$	$0 < y < \pi$
$\operatorname{arcsec} x$	$\sec^{-1} x$	$\sec^{-1} x = y$ if and only if $x = \sec y$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$\operatorname{arccsc} x$	$\csc^{-1} x$	$\csc^{-1} x = y$ if and only if $x = \csc y$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

The inverse trigonometric functions are angles and so are multi-valued functions and consequently one must define an interval over which single-valuedness occurs. The defined interval is called **a branch of the inverse trigonometric function.** Whenever a particular branch is required for certain problems, then these branches are called **principal branches.** The following table gives principal values for the inverse trigonometric functions.

Principal Values for Regions Indicated	
$x < 0$	$x \ge 0$
$-\frac{\pi}{2} \le \sin^{-1} x < 0$	$0 \le \sin^{-1} x \le \frac{\pi}{2}$
$\frac{\pi}{2} \le \cos^{-1} x \le \pi$	$0 \le \cos^{-1} x \le \frac{\pi}{2}$
$-\frac{\pi}{2} \le \tan^{-1} x < 0$	$0 \le \tan^{-1} x < \frac{\pi}{2}$
$\frac{\pi}{2} < \cot^{-1} x < \pi$	$0 < \cot^{-1} x \le \frac{\pi}{2}$
$\frac{\pi}{2} \le \sec^{-1} x \le \pi$	$0 \le \sec^{-1} x < \frac{\pi}{2}$
$-\frac{\pi}{2} \le \csc^{-1} x < 0$	$0 < \csc^{-1} x \le \frac{\pi}{2}$

The inverse trigonometric functions are obtained by taking the mirror image of the trigonometric functions about the line $y = x$. The trigonometric functions and the inverse trigonometric functions are illustrated in the figures 7-9 and 7-10.

Unless stated otherwise it is implicitly implied that principal values will be used in dealing with inverse trigonometric functions.

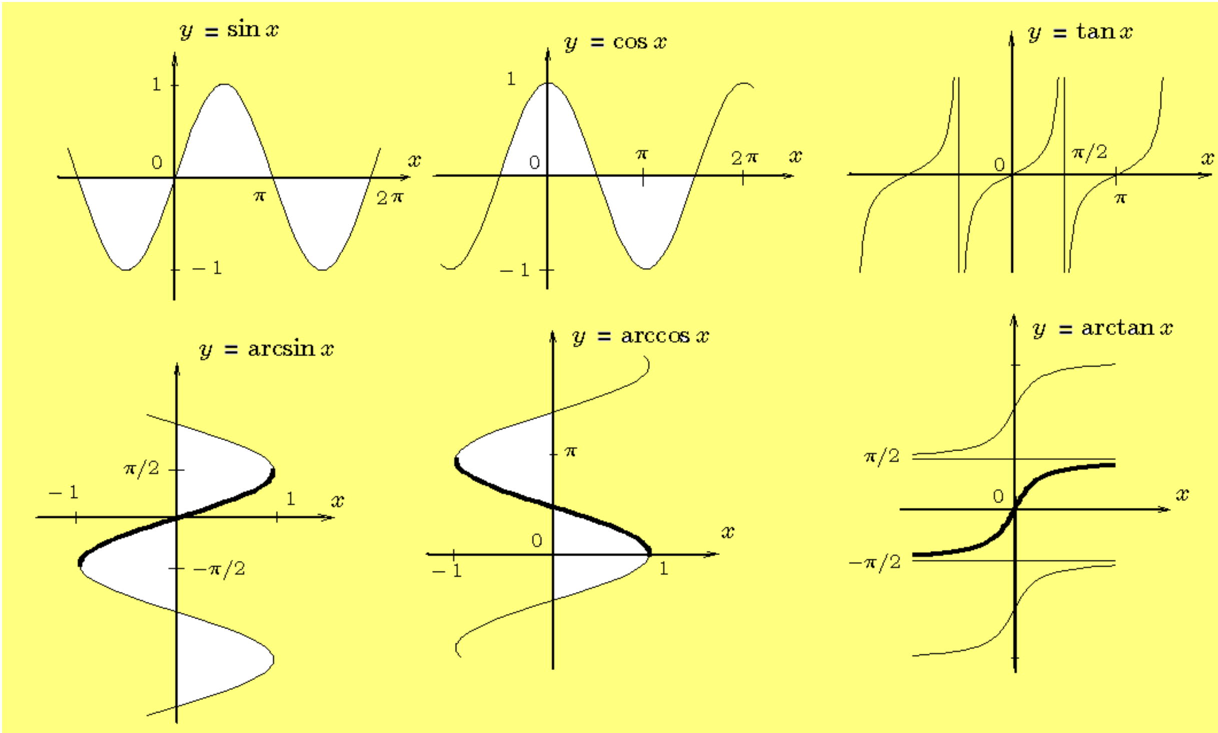


Figure 7-14. The inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$

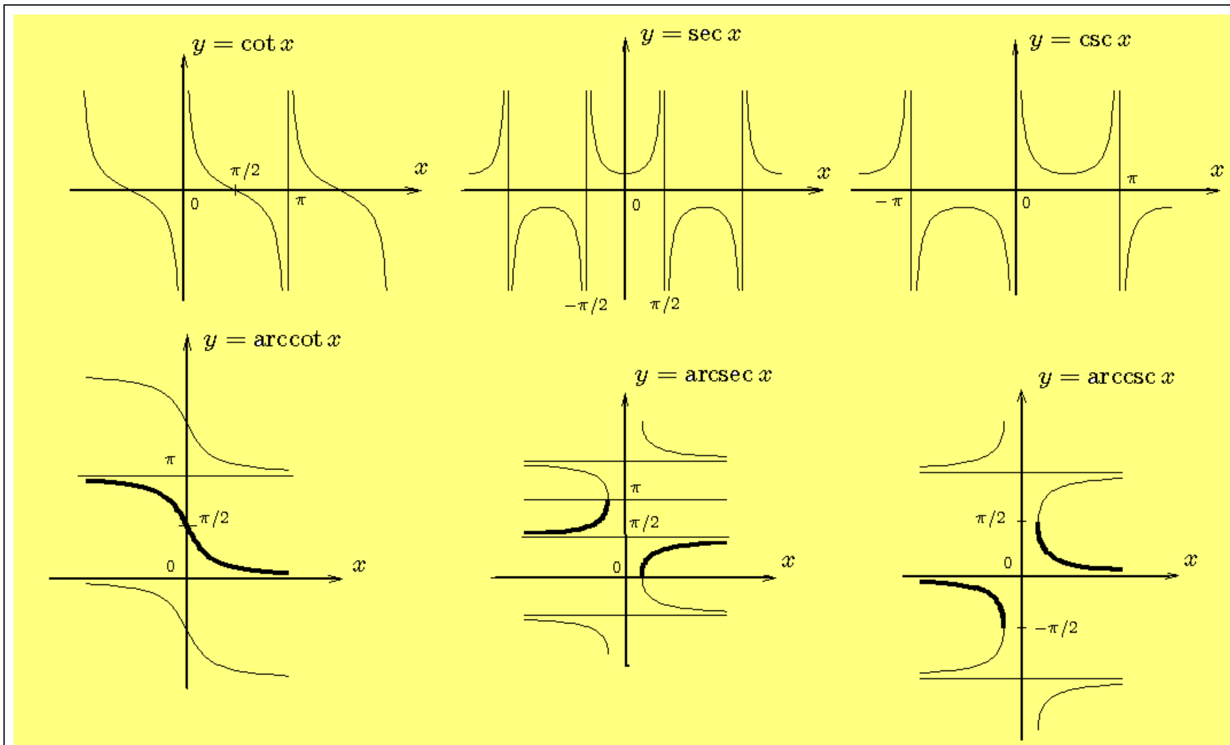
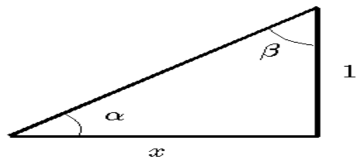


Figure 7-15. The inverse trigonometric functions $\cot^{-1} x$, $\sec^{-1} x$ and $\csc^{-1} x$

Principal Value Properties



In reference to the **right triangle illustrated**, observe that if $\alpha = \cot^{-1} x$, then $\cot\alpha = x$ and $\tan\alpha = \frac{1}{x}$. Consequently, $\alpha = \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$. Also observe that $\alpha + \beta = \frac{\pi}{2}$ and since $\tan\beta = x$ and $\cot\alpha = x$, there results

$$\alpha + \beta = \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \tag{7.59}$$

Other inverse trigonometric relationships can be determined in a similar manner and are listed below for $x > 0$.

$$\begin{aligned} \cot^{-1} x &= \tan^{-1}\left(\frac{1}{x}\right) & \qquad \cot^{-1} x + \tan^{-1} x &= \frac{\pi}{2} \\ \sec^{-1} x &= \cos^{-1}\left(\frac{1}{x}\right) & \qquad \cos^{-1} x + \sin^{-1} x &= \frac{\pi}{2} \\ \csc^{-1} x &= \sin^{-1}\left(\frac{1}{x}\right) & \qquad \csc^{-1} x + \sec^{-1} x &= \frac{\pi}{2} \end{aligned} \tag{7.60}$$

If $\theta = \sin^{-1}(-x)$, then $\sin\theta = -x$, and using the result $\sin(-\theta) = -\sin\theta = -(-x) = x$, one finds $\theta = -\sin^{-1} x$, and consequently, $\sin^{-1}(-x) = -\sin^{-1} x$. In a similar manner write $\theta = \cos^{-1}(-x)$, then $\cos\theta = -x$, and using $\cos(\pi - \theta) = -\cos\theta = -(-x) = x$ there results $\cos^{-1} x = \pi - \theta = \pi - \cos^{-1}(-x)$. Similar relationships can be derived and are summarized below.

$$\begin{aligned} \boldsymbol{\sin^{-1}(-x)} &= \boldsymbol{-\sin^{-1} x} & \qquad \boldsymbol{\cot^{-1}(-x)} &= \boldsymbol{\pi - \cot^{-1} x} \\ \boldsymbol{\cos^{-1}(-x)} &= \boldsymbol{\pi - \cos^{-1} x} & \qquad \boldsymbol{\sec^{-1}(-x)} &= \boldsymbol{\pi - \sec^{-1} x} \\ \boldsymbol{\tan^{-1}(-x)} &= \boldsymbol{-\tan^{-1} x} & \qquad \boldsymbol{\csc^{-1}(-x)} &= \boldsymbol{-\csc^{-1} x} \end{aligned} \tag{7.61}$$

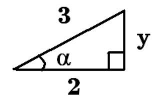
The notation for using principal values of the inverse trigonometric functions is to use capital letters in representing the function. For example, one would write either

$$\begin{aligned} &\mathrm{Sin}^{-1}\,\mathrm{x} \qquad \text{or} \qquad \mathrm{Arcsin}\,\mathrm{x} \\ &\mathrm{Cos}^{-1}\,\mathrm{x} \qquad \text{or} \qquad \mathrm{Arccos}\,\mathrm{x} \\ &\mathrm{Tan}^{-1}\,\mathrm{x} \qquad \text{or} \qquad \mathrm{Arctan}\,\mathrm{x} \end{aligned}$$

Example 7-4. Find $\tan(\cos^{-1}\frac{2}{3})$

Solution:

Remember that the inverse functions are angles. Let $\alpha = \cos^{-1}\frac{2}{3}$ so that $\cos\alpha = \frac{2}{3}$. By the Pythagorean theorem one can find the third side of this right triangle. If y is the third side, then $y^2 + 2^2 = 3^2$ or $y = \sqrt{5}$. Therefore, $\tan\alpha = \dfrac{y}{2} = \dfrac{\sqrt{5}}{2}$.



■

Example 7-5. The n-gon

For a regular polygon of n-sides, called a n-gon, let a denote the apothem, which is the perpendicular distance from the center of the regular n-gon to any side. This is also the radius of the inscribed circle. Let r denote the radius of the n-gon, which is the distance from the center of the n-gon to any vertex. The distance r also represents the radius of the circumscribed circle. Let $\alpha = \frac{2\pi}{n}$ denote the central angle associated with the n-gon and let s_n denote the length of any side of the n-gon. The above defined quantities are illustrated in the figure 7-16. Dropping a perpendicular from the center of the n-gon to a side forms a right triangle from which one can obtain the relations

$$\cos\frac{\alpha}{2} = \frac{a_n}{r}, \qquad \sin\frac{\alpha}{2} = \frac{s_n/2}{r}, \qquad \tan\frac{\alpha}{2} = \frac{s_n/2}{a_n} \tag{7.62}$$

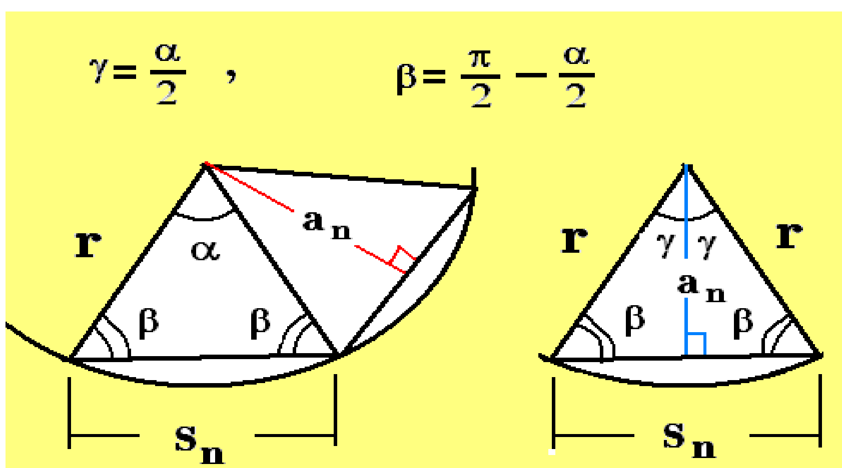


Figure 7-16.
Section of an n-gon

Using these relations the total area of a regular n-gon can be expressed in any of the following forms

$$\text{Area} = n\left[\frac{1}{2}a_n s_n\right] = n\left[\frac{1}{4}s_n^2 \cot\frac{\alpha}{2}\right] = n\left[a_n^2 \tan\frac{\alpha}{2}\right] = n\left[\frac{1}{2}r^2 \sin\alpha\right] \tag{7.63}$$

which are all representations of n times the area of one triangle.

The first area formula is just n times $\frac{1}{2}$ the base times the height, where the height of the triangle section is the apothem (a_n) and the base is (s_n). Into this first area formula substitute $a_n = \frac{s_n/2}{\tan\frac{\alpha}{2}}$ to obtain the second area formula. Substitute into the second area formula for equation (7.63) $s_n = 2a_n \tan\frac{\alpha}{2}$ to obtain the third area formula. By substituting $a_n = r\cos\frac{\alpha}{2}$ into the third area formula one obtains the fourth area formula. All the above substitutions used to change the forms for the area expressions come from the equations (7.62).

In a similar fashion one can construct various ways to represent the perimeter P of the n-gon. One can express the perimeter P in any of the forms

$$\text{Perimeter } P = ns_n = n\left[2r\sin\frac{\alpha}{2}\right] = n\left[2a_n \tan\frac{\alpha}{2}\right] \tag{7.64}$$

Note that the various forms are all constructed from the identities of equation (7.62).

■

The parallelogram

The parallelogram is a special quadrilateral having both pairs of opposite sides parallel as illustrated in the top of figure 7-17. One can say that both the opposite sides and the opposite angles associated with a parallelogram are equal. Note that by extending the sides of the parallelogram the angles α and β become alternate interior angles and they are supplementary angles so that $\alpha + \beta = \pi$ or $\alpha + \beta = 180^\circ$ depending upon the units you use.

The diagonals of a parallelogram will bisect each other. We have demonstrated this using Euclidean geometry in an earlier chapter. Lets demonstrate this fact again using Cartesian coordinates. If a parallelogram has the Cartesian coordinates as illustrated in the bottom figure 7-17, with top and bottom sides having slope zero and the sides having slope $\frac{k}{h}$, then one can quickly calculate the midpoints of the diagonals $\overline{\text{BD}}$ and $\overline{\text{AC}}$ as

$$\begin{aligned}\text{midpoint of diagonal } \overline{\text{BD}} =& \frac{1}{2}(x_1 + h + x_2, y_1 + y_1 + k)\\ \text{midpoint of diagonal } \overline{\text{AC}} =& \frac{1}{2}(x_1 + h + x_2, y_1 + y_1 + k)\end{aligned} \tag{7.65}$$

These midpoints are the same and are labeled as point P in the figure 7-17. Consequently, one can write

$$\overline{\text{AP}} = \overline{\text{PC}} \qquad \text{and} \qquad \overline{\text{DP}} = \overline{\text{PB}} \tag{7.66}$$

indicating that the diagonals intersect one another at a common point of intersection. This was a quick way to demonstrate the diagonals of a parallelogram bisect one another. Note that the midpoint of the diagonals defines the centroid of the parallelogram.

The area of the parallelogram can also be calculated knowing the Cartesian coordinates of the corners. One can drop perpendicular lines from points D and B to the opposite sides to form two triangles, then the area of the parallelogram is the summation of the area of the left triangle plus the area of a rectangle plus the area of another triangle. This gives

$$\frac{1}{2}hk + (x_2 - (x_1 + h))k + \frac{1}{2}hk = (x_2 - x_1)k = (\text{base})(\text{height})$$

Thus, the area of a parallelogram is also given by the base times the height. Here the base equals $(x_2 - x_1)$ and the height is k.

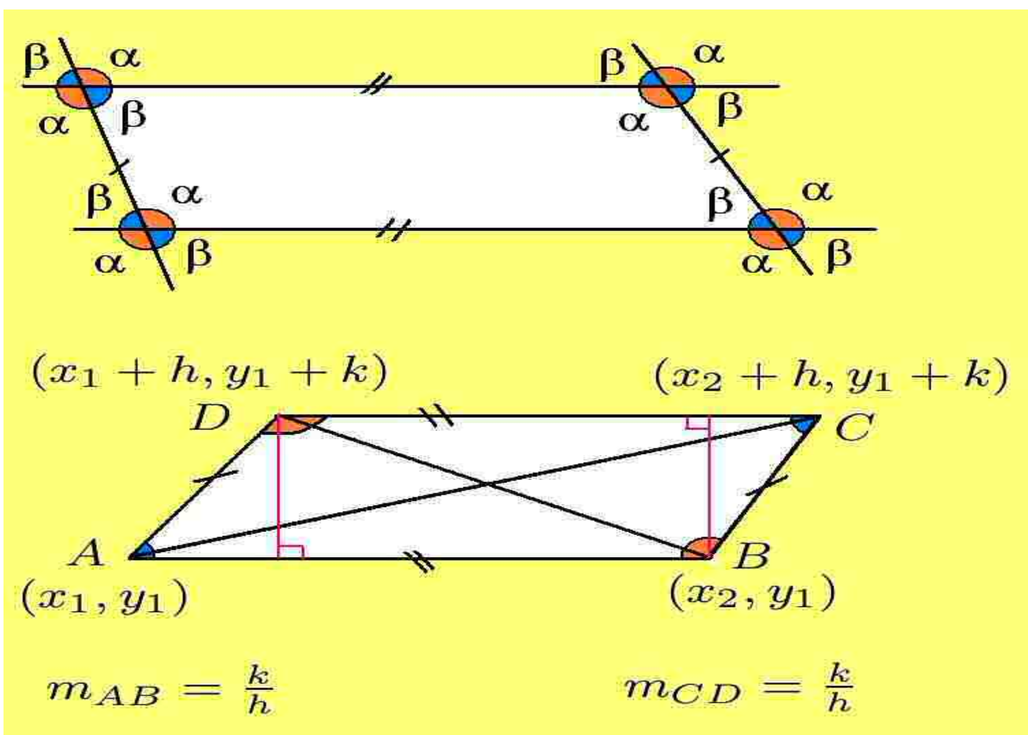
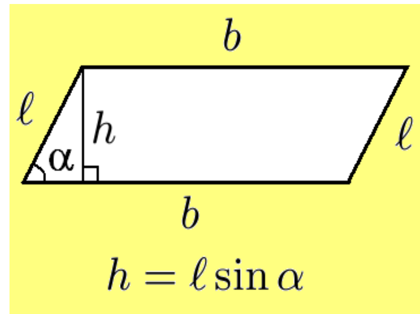


Figure 7-17.
Parallelogram represented in Cartesian coordinates

In summary, one can say that given a parallelogram with base b and height h one finds

$$\text{Area } A = bh = (base)(height) \qquad \text{Perimeter } P = 2(\ell + b)$$

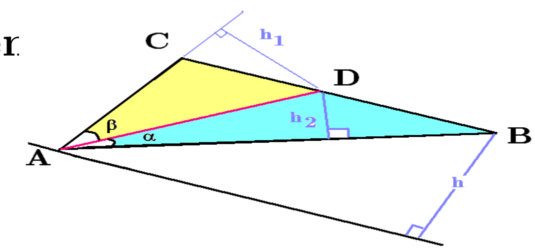


where ℓ is the length of the parallelogram side. Another form for representing the area of a parallelogram is

$$\text{Area} = b\ell \sin\alpha$$

where $\ell \sin\alpha = h$.

Generalization of angle bisector theorem



In triangle $\triangle ABC$ let $\overline{\text{AD}}$ denote an arbitrary cevian from vertex A which

(i) Divides the vertex angle $\angle A$ into two angles labeled α and β

(ii) Divides side $\overline{\text{BC}}$ into two parts $\overline{\text{CD}}$ and $\overline{\text{DB}}$

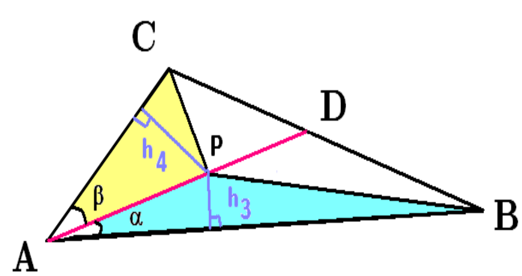
Construct a line through vertex A which is parallel to the side $\overline{\text{BC}}$ and let h denote the distance between the parallel lines. One can then calculate the areas of triangles $\triangle ADC$ and $\triangle ADB$ to obtain the ratio

$$\frac{[ADC]}{[ADB]} = \frac{\frac{1}{2}\overline{\text{CD}}\,h}{\frac{1}{2}\overline{\text{DB}}\,h} = \frac{\overline{\text{CD}}}{\overline{\text{DB}}} \tag{7.67}$$

This ratio of areas can also be calculated using the triangle bases $\overline{\text{AC}}$ and $\overline{\text{AB}}$ and the triangle heights $h_1 = \overline{\text{AD}}\sin\beta$ and $h_2 = \overline{\text{AD}}\sin\alpha$. One finds that

$$\frac{[ADC]}{[ADB]} = \frac{\overline{\text{CD}}}{\overline{\text{DB}}} = \frac{\frac{1}{2}\overline{\text{AC}}\,h_1}{\frac{1}{2}\overline{\text{AB}}\,h_2} = \frac{\overline{\text{AC}}\,\sin\beta}{\overline{\text{AB}}\,\sin\alpha} \tag{7.68}$$

Note in the special case that the cevian is an angle bisector, then $\alpha = \beta$ and the equation (7.68) reduces to the angle bisector theorem.



If P is any point on the cevian $\overline{\text{AD}}$, one can show that the ratio of the triangle areas $[APC]$

and $[APB]$ is the same as the ratio of the sides $\overline{\text{CD}}$ and $\overline{\text{DB}}$. Using the heights h_3 and h_4 defined by

$$\overline{\text{AP}}\sin\beta = h_4, \qquad \text{and} \qquad \overline{\text{AP}}\sin\alpha = h_3 \tag{7.71}$$

The areas of these triangles are

$$[APC] = \frac{1}{2}\overline{\text{AC}}\,h_4 = \frac{1}{2}\overline{\text{AC}}\,\overline{\text{AP}}\,\sin\beta, \qquad \text{and} \qquad [APB] = \frac{1}{2}\overline{\text{AB}}\,h_3 = \frac{1}{2}\overline{\text{AB}}\,\overline{\text{AP}}\,\sin\alpha \tag{7.70}$$

which can be used to produce the ratio

$$\frac{\overline{\text{CD}}}{\overline{\text{DB}}} = \frac{\overline{\text{AC}}\,\sin\beta\,\frac{1}{2}\overline{\text{AP}}}{\overline{\text{AB}}\,\sin\alpha\,\frac{1}{2}\overline{\text{AP}}} = \frac{[APC]}{[APB]} \tag{7.71}$$

The equation (7.71) is nothing more than the results from equation (7.68) where both the numerator and denominator have been multiplied by $\frac{1}{2}\overline{\text{AP}}$. The ratio of areas (7.71) associated with an arbitrary point P on the cevian $\overline{\text{AD}}$ is known as the area theorem for cevians.

Astronomy

In astronomy, angle measurements are used to determine the large distances between the Earth and **objects very far out beyond our solar system.** Consider the figure 7-18 which illustrates the Earths orbit about the Sun and the distance r to be measured. The figure shown **is not to scale** because the distance r is very, very large and the angle α, called **the parallax angle**, is very, very small. The Earth orbits the Sun in approximately 365.25 days with the **average distance between the Earth and Sun** defined to be **one astronomical unit (AU)**, where

$$1 \text{ astronomical unit} = 1 \text{ AU} = 149\,597\,870\,700 \text{ meters}$$
$$1 \text{ AU} \approx 93 \text{ million miles}$$

The parallax angle α is measured in units of arc seconds and then converted to radians. This is accomplished by observing that

$$\begin{aligned} 1'' =& 1 \text{ arc sec} = 1 \text{ sec}\cdot\frac{1 \text{ min}}{60 \text{ sec}}\cdot\frac{1 \text{ deg}}{60 \text{ min}}\cdot\frac{\pi \text{ rad}}{180 \text{ deg}} \\ 1'' =& 1\cdot\frac{1}{60}\cdot\frac{1}{60}\cdot\frac{\pi}{180} = \frac{\pi}{648\,000} \text{ radians} \end{aligned} \tag{7.72}$$

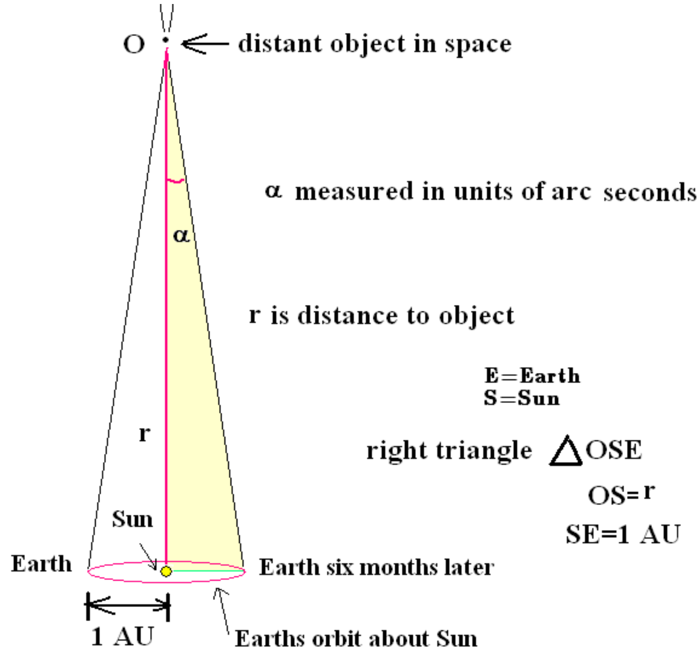
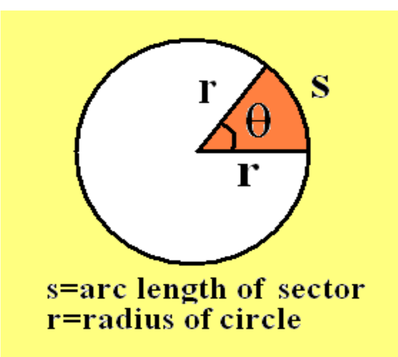


Figure 7-18. Geometry (not to scale) to measure distant object in space

Observe how the units associated with the various conversion factors cancel out as one changes from arc seconds to radians.



The arc length s associated with a sector of a circle with radius r is found by taking proportions.

$$\frac{\text{Arc length of sector}}{\text{Arc length of circle}} = \frac{\text{Angle of sector}}{\text{Angle of one rotation}}$$

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi} \quad \text{or} \quad s = r\theta \tag{7.73}$$

for the arc length s associated with a sector angle θ.

The formula for calculating the distance r to an object O very far away **is approximated by using equation (7.73) in the form**[4]

$$r = \frac{s}{\alpha}, \quad \alpha \text{ in radians, and distance } s = 1AU \tag{7.74}$$

[4] This form is also obtained using the very small angle approximation $\tan\alpha \approx \alpha$ for the trigonometric function tangent associated with the Earth-Sun-Object right triangle and the parallax angle α where $\tan\alpha = \frac{1\,AU}{r}$.

where the arc length s is approximated by $s = \overline{SE} = 1$ AU as the average distance between the Earth and the Sun. The angle θ in equation (7.73) is taken as the parallax angle α, when converted to units of radians. This distance formula is an approximation that comes from finding the radius r of a huge circle with far object (O) as the center of the circle. The parallax angle α is measured in arc seconds but must be converted to the units of radians before one can use equation (7.74).

By definition, when the angle α equals $1'' = 1$ arc sec, then the distance r is defined to be **1 parsec.** This produces the relation

$$\begin{aligned} 1 \ \text{parsec} =& \text{distance } r \text{ when parallax angle } \alpha = 1'' \\ =& \frac{1 \text{ AU}}{\pi} \cdot (60)(60)(180) \\ =& \frac{648\,800}{\pi} \text{ AU} \end{aligned} \tag{7.75}$$

One can then verify, using the proper conversion factors, the distance light travels in one year. Using the speed of light as $=299\,792\,458$ meters/second and 1 year $= 365.25\,\text{days} * 24\,\frac{\text{hours}}{\text{day}} * 60\,\frac{\text{minutes}}{\text{hour}} * 60\,\frac{\text{seconds}}{\text{minute}} = 3.15576\,10^7$ seconds one finds that

$$1 \text{ light year} = 299\,792\,458 \text{ meters/second} * 3.15576\,10^7 \text{ seconds} = 9.46073\,10^{15} \text{ meters}$$

The distance of 1 parsec can be converted into difference units

$$\begin{aligned} 1 \ \text{parsec} =& \frac{648\,000}{\pi} \text{ AU} = 206,264.81 \text{ AU} \\ =& \frac{648\,000}{\pi} \text{ AU} * 149\,597\,870\,700 \text{ meters/AU} = 3.08568\,(10)^{16} \text{ meters} \\ \approx& 19.173512\,(10)^{12} \text{ miles} \\ \approx& 3.26156 \text{ light years} \end{aligned} \tag{7.76}$$

Note that the parallax method is only good for measuring distances of stars less than about 1000 light years away. More advanced methods are used to measure distances greater than 1000 light years.

Defining π for computer usage

If you get involved with computers used for scientific computing you will find that all computers will truncate or chop off irrational numbers because every computer has a central processing unit which can only handle a fixed number of binary digits. This truncation of numbers is called computer round-off error. Computer manufacturers must do it, but they don't like to talk about it. To define π for computer usage let the computer save π to the best of its ability. Since $\tan\frac{\pi}{4}=1$, one can define $PI = 4.0 * \arctan(1.0)$. This way the computer will chop off the exact representation and give you a value for PI which is the best rational number approximation that will fit into the machine.

Additional trigonometric relations

$$
\begin{array}{lll}
\sin(-\theta) = -\sin\theta & \sin(\frac{\pi}{2}\pm\theta) = \cos\theta & \sin(\pi\pm\theta) = \mp\sin\theta \\
\cos(-\theta) = \cos\theta & \cos(\frac{\pi}{2}\pm\theta) = \mp\sin\theta & \cos(\pi\pm\theta) = -\cos\theta \\
\tan(-\theta) = -\tan\theta & \tan(\frac{\pi}{2}\pm\theta) = \mp\cot\theta & \tan(\pi\pm\theta) = \pm\tan\theta
\end{array}
$$

$$
\begin{array}{ll}
\sin(\frac{3\pi}{2}\pm\theta) = -\cos\theta & \sin(\pi\pm\theta) = \pm\sin\theta \\
\cos(\frac{3\pi}{2}\pm\theta) = \pm\sin\theta & \cos(\pi\pm\theta) = \cos\theta \\
\tan(\frac{3\pi}{2}\pm\theta) = \mp\cot\theta & \tan(\pi\pm\theta) = \pm\tan\theta
\end{array}
$$

If the angle θ is restricted to quadrant I, then the following relations are valued.

$$
\begin{array}{lll}
\text{If } u = \sin\theta, & \sqrt{1-u^2} = \cos\theta, & \dfrac{u}{\sqrt{1-u^2}} = \tan\theta \\
\text{If } u = \cos\theta, & \sqrt{1-u^2} = \sin\theta, & \dfrac{\sqrt{1-u^2}}{u} = \tan\theta \\
\text{If } u = \tan\theta, & \dfrac{u}{\sqrt{1+u^2}} = \sin\theta, & \dfrac{1}{\sqrt{1+u^2}} = \cos\theta
\end{array}
$$

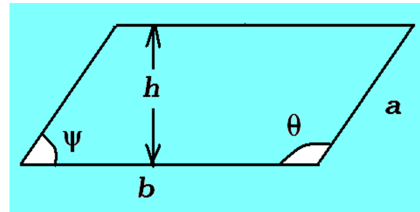
Exercises

▶ **7-1.**

Given the parallelogram with height h, base b, side a and angle θ.

(a) Find h, ψ and area if $a = 3$, $b = 5$, $\theta = 135°$

(b) Find h, ψ and area if $a = 4$, $b = 6$, $\theta = 120°$

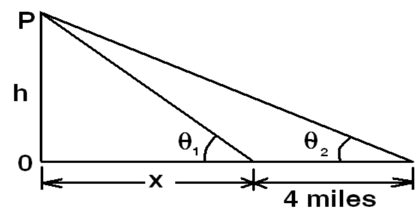


▶ **7-2.**

Point P is a mountain peak and point 0 is directly below P. At a distance x from point 0 the angle of elevation of point P is $\theta_1 = 26.0774°$. At a distance of (x+4) miles from point 0 the angle of elevation is $\theta_2 = 14.7564°$.

(a) Find the height h of the mountain.

(b) Find the distance x.



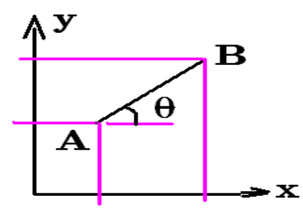
▶ **7-3.** Find the sine, cosine, tangent of the following angles

$$(a)\ \frac{\pi}{2} \pm \theta, \quad (b)\ \pi \pm \theta, \quad (c)\ \frac{3\pi}{2} \pm \theta, \quad (d)\ 2\pi \pm \theta$$

▶ **7-4.**

(a) Show the projection of $\overline{\text{AB}}$ onto the x-axis is $\overline{\text{AB}} \cos\theta$

(b) Show the projection of $\overline{\text{AB}}$ onto the y-axis is $\overline{\text{AB}} \sin\theta$ or $\overline{\text{AB}} \cos(\frac{\pi}{2} - \theta)$



▶ **7-5.** Prove the following identities.

$$(a)\ \sec\theta\csc\theta = \tan\theta + \cot\theta \qquad (d)\ \frac{1}{\tan\theta + \sec\theta} = \sec\theta - \tan\theta$$

$$(b)\ \tan\alpha\sin\alpha + \cos\alpha = \sec\alpha \qquad (e)\ \frac{\sin\theta + \tan\theta}{1 + \sec\theta} = \sin\theta$$

$$(c)\ \frac{1 + \sec\theta}{1 - \sec\theta} = \frac{\cos\theta + 1}{\cos\theta - 1} \qquad (f)\ \frac{\sin\alpha + \cos\beta}{\sin\alpha - \cos\beta} = \frac{\sec\beta + \csc\alpha}{\sec\beta - \csc\alpha}$$

▶ **7-6.** Evaluate the following.

$$(a)\ \sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6} \qquad (c)\ \cos\frac{\pi}{12}\cos\frac{\pi}{4} - \sin\frac{\pi}{12}\sin\frac{\pi}{4}$$

$$(b)\ \frac{\tan\frac{\pi}{12} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{12}\tan\frac{\pi}{4}} \qquad (d)\ \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{12}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{12}}$$

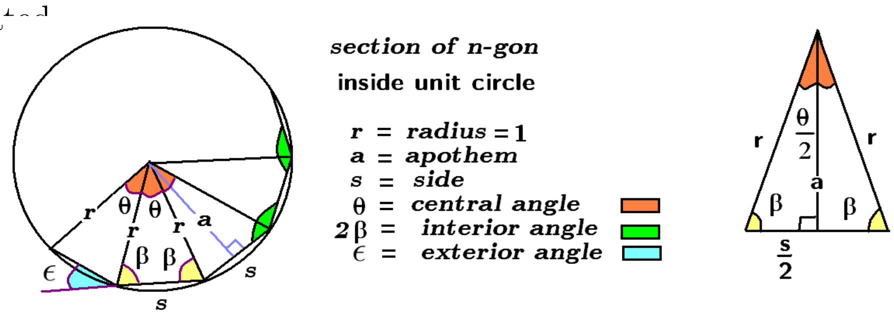
▶ **7-7.** Prove the following identities.

$$(a)\ \sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta} \qquad (d)\ \frac{\cos^2\alpha}{1-\sin\alpha} = 1+\sin\alpha$$

$$(b)\ \frac{\sin\alpha+\tan\alpha}{\cot\alpha+\csc\alpha} = \sin\alpha\tan\alpha \qquad (e)\ \frac{\tan^2\beta}{\sec^2\beta}+\frac{\cot^2\beta}{\csc^2\beta} = 1$$

$$(c)\ \sin\alpha(1+\cot^2\alpha) = \csc\alpha \qquad (f)\ \frac{\cos\beta}{\cos\beta-\sin\beta} = \frac{1}{1-\tan\beta}$$

▶ **7-8.** Consider a regular polygon with n sides inscribed within a unit circle as illustrated



Fill in the following table.

Regular polygons with					
sides n	central angle θ (*radians*)	isosceles base angle β (*radians*)	side s	apothem a	exterior angle (*radians*)
3					
4					
5					
6					
7					
8					
⋮	⋮	⋮	⋮	⋮	⋮
n					

▶ **7-9.** Use the given information to find $\sin\theta, \cos\theta, \tan\theta, \cot\theta, \sec\theta, \csc\theta$

$$(a)\ \sin\theta = \frac{2}{3} \qquad (b)\ \cot\theta = \frac{4}{3} \qquad (c)\ \sec\theta = \frac{12}{5} \qquad (d)\ \csc\theta = \frac{15}{8}$$

▶ **7-10.** Given a circle with unit radius with P a point on the circumference defining the angle $\angle P0B = \theta$. **Express all answers in terms of the angle θ.**

(a) Find the distance $\overline{\text{PB}}$.

(b) Find the distance $\overline{\text{0B}}$.

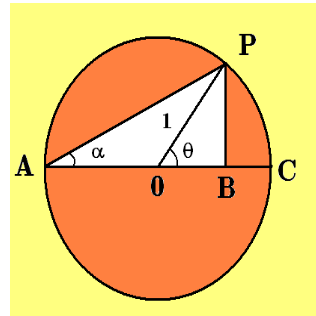
(c) Find the distance $\overline{\text{AB}}$.

(d) Find $\tan\theta$ and use the equation (7.39) to write $\tan\frac{\theta}{2}$

(e) Find the distance $\overline{\text{AP}}$.

(f) Find $\tan\alpha$

(g) Find α in terms of θ



▶ **7-11.** Given the triangle $\triangle ABC$ illustrated which is inscribed in a unit circle. **Express all answers in terms of the angle $\theta = \angle B0D$**

(a) Find $\overline{\text{0D}}$

(b) Find $\overline{\text{BD}}$

(c) Find $\overline{\text{BC}}$

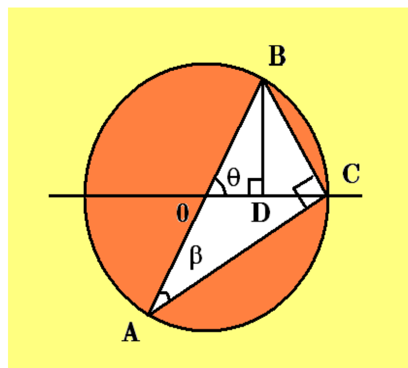
(d) Find $\overline{\text{AC}}$

(e) Find $\overline{\text{AB}}$

(f) Find $\cos\beta$

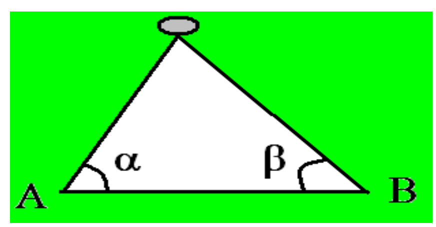
(g) Find $\sin\beta$

(h) Find β in terms of θ



▶ **7-12.**

Two observation stations A and B are 2 miles apart. They see a UFO overhead and simultaneously measure the angles of elevation α and β as illustrated. Find a formula for the height of the UFO.



▶ **7-13.** Find the area of

(a) An equilateral triangle inscribed within a unit circle.

(b) A square inscribed within a unit circle.

(c) A pentagon inscribed within a unit circle.

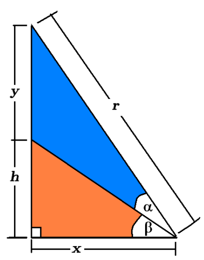
(d) A hexagon inscribed within a unit circle.

▶ **7-14.** Fill in the table below.

Exact trigonometric values for special angles							
angle θ *degrees*	angle θ *radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
15	$\frac{\pi}{12}$					$\sqrt{2}(-1+\sqrt{3})$	
36	$\frac{\pi}{5}$					$-1+\sqrt{5}$	
75	$\frac{5\pi}{12}$					$\sqrt{2}(1+\sqrt{3})$	
105	$\frac{7\pi}{12}$					$-\sqrt{2}(1+\sqrt{3})$	
120	$\frac{2\pi}{3}$					-2	
150	$\frac{5\pi}{6}$					$\frac{-2}{\sqrt{3}}$	
210	$\frac{7\pi}{6}$					$\frac{-2}{\sqrt{3}}$	
315	$\frac{7\pi}{4}$					$\sqrt{2}$	
345	$\frac{23\pi}{12}$					$-\sqrt{2}(1-\sqrt{3})$	

Hint: One method $15^\circ = 45^\circ - 30^\circ$. Another method is to draw a sketch.

▶ **7-15.**



In the right triangle given show that

$$y = x\,(\tan(\alpha+\beta) - \tan\beta)$$

▶ **7-16.** Simplify the following.

(i) $a\sin(\frac{\pi}{2}+\theta) + b\cos(\pi-\theta)$ (ii) $a\sin(2\pi+\theta) + b\cos(\frac{3\pi}{2}-\theta)$ (iii) $a\cos(\frac{\pi}{2}-\theta) - b\sin(\pi+\theta)$

▶ **7-17.** If $\tan 20^\circ = x$, find $y = \dfrac{\sin(90^\circ+20^\circ)+\cos(270^\circ-20^\circ)}{\csc(180^\circ-20^\circ)+\sec(360^\circ-20^\circ)}$ Hint: What are the signs associated with trigonometric functions in the four quadrants?

▶ **7-18.** What values of θ, $0 < \theta < 2\pi$, satisfy the given equations? If a value of θ satisfies the equation does the value $\theta + 2\pi$ also satisfy the equation?

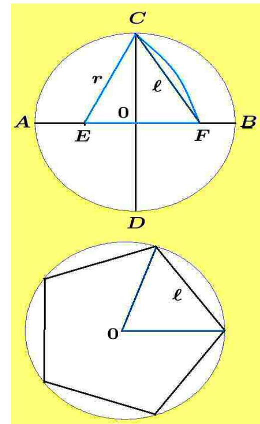
(a) $\sin\theta = -\dfrac{1}{2}$ $\quad$ (d) $\cot\theta = \sqrt{3}$

(b) $\cos\theta = -\dfrac{1}{2}$ $\quad$ (e) $\sec\theta = -\sqrt{2}$

(c) $\tan\theta = -\sqrt{3}$ $\quad$ (f) $\csc\theta = -2$

▶ **7-19.**

Prove the following construction associated with a pentagon inscribed within a given circle.
(i) Construct a circle with center 0.
(ii) Construct perpendicular diameters $\overline{\text{AB}} \perp \overline{\text{CD}}$.
(iii) Find the midpoint E of line segment $\overline{\text{A0}}$.
(iv) Set a drawing compass needle at point E and pencil point at point C and construct the arc $\widehat{\text{CF}}$ having radius $r = \overline{\text{EC}}$.
(v) Show the line segment $\overline{\text{CF}} = \ell$ is the side length of the inscribed pentagon.
Hint: $\sin\frac{\pi}{5} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$



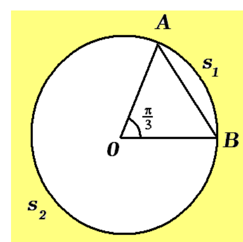
▶ **7-20.** Evaluate using principal values

$$(a)\ \cos^{-1}(-1) \qquad (b)\ \cos^{-1}(0) - \tan^{-1}(-\sqrt{3}) \qquad (c)\ \sin^{-1}(-\frac{1}{2}) + \cos^{-1}(\frac{\sqrt{3}}{2})$$

▶ **7-21.** Evaluate the following

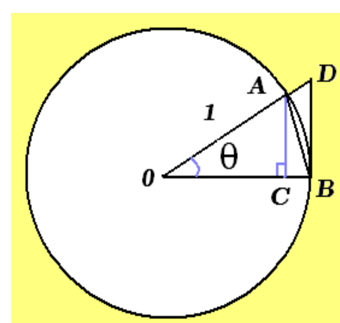
$$(a)\ \tan(\cot^{-1}\sqrt{3}) \qquad (b)\ \sin(\cos^{-1}(\frac{1}{2})) \quad (c)\ \cos(\sin^{-1}(\frac{1}{2}))$$

▶ **7-22.** Given a circle with radius 2.
(a) Find the area of the segment and sector.
(b) Find the minor arc s_1 and major arc s_2.



▶ **7-23.** Given a circle with radius $r = 1$. Let segment $\overline{\text{AC}} \perp \overline{\text{0B}}$, $\overline{\text{0D}}$ a ray at angle θ and $\overline{\text{BD}}$ tangent line to circle at point B. Show that
(i) $\overline{\text{AC}} = \sin\theta$ (ii) $\overline{\text{BD}} = \tan\theta$ (iii) $\theta = \widehat{\text{AB}}$
(iv) Use areas to show $\sin\theta < \theta < \tan\theta$
Fill in the following table

θ *degrees*	$\sin\theta$	θ *radians*	$\tan\theta$
2°			
1°			
$\frac{1}{2}^\circ$			
$\frac{1}{4}^\circ$			



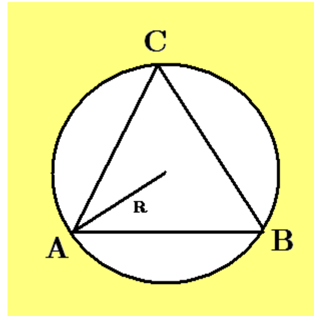
▶ **7-24.**

Given triangle $\triangle ABC$ with R the radius of the circumscribed circle. Show that

(a) $\overline{\text{BC}} = 2R\sin A$

(b) $\overline{\text{CA}} = 2R\sin B$

(c) $\overline{\text{AB}} = 2R\sin C$



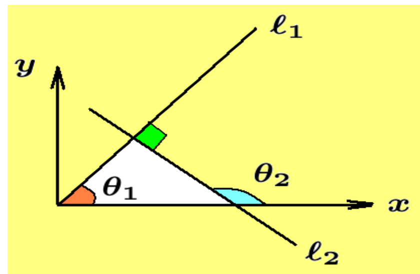
▶ **7-25.**

In the figure illustrated show that

(a) The slope of line ℓ_1 is $m_1 = \tan\theta_1$

(b) The slope of line ℓ_2 is $m_2 = \tan\theta_2$

(c) Prove that if line ℓ_1 is perpendicular to line ℓ_2, then $m_1 m_2 = -1$. Write out in words what this is telling you. Hint: How is an exterior angle related to the interior angles of a triangle.



▶ **7-26.** Find the area of a regular pentagon with edge of length s.
Hint: $\tan\left[36\frac{\pi}{180}\right] = \sqrt{5-2\sqrt{5}}$

▶ **7-27.** Find the area of a regular hexagon with edge of length s.

▶ **7-28.** Prove that $\sin\frac{\pi}{5} = \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$

▶ **7-29.** Prove that $\sin\frac{\pi}{10} = \frac{1}{4}\left(-1+\sqrt{5}\right)$

▶ **7-30.** Prove that $\sin\frac{3\pi}{10} = \frac{1}{4}\left(1+\sqrt{5}\right)$

▶ **7-31.** Prove that $\sin\frac{2\pi}{5} = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$

▶ **7-32.**

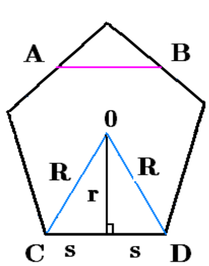
Given a regular pentagon with each side having length $\ell = 2s$ and angle $\angle C0D = \theta$.

(a) Find the circumradius R in terms of θ and s

(b) Find the inradius r in terms of θ and s

(c) If the line segment $\overline{\text{AB}}$ connects the midpoints of two sides. Find the length of $\overline{\text{AB}}$

(d) Find a relationship between r and R



Geometry

Chapter 8

Trigonometry II

Law of sines

In any triangle $\triangle ABC$ with angles α, β, γ and sides a, b, c one can write

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

This is a relationship between the interior angles of a triangle and the sides opposite these angles. This relationship is known as **the law of sines.**

In the top triangle construct the perpendicular line $\overline{AD}$ to side $\overline{BC}$ to form two right triangles

$$\text{In triangle } \triangle ADB, \quad \sin\beta = \frac{\overline{AD}}{c}$$

$$\text{In triangle } \triangle ADC, \quad \sin\gamma = \frac{\overline{AD}}{b}$$

Consequently, one can write

$$c\sin\beta = b\sin\gamma = \overline{AD}$$

or

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \tag{8.1}$$

In the lower triangle, construct the line $\overline{CD}$ which is perpendicular to side $\overline{AB}$. This forms two right triangles as illustrated.

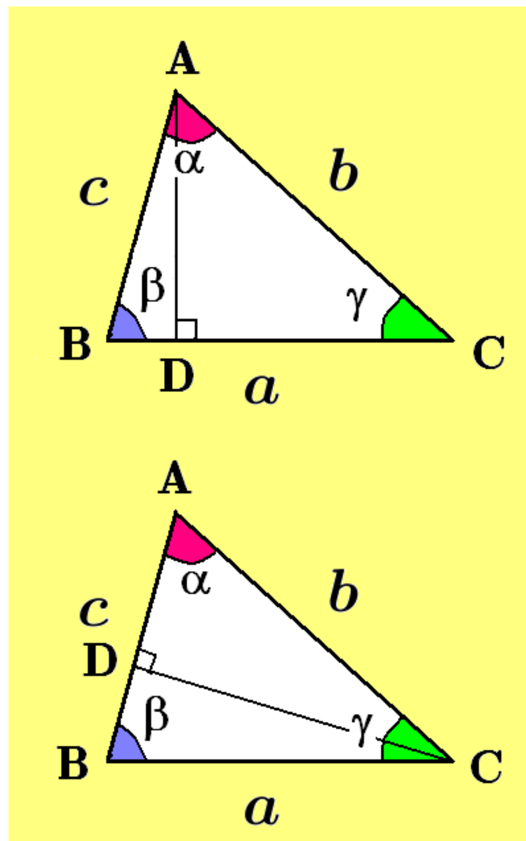


Figure 8-1.
Triangle $\triangle ABC$

$$\text{In triangle } \triangle CDB, \quad \sin\beta = \frac{\overline{DC}}{a}$$

$$\text{In triangle } \triangle CDA, \quad \sin\alpha = \frac{\overline{DC}}{b}$$

Consequently,

$$a\sin\beta = b\sin\alpha = \overline{DC} \qquad \text{or} \qquad \frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \tag{8.2}$$

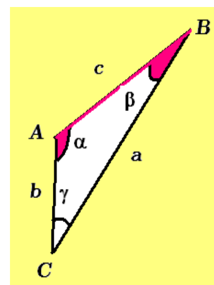
Using the fact that things equal to the same thing are equal to each other one finds

$$\boldsymbol{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}} \tag{8.3}$$

which is the law of sines.

In the above proof, there may arise the situation where in the triangle $\triangle ABC$ one of the angles β or γ is obtuse, then it will be necessary to extend side $\overline{BC}$ in order to drop a perpendicular to side $\overline{BC}$. In this case the proof given above must then be modified slightly.

Note that if you know two angles and included side associated with a triangle, then you can calculate the remaining angle and remaining sides of the triangle. For example, if angles α and β are known together with side c, then one can use the equation $\gamma = \pi - (\alpha + \beta)$ to find the remaining angle and then can use the law of sines to write



$$\frac{\sin(\pi - (\alpha+\beta))}{c} = \frac{\sin\beta}{b} \qquad \text{and} \qquad \frac{\sin(\pi - (\alpha+\beta))}{c} = \frac{\sin\alpha}{a}$$

from which the unknown sides a and b can be calculated. This technique is used over and over again in surveying where it is known as the method of triangulation.

Law of cosines

Consider the triangle $\triangle ABC$, illustrated in the figure 8-2, with

vertex angle A equal to α and side opposite the vertex angle $\overline{CB} = a$

vertex angle B equal to β and side opposite the vertex angle $\overline{AC} = b$

vertex angle C equal to γ and side oposite the vertex angle $\overline{AB} = c$

The **law of cosines** associated with triangle $\triangle ABC$ can be expressed

$$\begin{aligned} \boldsymbol{a^2 + b^2 - 2a\,b\cos\gamma} &= \boldsymbol{c^2} \\ \boldsymbol{b^2 + c^2 - 2b\,c\cos\alpha} &= \boldsymbol{a^2} \\ \boldsymbol{c^2 + a^2 - 2c\,a\cos\beta} &= \boldsymbol{b^2} \end{aligned} \tag{8.6}$$

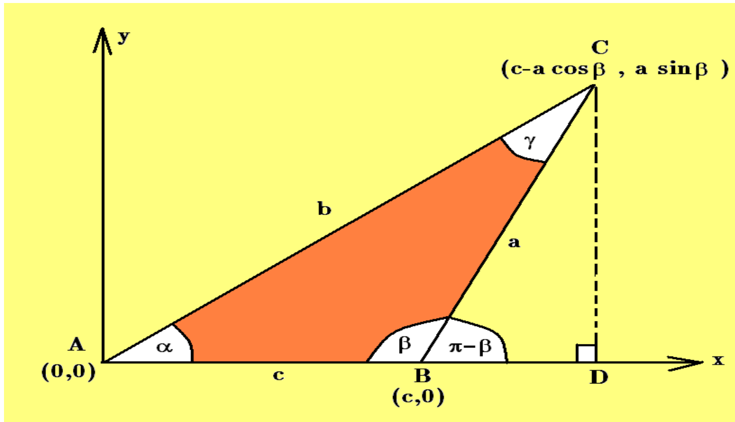
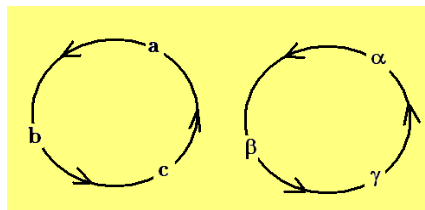


Figure 8-2.
Obtuse triangle $\triangle ABC$

Observe the **cyclic changing of the letters in the equations** (8.6). Use the circles illustrated to move each letter in an equation to the next ordering indicated by the arrows on the circles. Thus, replace a by b, b by c and c by a and use a similar replacement cycle for the angles α, β and γ.



Consider the figure 8-2 illustrating a triangle $\triangle ABC$ in a Cartesian coordinate system with coordinates for vertex A as $(0,0)$, the coordinates for vertex B as $(c,0)$ and the coordinates for vertex C as $(c - a\cos\beta, a\sin\beta)$.

To understand where these coordinates come from first drop a perpendicular line from vertex C to the extended line $\overline{AB}$ and note that for the right triangle $\triangle BDC$ one finds

$$\begin{aligned} \cos(\pi-\beta) =& \frac{\overline{BD}}{a} = -\cos\beta \\ \sin(\pi-\beta) =& \frac{\overline{CD}}{a} = \sin\beta \end{aligned} \qquad \text{or} \qquad \begin{aligned} \overline{BD} =& -a\cos\beta \\ \overline{CD} =& a\sin\beta \end{aligned}$$

The distance $\overline{AD} = c + \overline{BD} = c - a\cos\beta$ and the distance $\overline{CD} = a\sin\beta$ giving vertex C the coordinates $(c - a\cos\beta, a\sin\beta)$. Note in the figure that the angle β is an obtuse angle and the cosine of an angle in the second quadrant is negative.

Apply the Pythagorean theorem to triangle $\triangle ACD$ to obtain

$$(\overline{AD})^2 + (\overline{CD})^2 = (c - a\cos\beta)^2 + (a\sin\beta)^2 = b^2 \tag{8.5}$$

Expand the terms in equation (8.5) and show

$$\begin{aligned} c^2 - 2c\,a\cos\beta + a^2\cos^2\beta + a^2\sin^2\beta =& b^2 \\ \text{or} \qquad a^2 + c^2 - 2c\,a\cos\beta =& b^2 \end{aligned} \tag{8.6}$$

which states that in triangle $\triangle ABC$ when the sides a and c are given together with the angle β between the sides, then the square of the side opposite angle β must equal the sum of the squares of the other two sides minus twice the product of the other two sides times the cosine of the angle between the other two sides.

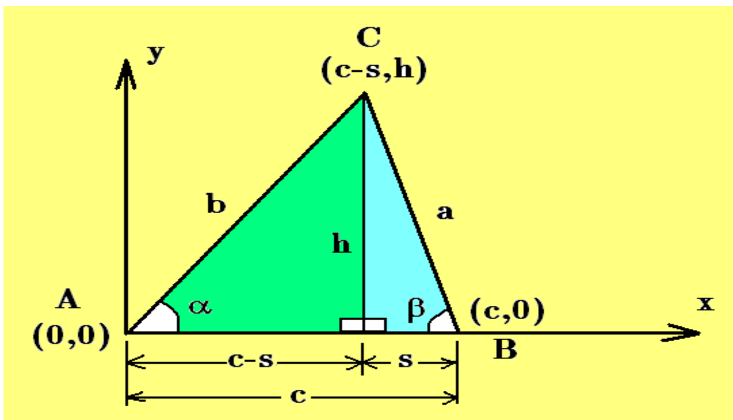


Figure 8-3.
Acute triangle $\triangle ABC$

In the case triangle $\triangle ABC$ is an acute triangle as in figure 8-3, then one can drop a perpendicular from vertex C to cut the base $c = \overline{\text{AB}}$ into two parts labeled $(c-s)$ and s such that $(c-s)+s=c$. If h is the length of the perpendicular line constructed, then one can apply the Pythagorean theorem to the two right triangles created to obtain

$$(c-s)^2 + h^2 = b^2 \qquad \text{and} \qquad s^2 + h^2 = a^2 \tag{8.7}$$

Expand the first equation and then substitute for h^2 from the second equation to obtain

$$(c^2 - 2cs + s^2) + h^2 = b^2 \qquad \Rightarrow \qquad (c^2 - 2cs + s^2) + (a^2 - s^2) = b^2$$

This last result can be simplified and rearranged into the form

$$b^2 = a^2 + c^2 - 2cs \tag{8.8}$$

Observe that the length s is the projection of the side a onto the base $\overline{\text{AB}} = c$ of the triangle and can be expressed as $s = a\cos\beta$ which when substituted into equation (8.8) gives the law of cosines

$$b^2 = a^2 + c^2 - 2a\,c\cos\beta \tag{8.9}$$

The above results can also be proven using Euclid's results on projections (See page 152). By rotating the triangle $\triangle ABC$ and assigning new coordinates to the vertices one can repeat the steps above to verify that

$$\begin{aligned} c^2 =& a^2 + b^2 - 2a\,b\cos\gamma \\ a^2 =& b^2 + c^2 - 2b\,c\cos\alpha \end{aligned} \tag{8.10}$$

These same results can be obtained by performing a cyclic rotation of the symbols in equation (8.9).

Note that if one of the triangle interior angles is 90° or $\frac{\pi}{2}$ radians, then the law of cosines reduces to the Pythagorean theorem.

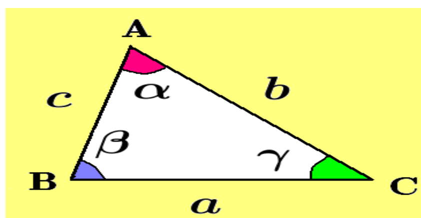
Law of tangents

The **law of tangents** associated with a general triangle $\triangle ABC$ can be expressed in any of the forms

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

$$\frac{c-a}{c+a} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$



Note the cyclic changing of the sides a,b,c and angles α, β, γ in the above formulas. To derive the first of these formulas one can use the law of sines and write

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \qquad \text{or} \qquad \frac{a}{b} = \frac{\sin\alpha}{\sin\beta} \tag{8.11}$$

Subtract 1 from both sides of equation (8.11) and write

$$\frac{a}{b} - 1 = \frac{\sin\alpha}{\sin\beta} - 1 \qquad \Rightarrow \qquad \frac{a-b}{b} = \frac{\sin\alpha - \sin\beta}{\sin\beta} \tag{8.12}$$

Add 1 to both sides of equation (8.11) and write

$$\frac{a}{b} + 1 = \frac{\sin\alpha}{\sin\beta} + 1 \qquad \Rightarrow \frac{a+b}{b} = \frac{\sin\alpha + \sin\beta}{\sin\beta} \tag{8.13}$$

Taking the ratio of the equations (8.12) and (8.13) one obtains

$$\frac{a-b}{a+b} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} \tag{8.14}$$

Using the previously derived addition and subtraction formulas for the sine function, the right hand side of equation (8.14) is reduced to

$$\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} \tag{8.15}$$

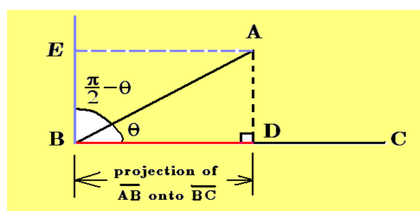
so that equation (8.14) becomes the tangent law

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} \tag{8.16}$$

Using a cyclic rotation of the symbols, the other tangent laws are derived in a similar manner.

Projections

Let θ denote the angle between the line segments $\overline{AB}$ and $\overline{BC}$ and then construct the line segment $\overline{AD}$ which is perpendicular to the line segment $\overline{BC}$ as illustrated. By definition



$$\cos\theta = \frac{adjacent\ side}{hypotenuse} = \frac{\overline{BD}}{\overline{AB}} \quad \text{or} \quad \overline{BD} = \overline{AB}\cos\theta = \text{projection of } \overline{AB} \text{ onto } \overline{BC}$$

The projection of $\overline{\text{AB}}$ onto the vertical line is obtained from

$$\cos(\frac{\pi}{2} - \theta) = \frac{adjacent\ side}{hypotenuse} = \frac{\overline{\text{EB}}}{\overline{\text{AB}}}$$

giving

$$\overline{\text{EB}} = \overline{\text{AB}}\cos(\frac{\pi}{2} - \theta) = \overline{\text{AB}}\sin\theta = \text{projection of } \overline{\text{AB}} \text{ onto vertical axis.}$$

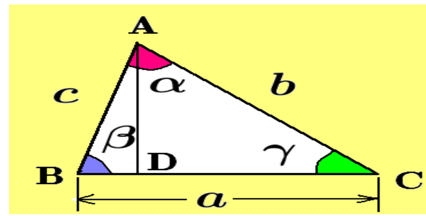
Example 8-1.

Show for a general triangle $\triangle ABC$ as illustrated, show that $a = b\cos\gamma + c\cos\beta$

Proof

The projection of side c onto the side $\overline{BC} = a$ is given by

$$\overline{BD} = c\cos\beta$$



The projection of side $\overline{AC} = b$ onto the side $\overline{BC} = a$ is

$$\overline{DC} = b\cos\gamma$$

Adding the line segments $\overline{BD}$ and $\overline{DC}$ gives

$$\overline{BD} + \overline{DC} = b\cos\gamma + c\cos\beta = a \qquad (8.17)$$

It is left as an exercise to perform a cyclic changing of symbols in the equation (8.17) to obtain

$$b = c\cos\alpha + a\cos\gamma$$

$$c = a\cos\beta + b\cos\alpha$$

and then interpret these equations as representing the summation of projections. Remember that if γ is the angle between two line segments a and b, then $b\cos\gamma$ is the projection of b onto the line segment a (See the above triangle).

■

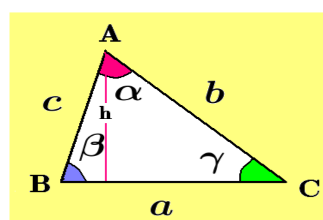
Find alternative formulas for the area of a triangle.

There are alternative ways to calculate the area of a triangle based upon your knowledge of the sides and angles associated with the triangle.

Two sides and included angle

The area (A) of the triangle $\triangle ABC$ is half the base (a) times the height (h) or

$$A = \frac{1}{2}ah \qquad (8.18)$$



In triangle $\triangle ABC$ $\sin\gamma = \dfrac{h}{b}$ or $h = b\sin\gamma$ so one alternative representation for the area of the triangle is

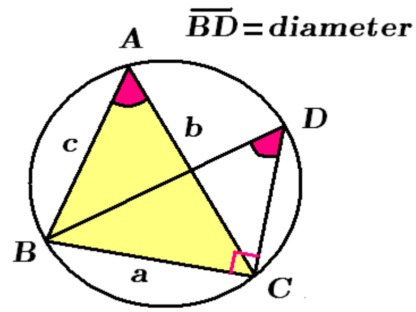
$$A = \frac{1}{2}ab\sin\gamma \quad \text{or} \quad A = \frac{1}{2}ab\sin C \qquad (8.19)$$

This represents half the product of two adjacent sides times the sine of the angle between the sides. Performing a cyclic rotation of the symbols in equation (8.19) one can obtain the area equations

$$A = \frac{1}{2}bc\,\sin\alpha,\;\; A = \frac{1}{2}ca\,\sin\beta \quad \text{or} \quad A = \frac{1}{2}bc\,\sin A,\;\; A = \frac{1}{2}ca\,\sin B \tag{8.20}$$

Extended law of sines

Given a triangle $\triangle ABC$ one can express the law of sines as



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of the given triangle. This can be shown to be true by constructing the diameter $\overline{\text{BD}}$ and chord $\overline{\text{DC}}$ and observing that the angles $\angle BAC$ and $\angle BDC$ both subtend the same arc $\overset{\frown}{\text{BC}}$. Therefore, $\angle BAC = \angle BDC$ and

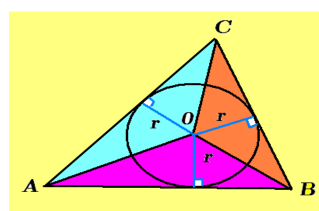
$$\sin\angle BAC = \sin A = \sin\angle BDC = \frac{a}{2R}$$

because the triangle $\triangle BDC$ is a right triangle. Using the equation (8.20) $A = \frac{1}{2}bc\,\sin A$ one can write

$$\text{Area of triangle } \triangle ABC = A = \frac{1}{2}bc\sin A = \frac{1}{2}bc\,\frac{a}{2R} = \frac{abc}{4R} \tag{8.21}$$

Area in terms of inradius

The area A of a triangle $\triangle ABC$ in terms of the inradius r is given by



$A = rs$, where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of triangle

The proof is as follows

$$A_1 = \text{Area } \triangle A0C = \frac{1}{2}rb$$
$$A_2 = \text{Area } \triangle A0B = \frac{1}{2}rc$$
$$A_3 = \text{Area } \triangle B0C = \frac{1}{2}ra$$

Add the above equations using the whole is the sum of its parts to show

$$\text{Area } \triangle ABC = r\, s \quad \text{where } s = \frac{1}{2}(a+b+c)$$

is the semiperimeter of the triangle.

Angle-side-angle area

Using the law of sines $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$ one can write $b = a\frac{\sin\beta}{\sin\alpha}$ and then substitute for b into the equation (8.19) to obtain

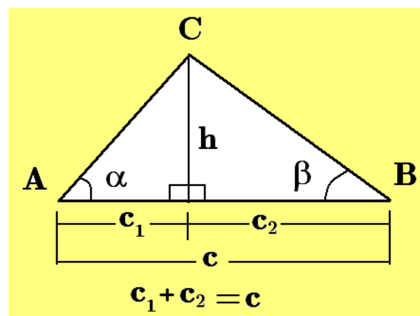
$$A = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha} = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin(\pi-(\beta+\gamma))} = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)} \tag{8.22}$$

Note that in the above equation the angle α has been replaced by $\pi-(\beta+\gamma)$ because the interior angles of any triangle must satisfy $\alpha+\beta+\gamma=\pi$. Making cyclic changes to the area formula (8.22) one finds the additional area formulas

$$A = \frac{1}{2}b^2\frac{\sin\gamma\sin\alpha}{\sin(\gamma+\alpha))} \qquad \text{and} \qquad A = \frac{1}{2}c^2\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)} \tag{8.23}$$

We know that for a triangle with height h and base c that

$$Area = \frac{1}{2}(base)(height) = \frac{1}{2}ch \tag{8.24}$$



By constructing the perpendicular line h from the vertex C one divides the base into two parts which are labeled c_1 and c_2 such that $c = c_1 + c_2$ as illustrated in the figure.

But

$$\tan\alpha = \frac{h}{c_1} \quad \text{or} \quad c_1 = \frac{h}{\tan\alpha}$$
$$\tan\beta = \frac{h}{c_2} \qquad\quad c_2 = \frac{h}{\tan\beta} \tag{8.25}$$

The equations (8.25) can be used to write

$$c = c_1 + c_2 = \frac{h}{\tan\alpha} + \frac{h}{\tan\beta} = h(\cot\alpha + \cot\beta)$$

so that

$$\begin{aligned} Area = \frac{1}{2}hc =& \frac{1}{2}h^2(\cot\alpha+\cot\beta) \\ =& \frac{1}{2}h^2\frac{(\cot\alpha+\cot\beta))^2}{(\cot\alpha+\cot\beta)} \\ =& \frac{1}{2}\frac{c^2}{\cot\alpha+\cot\beta} \end{aligned} \tag{8.26}$$

Thus, knowing two angles and included side one can determine the area of the triangle.

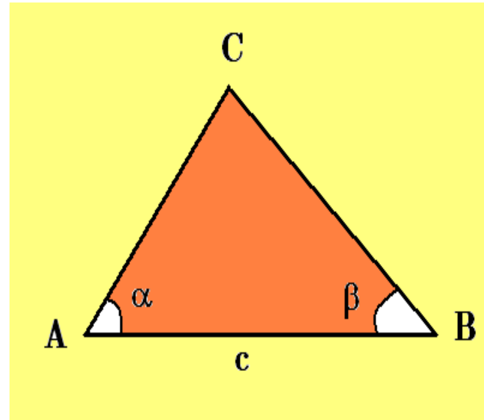
Example 8-2.

Apply the results from equations (8.26) and (8.23) to the figure given and obtain the representations

$$\text{Area } \triangle ABC = A = \frac{1}{2} c^2 \frac{\sin\alpha \sin\beta}{\sin(\alpha+\beta))}$$

$$\text{Area } \triangle ABC = A = \frac{1}{2} \frac{c^2}{\cot\alpha + \cot\beta}$$

Show these equations are equivalent.



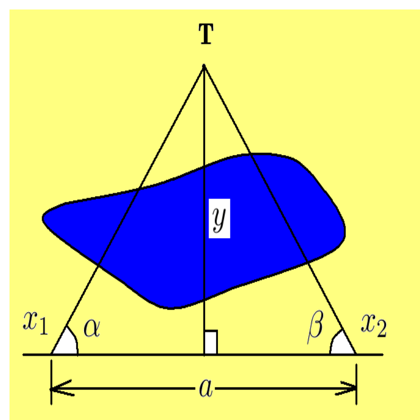
Solution

$$\frac{1}{\cot\alpha+\cot\beta} = \frac{1}{\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}} = \frac{1}{\frac{\cos\alpha\sin\beta+\cos\beta\sin\alpha}{\sin\alpha\sin\beta}} = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

■

Example 8-3. Find shortest distance.

As part of a cellular network a cell tower (T) is to be constructed at a site near a major highway which is assumed to be a straight line. A lake exists between the proposed cell tower and the highway. A surveyor uses a transit and measures the angles α and β as well as the distance $a = \overline{x_1x_2}$. Find the shortest distance (y) from the highway to the tower, using the known information.



Solution

Use the result from the equation (8.23) and show that the area of triangle $\triangle Tx_1x_2$ is given by

$$\text{Area } \triangle Tx_1x_2 = \frac{1}{2} ay = \frac{1}{2} a^2 \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)} \tag{8.27}$$

Simplify the equation (8.27) and show

$$y = a\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)} \tag{8.28}$$

If $a = 2$ miles, $\alpha = 60^\circ = \pi/3$ radians and $\beta = 70^\circ = 7\pi/18$ radians, then

$$y = 2\frac{\sin(\pi/3)\sin(7\pi/18)}{\sin(\pi/3 + 7\pi/18))} = 2.12467 \text{ miles} = (2.12467 \text{ miles})\left(5280 \ \frac{\text{ft}}{\text{mile}}\right) = 11,218.3 \text{ feet}$$

■

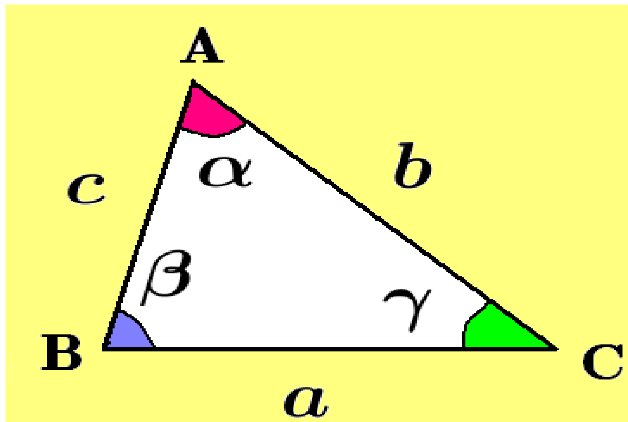
Example 8-4.

Show that in a general triangle $\triangle ABC$

$$\frac{a+b}{c} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{b+c}{a} = \frac{\cos\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\beta+\gamma}{2}\right)}$$

$$\frac{c+a}{b} = \frac{\cos\left(\frac{\gamma-\alpha}{2}\right)}{\cos\left(\frac{\gamma+\alpha}{2}\right)}$$



Solution

By the law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

so that one can write

$$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta}$$

Add 1 to both sides of this equation to show

$$\frac{a}{b} + 1 = \frac{\sin\alpha}{\sin\beta} + 1 \quad \Rightarrow \quad \frac{a+b}{b} = \frac{\sin\alpha + \sin\beta}{\sin\beta} \tag{8.29}$$

Rearrange the equation (8.29) and show

$$\frac{a+b}{\sin\alpha + \sin\beta} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

or

$$\frac{a+b}{c} = \frac{\sin\alpha + \sin\beta}{\sin\gamma} = \frac{\sin\alpha + \sin\beta}{\sin(\pi - (\alpha+\beta))} \tag{8.30}$$

Now use the equations (7.17)and (7.52) to express the equation (8.30) in the form

$$\frac{a+b}{c} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\pi\cos(\alpha+\beta) - \cos\pi\sin(\alpha+\beta)} \tag{8.31}$$

In equation (8.31) note that

$$\cos\pi = -1,\ \sin\pi = 0 \text{ and } \sin(\alpha+\beta) = \sin 2\left(\frac{\alpha+\beta}{2}\right)$$

so that by using the double-angle formula (7.23) one can alter equation (8.31) to have the form

$$\frac{a+b}{c} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}$$

which simplifies to the result

$$\frac{a+b}{c} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

Performing a cyclic changing of the symbols in the above equation will produce the other equations given above.

■

The circle

Associated with every circle is a radius of length r. Recall that as the radius rotates through an angle θ an arc length s is swept out as the radius rotates. When the angle θ is 2π the arc length $s = 2\pi r$, the circumference of the circle. Using proportions one can write

$$\frac{\theta}{2\pi} = \frac{s}{2\pi r} \quad \Rightarrow \quad s = r\,\theta \tag{8.32}$$

The arc length s is determined by the radius r and the central angle θ in radians. When the radius sweeps out an arc length s a sector of the circle is created. The minor sector of a circle is the area between the arc length s and the two radii r as illustrated in the figure 8-5.

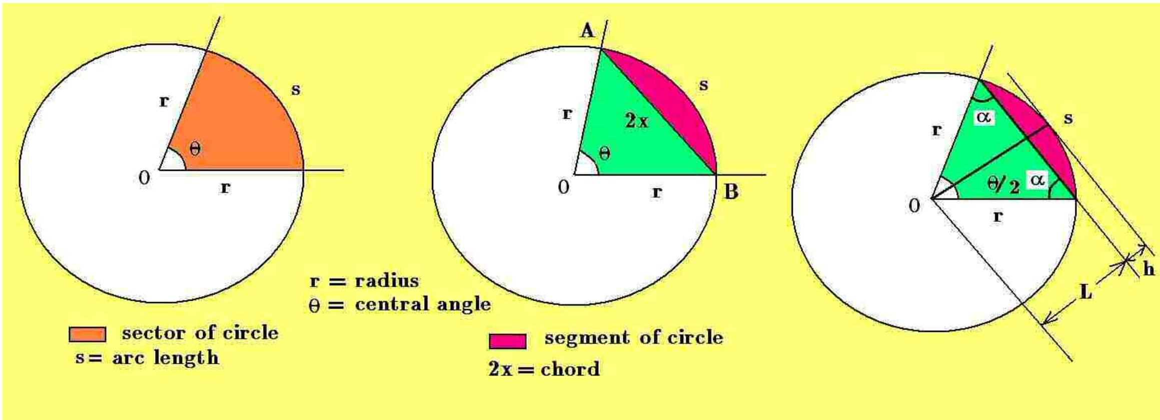


Figure 8-5. Sector, chord, arc length and segment of a circle.

The major sector is the area of the circle minus the area of the minor sector. To find the area of a minor sector one can again use proportions. The ratio of the minor sector area A_s to the angle θ is in the same ratio as the area of the circle is to 2π or

$$\frac{A_s}{\theta} = \frac{\pi r^2}{2\pi} \quad \Rightarrow \quad A_s = \frac{1}{2} r^2 \theta \tag{8.33}$$

where the central angle θ is in radians. One could also use the ratio of area of sector compared to area of circle being in the same ratio as the arc length s compared to the total circumference to obtain

$$\frac{A_s}{\text{Area of circle}} = \frac{s}{\text{circle circumference}} \qquad \frac{A_s}{\pi r^2} = \frac{r\theta}{2\pi r} \quad \Rightarrow \quad A_s = \frac{1}{2} r^2 \theta \tag{8.34}$$

The chord of a circle is any line segment whose endpoints lie on the circle circumference. The segment of a circle is the area between a chord of length $2x$ and the arc length (s) subtended by the chord. The segment of a circle is also illustrated in the figure 8-5. The maximum distance between the arc length s and the chord $2x$ is denoted by h and has the official name of **sagitta.** The distance $L = r - h$ is the height of the triangle $\triangle A0B$ having the chord for its base.

The area of the segment $A_{segment}$ is the area of the sector A_s minus the area of the triangle. Verify that in triangle $\triangle A0B$, with base $2x$ and height L, the following relations hold

$$2\alpha + \theta = \pi \qquad \Rightarrow \qquad \alpha = \frac{\pi}{2} - \frac{\theta}{2} \tag{8.35}$$

$$\sin\alpha = \frac{L}{r} \qquad \Rightarrow \qquad L = r\sin\alpha \tag{8.36}$$

$$\cos\alpha = \frac{x}{r} \qquad \Rightarrow \qquad x = r\cos\alpha \tag{8.37}$$

$$L^2 + x^2 = r^2 \qquad \Rightarrow \qquad L = \sqrt{r^2 - x^2} \tag{8.38}$$

$$\text{sagitta } h = r - L = r - \sqrt{r^2 - x^2} = r - r\sin\alpha = r\left(1 - \cos\frac{\theta}{2}\right) \tag{8.39}$$

$$\begin{aligned}\text{Area } \triangle A0B =& \frac{1}{2}(2x)L = \frac{1}{2}\left[2r\cos\alpha\right]\left[r\sin\alpha\right] \\ =& \frac{1}{2}r^2\sin 2\alpha = \frac{1}{2}r^2\sin(\pi - \theta) = \frac{1}{2}r^2\sin\theta \end{aligned} \tag{8.40}$$

$$A_{segment} = A_s - Area A0B = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}\left(\theta - \sin\theta\right)r^2 \tag{8.41}$$

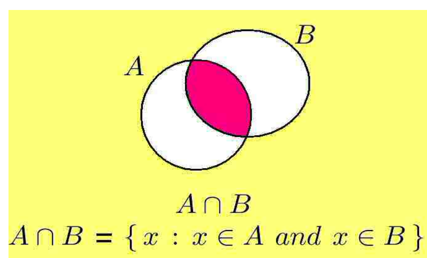
$$\begin{aligned}\text{chord length } 2x =& 2r\sin\frac{\theta}{2} = 2r\sqrt{1 - \cos^2\left(\frac{\theta}{2}\right)} \\ 2x =& 2r\sqrt{1 - \left(\frac{1+\cos\theta}{2}\right)} = r\sqrt{2 - 2\cos\theta}\end{aligned} \tag{8.42}$$

Intersection and Union of sets

The intersection of two sets A and B is denoted $A \cap B$ and is defined as the set of elements common to both sets A and B.

$$A \cap B = \{\, x \; : \; x \in A \; and \; x \in B \,\}$$

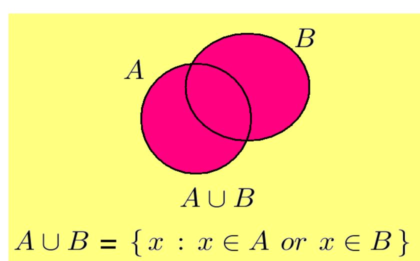
This is read as ≫ A intersection B is the set of elements x such that x is in A and x is in B.≪



The union of two sets A and B is denoted $A \cup B$ and is defined as the set of elements in set A, in set B or in both A and B.

$$A \cup B = \{\, x \; : \; x \in A \; or \; x \in B \,\}$$

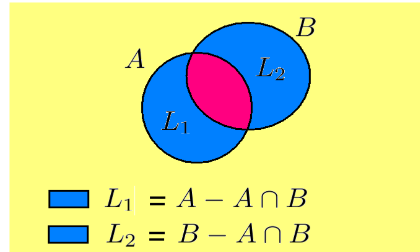
This is read ≫ A union B is the set of elements x such that x is in A or x is in B or x is in both A and B. ≪



The lune

A lune is the area bounded by two circular arcs.

Convex-convex[1] area called a lens.
Concave-convex area called a lune.



In terms of set theory language, the lunes illustrated in the figure are defined

$$L_1 = A - A \cap B, \qquad \textit{and} \qquad L_2 = B - A \cap B$$

The area associated with certain special lunes are easily calculated. Consider the figure 8-6 where one draws a chord $\overline{AB}$ of length $2x$ inside a circle. One can then use a drawing compass and find the midpoint of the chord and label the length of half the chord as the distance x as illustrated. Widen the drawing compass and place the needle of the compass at the midpoint and the pencil point on either point A or B. One can then construct the red semi-circle with area $\frac{1}{2}\pi x^2$ as illustrated.

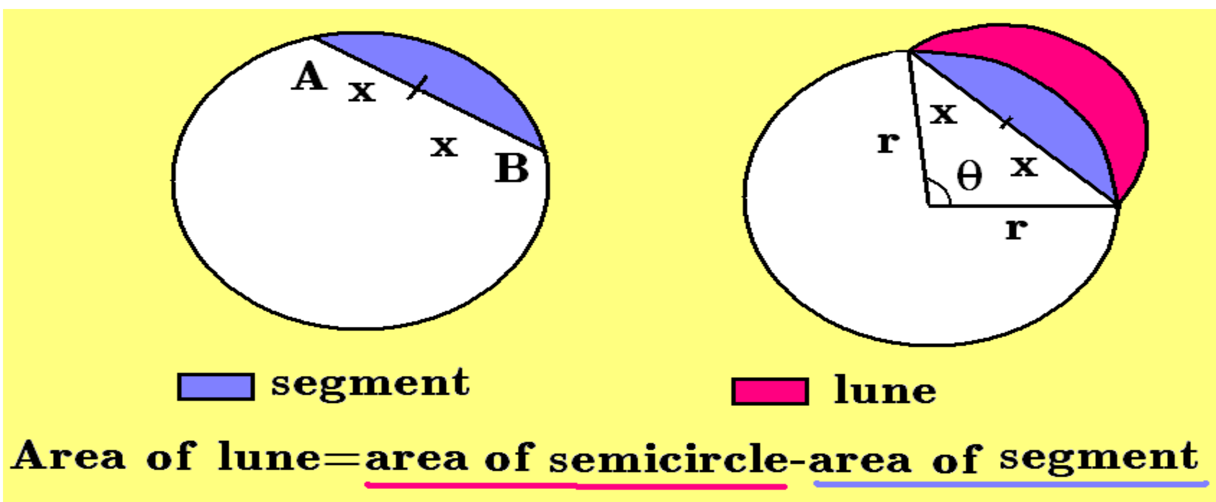


Figure 8-6. Construction of special lune.

An examination of the figure 8-6 one finds

$$\text{Area of lune} = \text{Area of semicircle} - \text{Area of segment}$$
$$A_{lune} = \frac{1}{2}\pi x^2 - \frac{1}{2}(\theta - \sin\theta)r^2$$

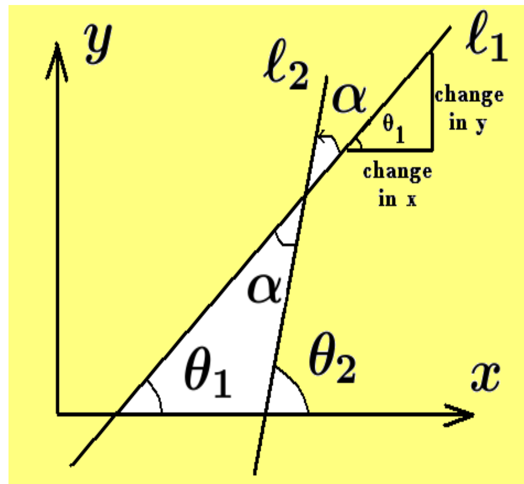
[1] If you are inside an area, then one says a boundary curve is convex if the curve extends outward. The boundary curve is called concave if it extends inward.

where r is the radius of the original circle, $2x$ is length of chord and diameter of constructed semi-circle and the area of the segment is obtained from the previous equation (8.41).

Slopes and angle between intersecting lines

Consider the intersection of two lines ℓ_1 and ℓ_2 as illustrated. Recall the definition of the slope of the line ℓ_1 which is

$$\text{Slope of line } \ell_1 \; = m_1 = \frac{\text{change in y}}{\text{change in x}}$$



This is also the definition associated with the tangent of the angle θ_1 which is the angle between the line ℓ_1 and the x-axis. Consequently, one can write

$$m_1 = \text{Slope of line } \ell_1 = \tan\theta_1 \tag{8.43}$$

In a similar fashion one can show the slope of the line ℓ_2 is given by $m_2 = \tan\theta_2$.

The angle α represents the angle formed **by rotating the line ℓ_1 in a counterclockwise direction about the point of intersection**, to the line ℓ_2. The angle of rotation α is defined as **the angle of intersection** associated with the intersection of the two lines ℓ_1 and ℓ_2. The angle of intersection α is related to the slope angles θ_1 and θ_2 by the relation

$$\theta_2 = \theta_1 + \alpha \tag{8.44}$$

because the exterior angle θ_2 is equal to the sum of the two opposite interior angles of a triangle. Therefore, $\alpha = \theta_2 - \theta_1$ and

$$\tan\alpha = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_1 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_2 - m_1}{1 + m_1 m_2} \tag{8.45}$$

One way to remember this result is to write it in the form

$$\tan\alpha = \frac{\overleftarrow{m_2 - m_1}}{1 + m_1 m_2} \tag{8.46}$$

where the left arrow is to remind you that line ℓ_1 with slope m_1 is being rotated in a counterclockwise direction about the point of intersection toward line ℓ_2 with slope m_2 to form the angle of intersection α.

Example 8-5. Three noncollinear points form a triangle. Consider the triangle formed by the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ as in the figure below, where the lines ℓ_1, ℓ_2 and ℓ_3 make up the sides of the triangle. Find the interior angles α, β and γ of the triangle.

Solution

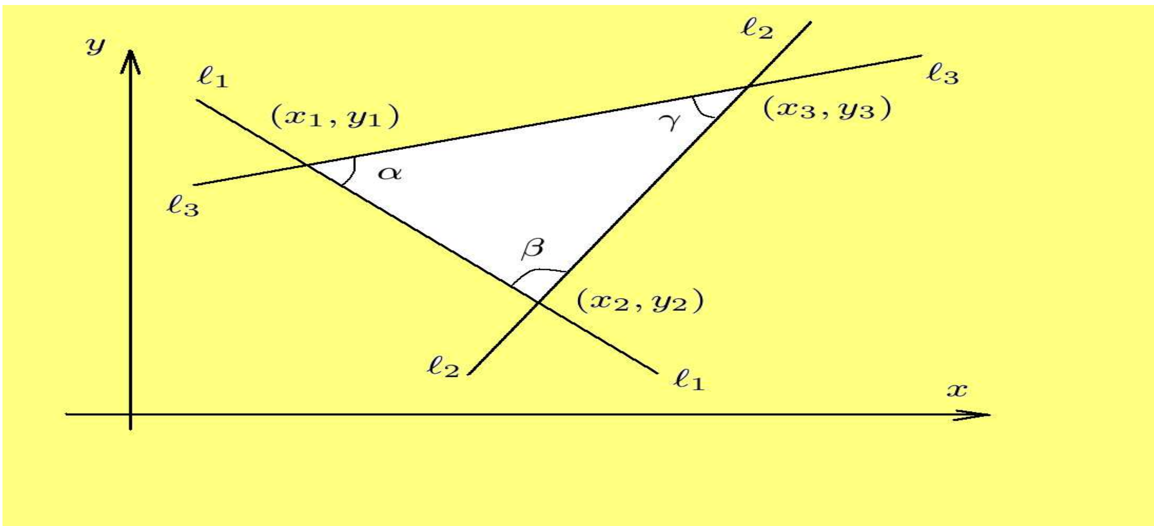


Figure 8-7. Triangle formed from three given points.

The equation of line ℓ_1 is $y - y_1 = m_1(x - x_1)$

The equation of line ℓ_2 is $y - y_2 = m_2(x - x_2)$

The equation of line ℓ_3 is $y - y_3 = m_3(x - x_3)$

where the slopes m_1, m_2, m_3 of the lines ℓ_1, ℓ_2, ℓ_3 are given by the equations

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} \qquad m_2 = \frac{y_3 - y_2}{x_3 - x_2} \qquad m_3 = \frac{y_3 - y_1}{x_3 - x_1} \tag{8.47}$$

which represent the change in y-values divided by the change in x-values while moving from one vertex to another of the triangle. The angles formed by the intersection of the lines defines the interior angles of the triangle. One can write

$$\tan\alpha = \frac{m_3 - m_1}{1 + m_1 m_2} \qquad \tan\beta = \frac{m_1 - m_2}{1 + m_1 m_2} \qquad \tan\gamma = \frac{m_2 - m_3}{1 + m_2 m_3} \tag{8.48}$$

and from these equations the angles α, β, γ can be determined.

For example, given the points

$$(x_1, y_1) = (1, 4) \qquad (x_2, y_2) = (4, 1) \qquad (x_3, y_3) = (4 + 3\frac{\sqrt{3}}{2}, \frac{11}{2})$$

one finds the slopes of the lines connecting these points are

$$m_1 = \frac{1-4}{4-1} = -1 \qquad m_2 = \frac{\frac{11}{2}-1}{4+3\frac{\sqrt{3}}{2}-4} = \frac{3}{\sqrt{3}} = \sqrt{3} \qquad m_3 = \frac{\frac{11}{2}-4}{4+3\frac{\sqrt{3}}{2}-1} = 2-\sqrt{3} \quad (8.49)$$

Using these slopes in the equations (8.48) one finds

$$\tan\alpha = \frac{2-\sqrt{3}-(-1)}{1+(-1)(2-\sqrt{3})} = \sqrt{3} \text{ so that } \alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3} \tag{8.51}$$

The angle γ is determined from the equation

$$\tan\gamma = \frac{\sqrt{3}-(2-\sqrt{3})}{1+\sqrt{3}(2-\sqrt{3})} = 1 \text{ so that } \gamma = \tan^{-1}(1) = \frac{\pi}{4} \tag{8.51}$$

We know the sum of the interior angles of a triangle must sum to π radians so that

$$\alpha+\beta+\gamma=\pi \quad \text{or} \quad \beta = \pi - \frac{\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12} \tag{8.52}$$

■

Normal form for equation of a line

The normal form for the equation of a line ℓ with slope m is determined by the perpendicular distance $p = \overline{0B}$ from the origin to the line. This distance is called the normal distance from the origin to the line. The line ℓ_N through the origin and normal to line ℓ is called the normal line. The normal line ℓ_N creates an angle α with the x-axis and has slope $\frac{-1}{m} = \tan\alpha$ and is illustrated in the figure 8-8.

The normal form for the equation of a line is constructed by selecting a variable point (x,y) on the line ℓ and then projecting the distances x and y onto the line ℓ_N. Note that

$$\begin{aligned} \overline{0A} &= x\cos\alpha = \text{projection of } x \text{ onto line } \ell_N \\ \overline{AB} &= y\cos(\frac{\pi}{2}-\alpha) = y\sin\alpha = \text{projection of } y \text{ onto line } \ell_N \end{aligned} \tag{8.53}$$

These projections are highlighted in red in the figure 8-8.

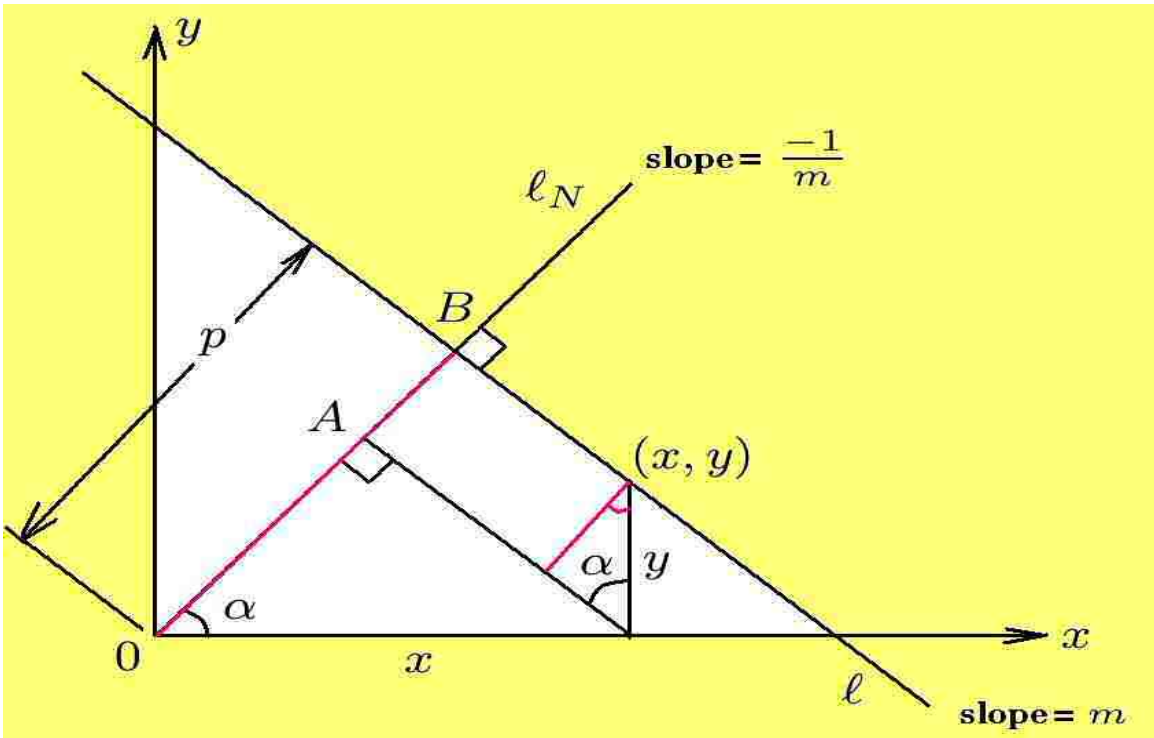


Figure 8-8. Normal line ℓ_N perpendicular to line ℓ

The sum of these projections gives

$$\begin{aligned} \overline{0A} + \overline{AB} &= \overline{0B} \\ \text{or} \qquad x\cos\alpha + y\sin\alpha &= p \end{aligned} \tag{8.54}$$

Here p is always positive unless the line ℓ passes through the origin. The equation (8.54) is known as the **normal form for the equation of a line** ℓ. The normal form for the equation of a line can be used to find the perpendicular distance d from a general point (x_1, y_1) to the line ℓ. The procedure for calculating this distance is illustrated in the following example.

Example 8-6.

Find perpendicular distance d from point (x_1, y_1) to a given line ℓ.

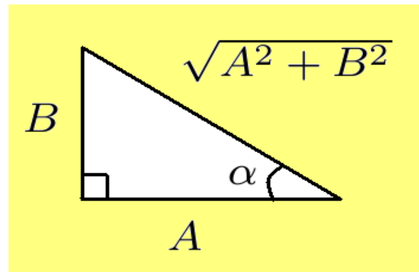
Solution

The general form for the equation of a line ℓ is

$$Ax + By + C = 0 \tag{8.55}$$

where A, B, C are constants.

Construct a right triangle with sides A and B and hypotenuse $\sqrt{A^2 + B^2}$ and label the angle between side A and the hypotenuse as angle α as illustrated in the figure.



Define the quantities

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2 + B^2}}, \qquad \sin\alpha = \frac{B}{\pm\sqrt{A^2 + B^2}}, \qquad p = \frac{C}{\mp\sqrt{A^2 + B^2}} \tag{8.56}$$

Note that if equation (8.55) is multiplied by $\frac{1}{\pm\sqrt{A^2+B^2}}$ one obtains

$$\frac{A}{\pm\sqrt{A^2 + B^2}}\,x + \frac{B}{\pm\sqrt{A^2 + B^2}}\,y + \frac{C}{\mp\sqrt{A^2 + B^2}} = 0 \tag{8.57}$$

or

$$x\,\cos\alpha + y\,\sin\alpha - p = 0 \tag{8.58}$$

where the sign $+$ or $-$ assigned to the square root function is selected so that p is a positive quantity. The sign selected will always be opposite of the sign of the constant term C in the original equation.

Select an arbitrary point (x_1, y_1) not on the line ℓ as in figure 8-9. In the figure

$$\begin{aligned} \overline{\text{0A}} =& x_1 \cos\alpha = \text{projection of } x_1 \text{ on line } \ell_N \\ \overline{\text{AQ}} =& y_1 \cos(\frac{\pi}{2} - \alpha) = y_1 \sin\alpha = \text{projection of } y_1 \text{ on line } \ell_N \end{aligned}$$

Examination of figure 8-9 shows that

$$\overline{\text{0A}} + \overline{\text{AQ}} = \overline{\text{0Q}} = p + d = x_1 \cos\alpha + y_1 \sin\alpha \tag{8.59}$$

This equation can be rewritten to obtain the perpendicular distance d of the point (x_1, y_1) from the line ℓ as

$$d = x_1 \cos\alpha + y_1 \sin\alpha - p \tag{8.60}$$

Using equation (8.57) this can be written as

$$d = \frac{Ax_1 + By_1}{\pm\sqrt{A^2+B^2}} + \frac{C}{\mp\sqrt{A^2+B^2}} \tag{8.61}$$

where the sign on the square root is selected opposite to that of the constant C.

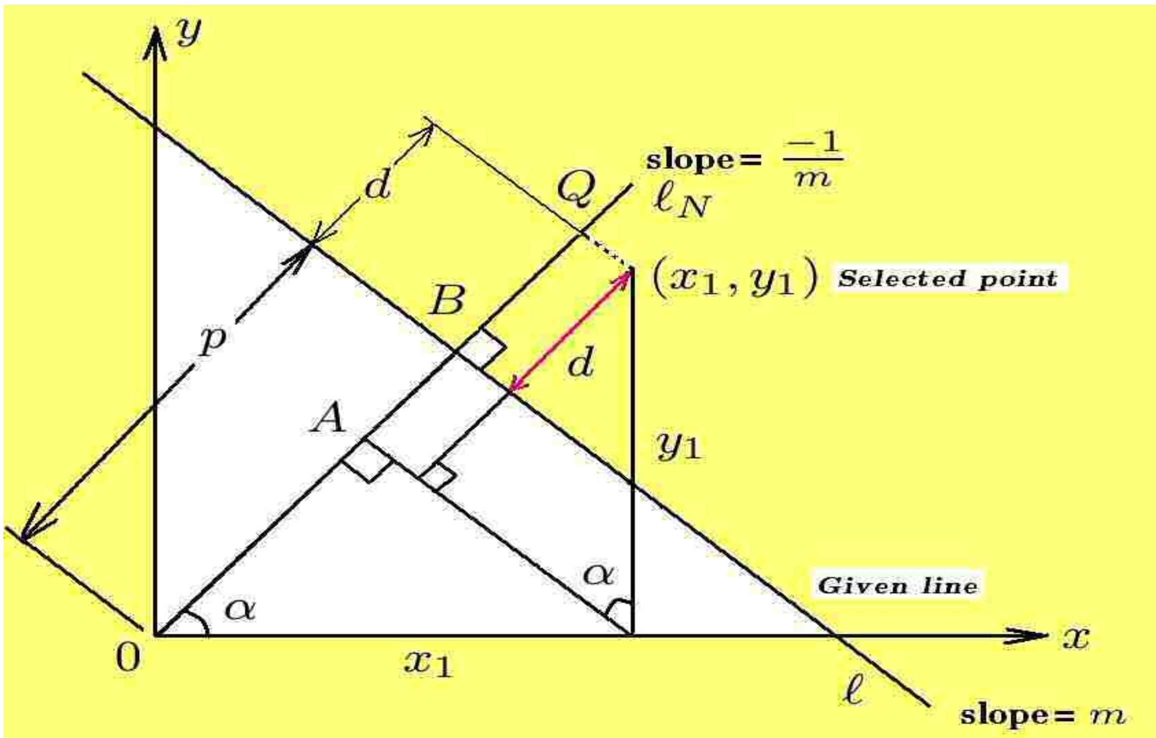


Figure 8-9. Normal line ℓ_N perpendicular to line ℓ with (x_1, y_1) not on line ℓ

The normal form for the equation of a line has the representation

$$x\cos\alpha + y\sin\alpha - p = 0 \tag{8.62}$$

and has the property
(a) If you substitute the values $x = x_1$ and $y = y_1$ into equation (8.62) and obtain zero, then the point (x_1, y_1) is on the line.
(b) If you substitute the values $x = x_1$ and $y = y_1$ in equation (8.62) and obtain a nonzero value d, then this value represents the normal distance from the point (x_1, y_1) to the line ℓ. The above result agrees with our previous result from chapter 4.

■

Example 8-7. Find the equation of the angle bisectors associate with the vertex angle A of triangle $\triangle ABC$ illustrated in the figure 8-10.

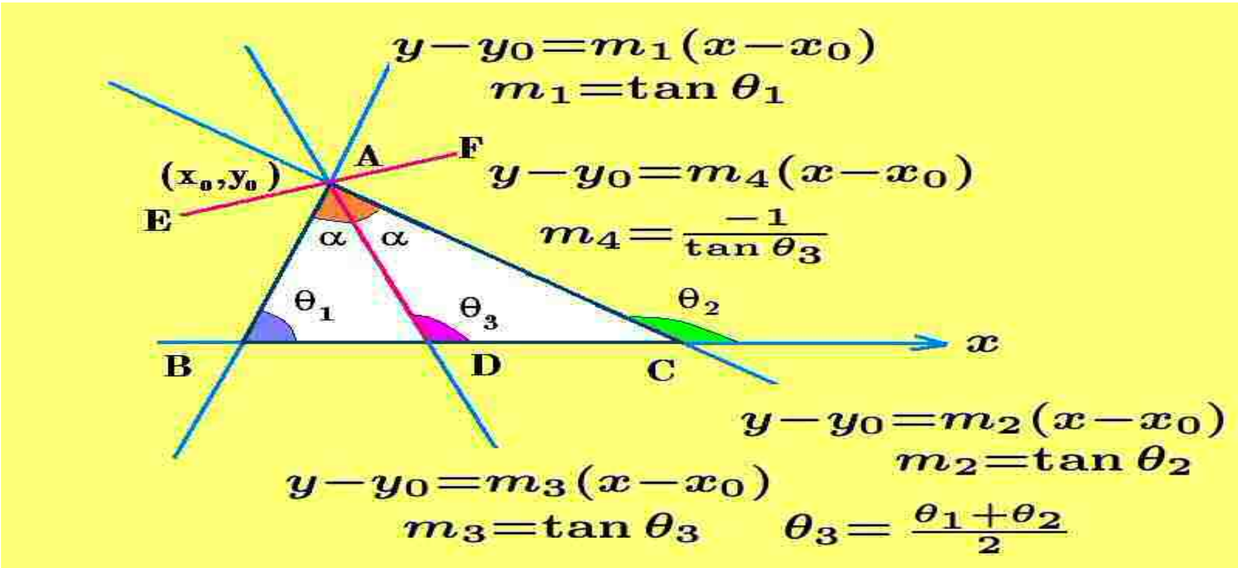


Figure 8-10.
Finding equations for angle bisectors.

The given triangle $\triangle ABC$ has the vertex A at the position (x_0, y_0) and the lines through the sides of the triangle which pass through the vertex A have the equations $y - y_0 = m_1(x - x_0)$ and $y - y_0 = m_2(x - x_0)$ where $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

The angle bisector of vertex angle A produces $m\angle A = 2\alpha$ and intersects the triangle base at an angle $\theta_3 = \theta_1 + \alpha$ since the exterior angle of the triangle $\triangle ABD$ must equal the sum of the two opposite interior angles. Using the fact that the sum of the interior angles of triangle $\triangle ABC$ must equal π radians one finds

$$2\alpha + \theta_1 + (\pi - \theta_2) = \pi \quad \Rightarrow \alpha = \frac{(\theta_2 - \theta_1)}{2} \quad \Rightarrow \theta_3 = \frac{\theta_1 + \theta_2}{2}$$

Hence the bisector of the vertex angle A has the equation $y - y_0 = m_3(x - x_0)$ where $m_3 = \tan\theta_3$. The angle $\pi - A$ is an exterior angle at vertex A with an angle bisector $\overline{\text{EF}}$ which is perpendicular to the interior angle A bisector and therefore has the equation $y - y_0 = m_4(x - x_0)$ where $m_4 = \frac{-1}{\tan\theta_3}$ because if the bisector lines are perpendicular the product of their slopes must equal -1.

An alternative form for $m_3 = \tan\theta_3 = \tan(\frac{\theta_1+\theta_2}{2})$ is obtained using some trigonometry. Observe that

$$m_3 = \tan\left(\frac{\theta_1+\theta_2}{2}\right) = \frac{1-\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)} = \frac{1-[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2]}{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2}$$

Divide both the numerator and denominator by $\cos\theta_1\cos\theta_2$ to show

$$m_3 = \frac{\frac{1}{\cos\theta_1\cos\theta_2} - 1 + \tan\theta_1\tan\theta_2}{\tan\theta_1+\tan\theta_2} = \frac{\sqrt{\sec^2\theta_1}\sqrt{\sec^2\theta_2} - 1 + \tan\theta_1\tan\theta_2}{\tan\theta_1+\tan\theta_2}$$

Using the Pythagorean identity $1+\tan^2\theta = \sec^2\theta$ and the substitutions $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$ one finds

$$m_3 = \frac{m_1m_2 - 1 + \sqrt{1+m_1^2}\sqrt{1+m_2^2}}{m_1+m_2}$$

This is a relation for the slope of the bisector line knowing the slopes m_1 and m_2 of the intersecting lines which create the angle being bisected. The reciprocal $\frac{-1}{m_3}$ gives the slope of the bisector line for the exterior angle.

■

Sum of interior angles of simple polygons

We know the sum of the interior angles of a triangle is 180° or π radians. Let us investigate the sum of the interior angles of polygons with 4,5,6,... sides. Let us begin by examining the interior angles of a quadrilateral as illustrated in the figure 8-11.

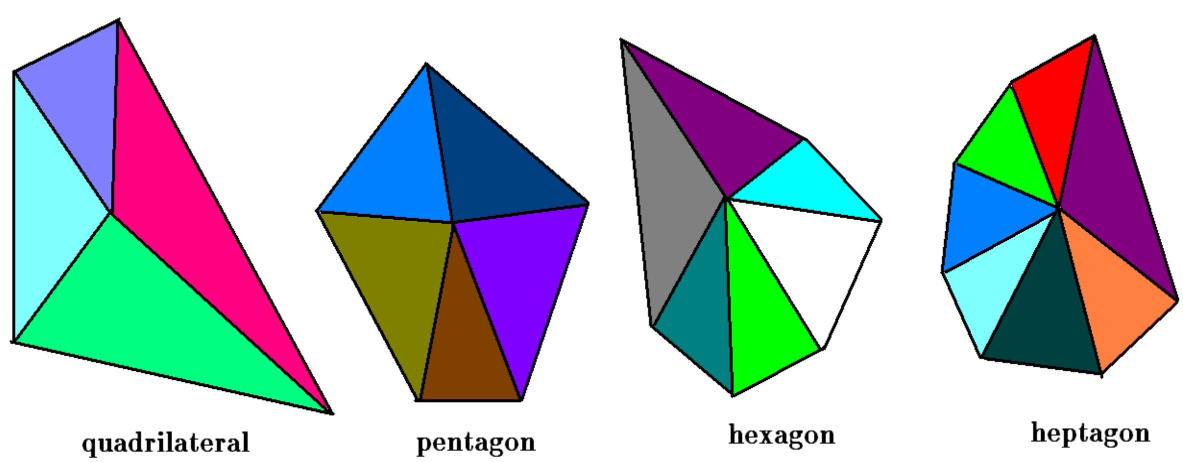


Figure 8-11.
Triangles inside selected convex polygons.

Select a point inside the quadrilateral and then draw lines from the selected point to each vertex of the quadrilateral. This creates four triangles inside the quadrilateral. The interior angles of each triangle is 180° or π radians. The sum of the angles from each triangle is $4(180^\circ)$ or 4π radians. The central angles around the point selected inside the triangle sum to $360^\circ = 2(180^\circ)$ or 2π radians. Therefore, the sum of the interior angles of a quadrilateral is

$$\text{Sum of angles of triangles} - \text{Sum of angles around selected central point} \tag{8.63}$$

This gives

$$4(180^\circ) - 2(180^\circ) = 2(180^\circ) \quad \text{or} \quad 4\pi - 2\pi = 2\pi \text{ radians}$$

as the sum of the interior angles of a quadrilateral.

In a similar fashion select a point inside a pentagon and construct lines from this point to each of the five vertices. This creates five triangles. The sum of the interior angles of a pentagon can be determined by the use of equation (8.63) which produces the relation

$$5(180^\circ) - 2(180^\circ) = 3(180)^\circ \quad \text{or} \quad 5\pi - 2\pi = 3\pi \text{ radians}$$

as representing the sum of the interior angles of a pentagon.

Note that a pattern is developing. By selecting an interior point inside a hexagon, then one can create six triangles so that the equation (8.63) can be used to determine the sum of the interior angles of a hexagon. This gives

$$6(180^\circ) - 2(180^\circ) = 4(180^\circ) \quad \text{or} \quad 6\pi - 2\pi = 4\pi \text{ radians} \tag{8.64}$$

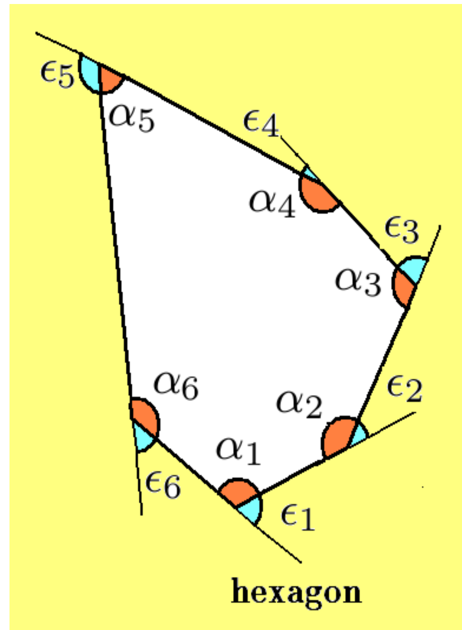
This pattern can be extended to determine the sum of the interior angles of an **n-gon** as

$$n(180^\circ) - 2(180^\circ) = (n-2)(180^\circ) \quad \text{or} \quad n\pi - 2\pi = (n-2)\pi \text{ radians}$$

In summary, one can state that **the summation of the interior angles of an n-gon will be $(n-2)\pi$ radians.**

Sum of exterior angles of simple polygons

One can set up a pattern for calculating **the summation of the exterior angles of a n-gon.** Examine the hexagon illustrated with the exterior angles ϵ_i, $i = 1,2,3,4,5,6$ and interior angles α_i, $i = 1,2,3,4,5,6$. Note the summation of an exterior angle with its corresponding interior angle will always produce the result π because these angles are supplementary. Consequently one can write



$$
\begin{aligned}
\alpha_1 + \epsilon_1 =& \pi \\
\alpha_2 + \epsilon_2 =& \pi \\
\alpha_3 + \epsilon_3 =& \pi \\
\alpha_4 + \epsilon_4 =& \pi \\
\alpha_5 + \epsilon_5 =& \pi \\
\alpha_6 + \epsilon_6 =& \pi
\end{aligned}
\tag{8.65}
$$

and a summation of the terms on the left-hand and right-hand sides of equation (8.65) produces the result

$$
\sum_{i=1}^{6} \alpha_i + \sum_{i=1}^{6} \epsilon_i = 6\pi \tag{8.66}
$$

We know $\sum_{i=1}^{6} \alpha_i$ represents the summation of the interior angles of the hexagon and this result is $(6-2)\pi$ from equation (8.64). Therefore, equation (8.66) becomes

$$
\sum_{i=1}^{6} \epsilon_i = 6\pi - (6-2)\pi = 2\pi \tag{8.67}
$$

The equation (8.67) can be generalized in the case of an n-gon instead of a hexagon. Just replace 6 in equation (8.67) by n to obtain

$$
\sum_{i=1}^{n} \epsilon_i = n\pi - (n-2)\pi = 2\pi \tag{8.68}
$$

which tells us that **the summation of the exterior angles of an n-gon must equal 2π.**

In summary, we have discovered that **the summation of the exterior angles of an n-gon will always equal 2π.**

Area of regular polygons

Consider the regular polygons illustrated in the figure 8-12. These polygons have equal sides and equal interior angles. The symmetry associated with regular polygons allows one to take the summation of the interior angles of an n-gon $((n-2)\pi)$ and then divide by n to determine the angular value at a single vertex. The **center point** of a regular polygon is the point inside the boundaries which is equidistant from each vertex. The **apothem** of a regular polygon is the distant from the center point to the midpoint of a side. The center point and apothem for selected regular polygons are illustrated in the figure 8-12. Note that the distance from the center of a regular polygon to a vertex gives the radius (r) of a circumscribed circle and the distance from the center to the midpoint of a side gives the apothem (a) which represents the radius of the inscribed circle. Each regular polygon has an isosceles triangle consisting of a side $s_i,\ i=4,5,5,7,\ldots$ and two radii $r_i,\ i=4,5,6,7,\ldots$. Each regular polygon has an apothem $a_i,\ i=4,5,6,7\ldots$ which represents the height of the single triangles illustrated in the figure 8-12. Let $A_i,\ i=4,5,6,7,\ldots$ denote the area of a single triangle associated with a regular polygon of i-sides and let A_{ti} denote the total area associated with a regular polygon of i-sides. Also let P_n denote the perimeter of a regular polygon having n equals sides. Using a summation of triangle areas one can verify

	Area of single triangle	Total area of polygon
square $n=4$	$A_4=\frac{1}{2}a_4s_4$	$A_{t4}=4A_4=4\frac{1}{2}\left(\frac{s_4}{2}\right)s_4=s_4^2$
pentagon $n=5$	$A_5=\frac{1}{2}a_5s_5$	$A_{t5}=5A_5=\frac{1}{2}a_5(5s_5)=\frac{1}{2}a_5P_5$
hexagon $n=6$	$A_6=\frac{1}{2}a_6s_6$	$A_{t6}=6A_6=\frac{1}{2}a_6(6s_6)=\frac{1}{2}a_6P_6$
heptagon $n=7$	$A_7=\frac{1}{2}a_7s_7$	$A_{t7}=7A_7=\frac{1}{2}a_7(7s_7)=\frac{1}{2}a_7P_7$
	$\vdots$	$\vdots$
n-gon	$A_n=\frac{1}{2}a_ns_n$	$A_{tn}=nA_n=\frac{1}{2}a_n(ns_n)=\frac{1}{2}a_nP_n$

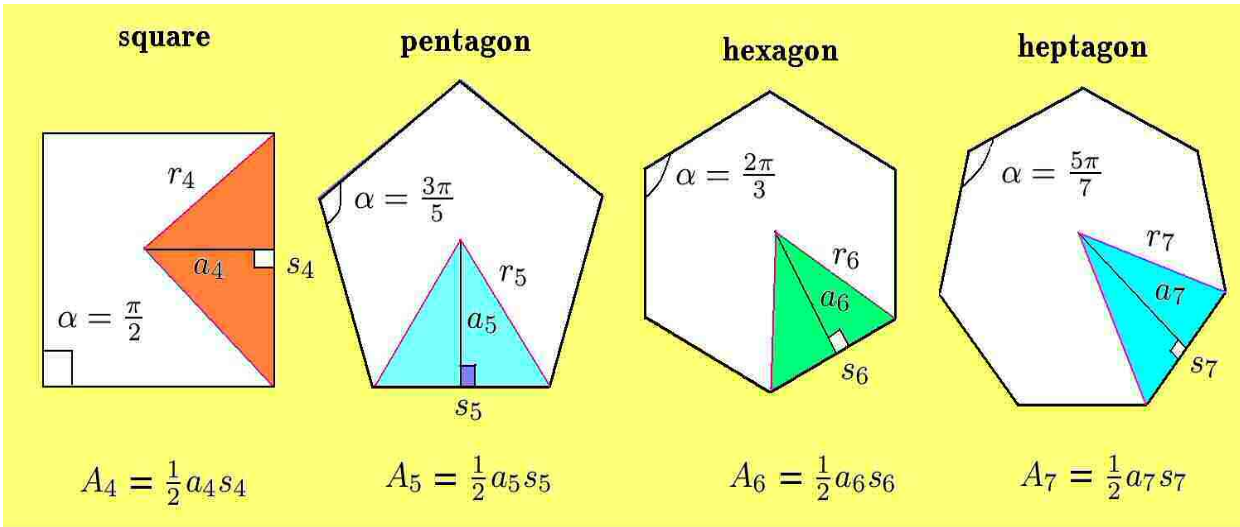


Figure 8-12.
Selected regular polygons

Observe that in the special case of a square the apothem is given by $a_4 = \frac{s_4}{2}$ so that the total area for the square is the side squared. Generalizing the area results for the other polygons one finds

$$\text{Area of regular polygon} = \frac{1}{2}(\text{apothem})(\text{perimeter})$$

which states the general formula for the area of a regular polygon is half the product of the apothem times the perimeter of the regular polygon.

The n-gon

For a regular polygon of n-sides, called a n-gon, let a denote the apothem, which is the perpendicular distance from the center of the regular n-gon to any side. This is also the radius of the inscribed circle. Let R denote the radius of the n-gon, which is the distance from the center of the n-gon to any vertex. The distance R also represents the radius of the circumscribed circle. Let $\alpha = \frac{2\pi}{n}$ denote the central angle associated with the n-gon and let s_n denote the length of any side of the n-gon. The above defined quantities are illustrated in the figure 8-13.

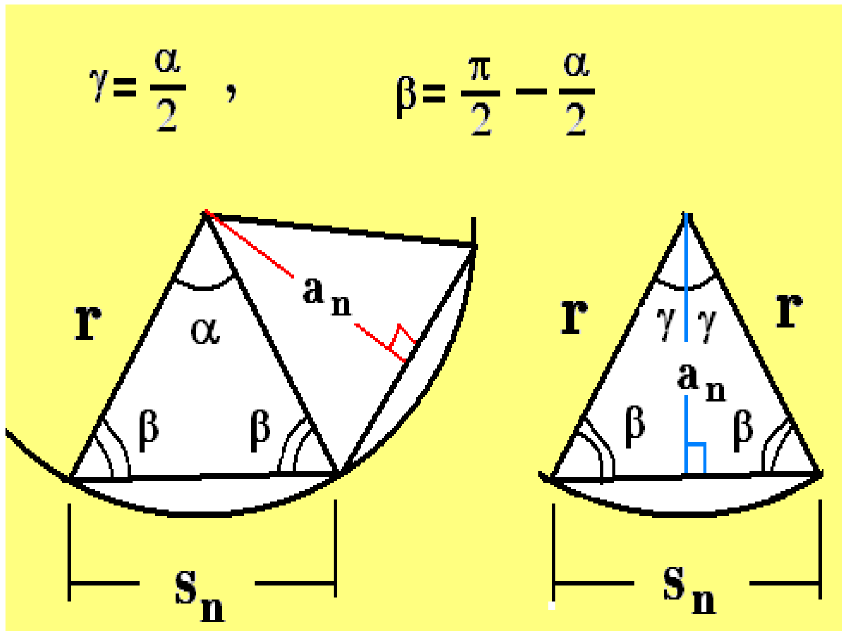


Figure 8-13.

Section of an n-gon

Dropping a perpendicular from the center of the n-gon to a side of length forms a right triangle from which one can obtain the relations

$$\cos\frac{\alpha}{2} = \frac{a}{R}, \qquad \sin\frac{\alpha}{2} = \frac{s_n/2}{R}, \qquad \tan\frac{\alpha}{2} = \frac{s_n/2}{a} \tag{8.69}$$

Using these relations the total area of a regular n-gon can be expressed in any of the following forms

$$\text{Area} = n\left[\frac{1}{2}as_n\right] = n\left[\frac{1}{2}s_n^2\cot\frac{\alpha}{2}\right] = n\left[a^2\tan\frac{\alpha}{2}\right] = n\left[\frac{1}{2}R^2\sin\alpha\right] \tag{8.70}$$

The first area formula is just the height of the triangle section, the apothem (a) times the base (s_n). Into this first area formula substitute $a = \frac{s_n/2}{\tan\frac{\alpha}{2}}$ to obtain the second area formula. Substitute into the second area formula for equation (8.70) $s_n = 2a\tan\frac{\alpha}{2}$ to obtain the third area formula. By substituting $a = R\cos\frac{\alpha}{2}$ into the third area formula one obtains the fourth area formula. All the above substitutions use to change the forms for the area come from the equations (8.69).

In a similar fashion one can construct various ways to represent the perimeter P of the n-gon. One can express the perimeter P in any of the forms

$$\text{Perimeter } P = ns_n = n\left[2R\sin\frac{\alpha}{2}\right] = n\left[2a\tan\frac{\alpha}{2}\right] \tag{8.71}$$

Note that the various forms are all constructed from the identities of equation (8.69).

Area of irregular polygons

The figure 8-14 illustrates selected irregular shaped polygons. The problem is to find the area and perimeter associated with these plane figures. It will be assumed that the Cartesian coordinates of the polygon vertices are known.

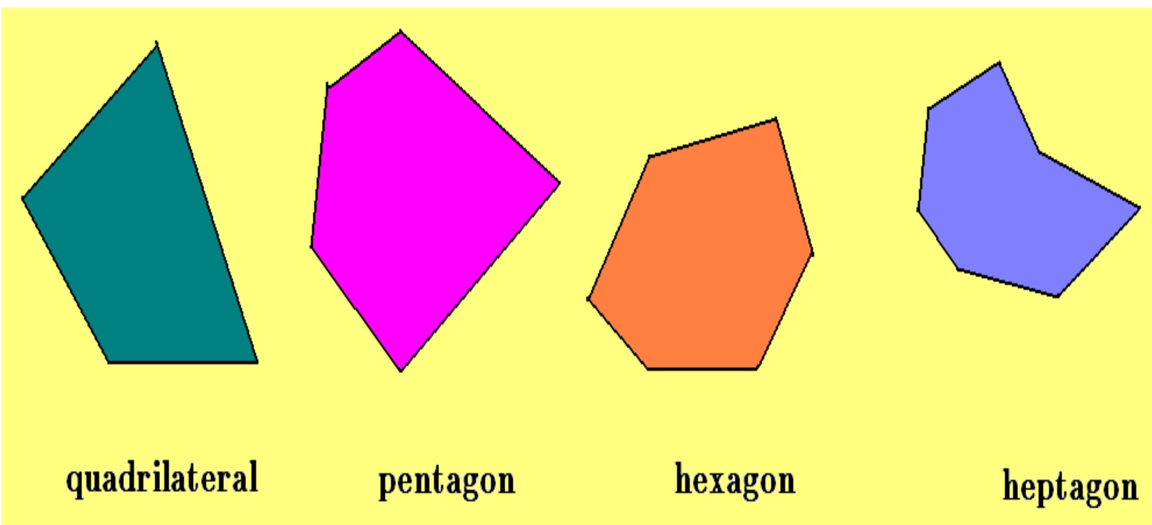


Figure 8-14. Irregular shaped polygons

To find the area of irregular shaped polygons such as the ones illustrated in the figure 8-14 one can proceed using one of the following methods.

Construct triangles

Start at a vertex of the given irregular polygon and construct triangles by drawing lines to the other vertices as illustrated in the figure 8-15(a). The problem is then reduced to finding the area of each triangle. The total area of the polygon is then a summation of these areas.

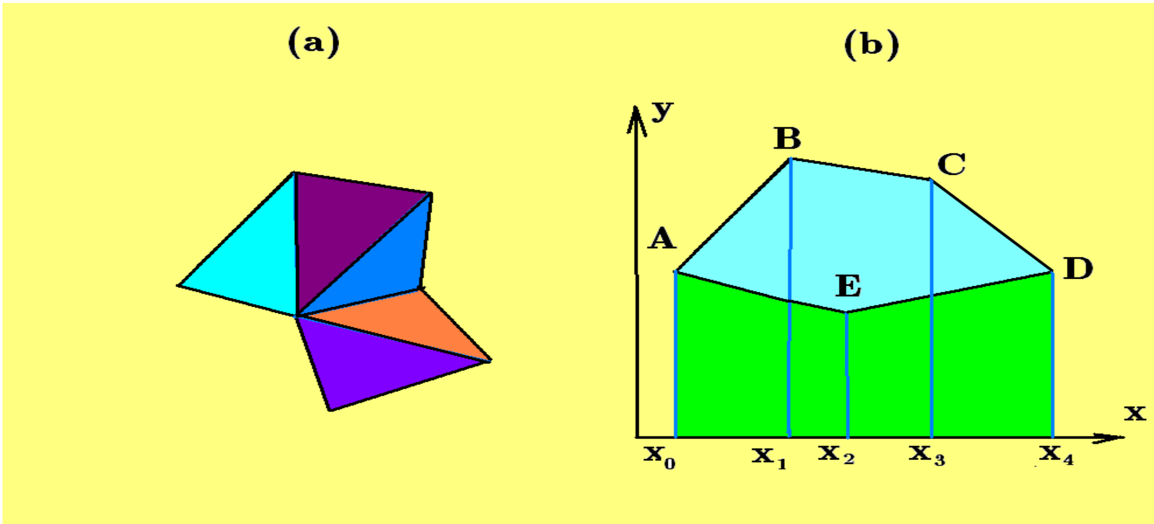
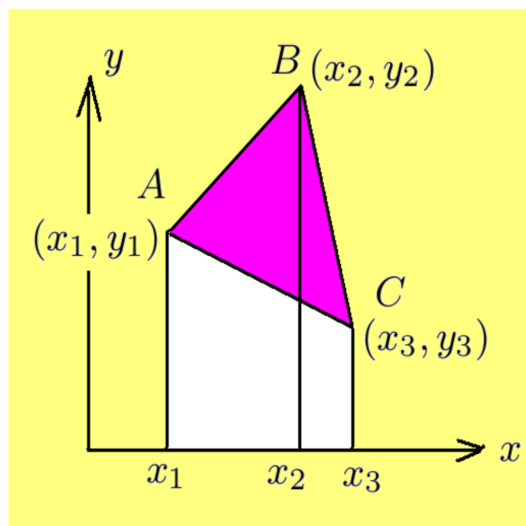


Figure 8-15. Area associated with irregular shaped polygons

To find the area of a single triangle with known coordinates, one can drop perpendicular lines from each vertex to the x-axis as illustrated in the accompanying figure. The area of the triangle $\triangle ABC$ is then given by



$$\begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{AB}x_2x_1 \end{bmatrix} + \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{BC}x_3x_2 \end{bmatrix} - \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{AC}x_3x_1 \end{bmatrix}$$

The area of the trapezoids is the average height times the base so that one can write

$$\text{Area}\,\triangle ABC \;=\frac{1}{2}(y_2+y_1)(x_2-x_1)+\frac{1}{2}(y_2+y_3)(x_3-x_2)-\frac{1}{2}(y_1+y_3)(x_3-x_1)$$

Construct trapezoids

To find the area of the irregular polygon in figure 8-15(b) one can drop perpendicular lines from each vertex to the x-axis as illustrated. The area of the polygon is then obtained as an addition followed by a subtraction of areas associated with trapezoids. For example, in the figure 8-15(b) the area of the polygon ABCDE is calculated using

$$\begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{AB}x_1x_0 \end{bmatrix} + \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{BC}x_3x_1 \end{bmatrix} + \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{CD}x_4x_3 \end{bmatrix} - \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{AE}x_2x_0 \end{bmatrix} - \begin{bmatrix} \text{Area of} \\ \text{trapezoid} \\ \text{ED}x_4x_3 \end{bmatrix}$$

Exercises

▶ **8-1.**

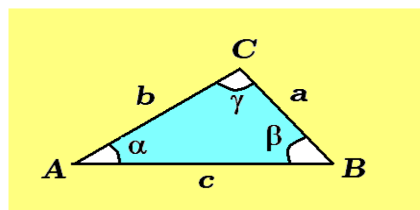
Given the triangle $\triangle ABC$

(a) If $\alpha = 30^\circ,\ a = 5,\ c = 7$, find angle γ.

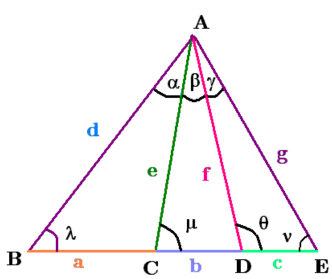
(b) If $\beta = 45^\circ,\ b = 7,\ a = 4$, find angle α.

(c) If $\gamma = 120^\circ,\ c = 12,\ \alpha = 30^\circ$, find side a.

(d) If $\alpha = 35^\circ,\ \gamma = 100^\circ,\ a = 6$, find side c.



▶ **8-2.**



Write out the law of sines for

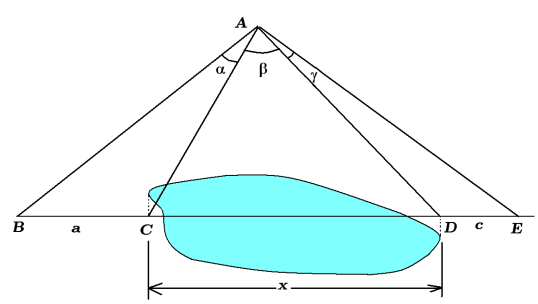
(a) $\triangle ADE$ (b) $\triangle ACD$ (c) $\triangle ACE$

(d) $\triangle ABC$ (e) $\triangle ABD$

(f) Show $\dfrac{\sin\gamma}{c}\dfrac{\sin\alpha}{a} = \dfrac{\sin(\alpha+\beta)}{a+b}\dfrac{\sin(\beta+\gamma)}{b+c}$

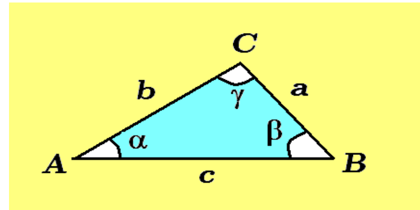
▶ **8-3.**

The previous problem developed an equation that can be used by a surveyor to find the length x of a lake if the angles α, β, γ and distances a, c are known. Show how one can solve for the distance x.



► **8-4.**
For triangle $\triangle ABC$ the law of sines states

$$\frac{b}{a} = \frac{\sin\beta}{\sin\alpha}$$



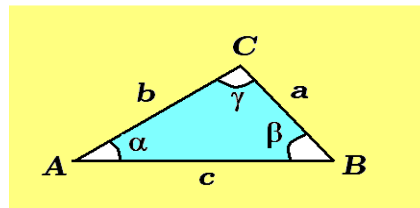
(a) Use the theory of proportions to show

$$\frac{a-b}{a+b} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$$

(b) Use the sum and difference formulas to show $\dfrac{a-b}{a+b} = \dfrac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$

► **8-5.**
The law of cosines for triangle $\triangle ABC$ is

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$



(a) Show that $\cos\alpha = \dfrac{b^2+c^2-a^2}{2bc}$

(b) Show that $1-\cos\alpha = \dfrac{(a-(b-c))(a+(b-c))}{2bc}$

(c) Let $s = \frac{1}{2}(a+b+c)$ denote the semiperimeter of the triangle $\triangle ABC$ and show

(i) $a-b+c = 2(s-b)$

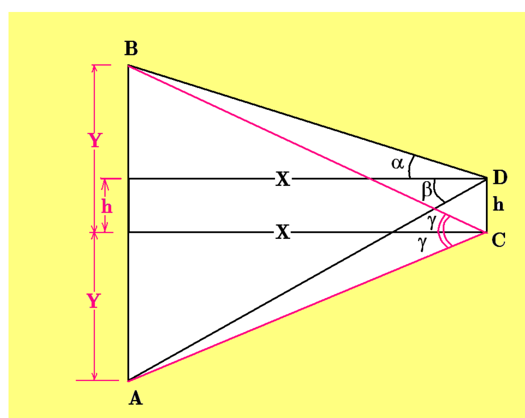
(ii) $a+b-c = 2(s-c)$

(iii) Show part (b) above can be written $\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

(iv) Show that by using the cyclic rotation $a->b->c->a$, one finds

$$\sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}, \qquad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

► **8-6.**
In the triangle $\triangle ABD$ only $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{h}$ are known. In the triangle $\triangle ABC$ only $\boldsymbol{\gamma}$ and $\boldsymbol{h}$ are known. Express the following in terms of **known quantities**.



(a) Find the distance Y

(b) Find the area of triangle $\triangle ABD$

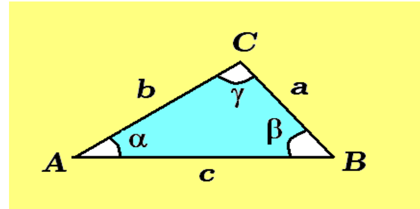
(c) Find the area of triangle $\triangle ABC$

(d) Find the distance X

A good surveyor knows how to solve this problem.

▶ **8-7.**

Explain how to find the area of the given triangle if you are given



(a) the angles α, β and side c. For example, $\alpha = 25°, \beta = 30°,\ c = 10$

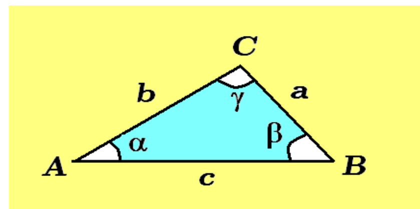
(b) the three sides a, b, c. For example, $a = 10, b = 5, c = 8$

(c) the sides a and c along with angle β. For example, $a = 10, c = 12, \beta = 30°$

(d) the angles α, γ and side b. For example, $\alpha = 25°, \gamma = 68°, b = 12$

▶ **8-8.**

For triangle $\triangle ABC$ the law of cosines is



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

(a) Show that $\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

(b) Show that $1 + \cos\alpha = \dfrac{(b + c - a)(b + c + a)}{2bc}$

(c) Let $s = \dfrac{1}{2}(a + b + c)$ denote the semiperimeter of triangle $\triangle ABC$

(i) Show $(b + c - a) = 2(s - a)$

(ii) Show $(b + c + a) = 2s$

(iii) Show $\cos\dfrac{\alpha}{2} = \sqrt{\dfrac{s(s-b)}{bc}}$

(iv) Show using a cyclic rotation of the symbols

$$\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

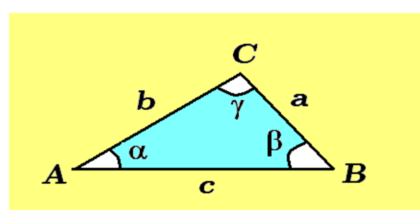
$$\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

▶ **8-9.** Use the results from the two previous problems to show

$$\tan\frac{\alpha}{2} = \sqrt{(s-b)(s-c)}s(s-a), \qquad \tan\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \qquad \tan\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Check that these equations follow the pattern associated with a cyclic rotation of the symbols.

▶ **8-10.** In the triangle $\triangle ABC$

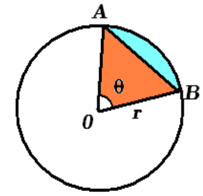


(a) If $b = 10,\ \alpha = 58.3°,\ \beta = 48.7°$, then find side a.

(b) If $a = 10,\ \alpha = 78.2°,\ \gamma = 31.5°$, then find side c.

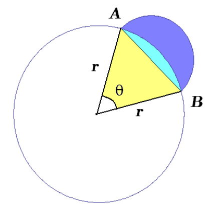
▶ **8-11.** Assume $\frac{\pi}{2} \le \theta \le \pi$. Let ϕ denote the supplementary angle associated with θ. Verify the following. $\sin\theta = \sin\phi,\ \cos\theta = -\cos\phi,\ \tan\theta = -\tan\phi,\ \cot\theta = -\cot\phi$

► **8-12.**
For each of the circles given, find (i) the chord length, (ii) the area of the segment, (iii) area of the sector and (iv) the minor arc length.



(a) $\theta = \frac{\pi}{3},\ r = 2,$ (b) $\theta = \frac{\pi}{4},\ r = 4,$ (c) $\theta = \frac{\pi}{6},\ r = 6$

► **8-13.**
Find the area of the lune illustrated when the following conditions exist.



(a) $r = 2,\ \theta = \frac{\pi}{3},$ (b) $r = 1,\ \theta = \frac{\pi}{6},$ (c) $r = 3,\ \theta = \frac{5\pi}{12}$

► **8-14.** Find the area of the given polygons.
(a) Pentagon with vertices $(0,0),(1,3),(3,4),(5,-1),(2,-1)$
(b) Hexagon with vertices $(1,1),(2,3),(0,4),(-2,3),(-3,2),(-1,1)$
(c) Heptagon with vertices $(1,5),(2,7),(4,8),(6,7),(5,4),(4,2),(3,1)$

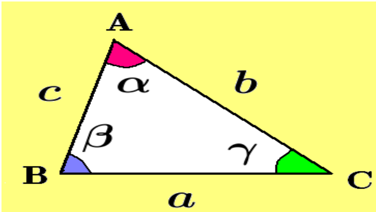
► **8-15.** Find the angle at point of intersection for the given lines.
(a) $\ell_1:\ y = 3x - 7$ $\ell_2:\ y = m_2x + 16,$ $m_2 = -2 - 5/\sqrt{3}$
(b) $\ell_1:\ y = 3x - 7$ $\ell_2:\ y = m_3x + 16,$ $m_3 = -2$
(c) $\ell_1:\ y = 3x - 7$ $\ell_2:\ y = m_4x + 16,$ $m_4 = \dfrac{1}{13}(-6 - 5\sqrt{3})$

► **8-16.** Find the internal angles of the triangle having the vertices at the given points.
(a) $(1,1),\ (3,5),\ (6,-1),$ (b) $(-1,-1),\ (1,5),\ (4,3),$ (c) $(2,-4),\ (7,1),\ (3,5)$

► **8-17.** Find the normal form for the equation of the given line.
(a) $\dfrac{x}{3} + \dfrac{y}{4} = 1$ (b) $y = 3x - 7$ $6x + 8y = 10$

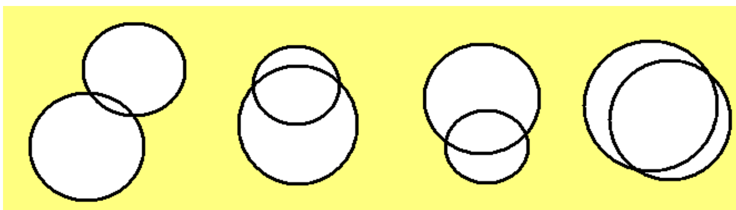
► **8-18.** Find the perpendicular distance from the point $(100,100)$ to the given line.
(a) $y = 3x,$ (b) $3x + 4y = 10,$ (c) $5x + 12y = 13$

▶ **8-19.**



(a) Given $\alpha = 31^\circ$, $\beta = 40^\circ$, $b = 4$, solve for side a.
(b) Given $\beta = 41^\circ$, $\gamma = 25^\circ$, $b = 4$, $c = 10$, solve for side b.
(c) Given $\alpha = 32^\circ$, $\gamma = 24^\circ$, $c = 10$ solve for side a.

▶ **8-20.** Each sketch has two lunes associated with it. Shade these lunes.

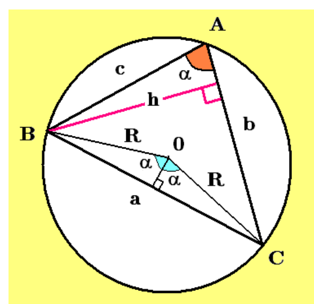


▶ **8-21.** Use trapezoids to find the area of the triangles having the vertices given.

(a) $(1,1),\ (4,5),(6,3)$
(b) $(-2,2),\ (0,8),\ (3,5)$
(c) $(5,5),\ (8,10),\ (6,1)$

▶ **8-22.** Given triangle $\triangle ABC$ inside circle.

(a) Show $h = c \sin \alpha$
(b) Show $\sin \alpha = \frac{a}{2R}$
(c) Show $[ABC] = \frac{1}{2}bh = \frac{abc}{4R}$

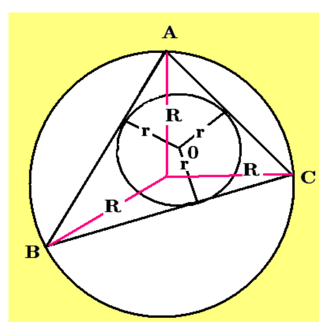


▶ **8-23.**

Given a triangle $\triangle ABC$ with r the radius of the inscribed circle, R the radius of the circumscribed circle and a, b, c representing the sides of the triangle. Show that

$$rR = \frac{abc}{2(a+b+c)}$$

Hint: Show $[ABC] = [A0B] + [B0C] + [C0A]$



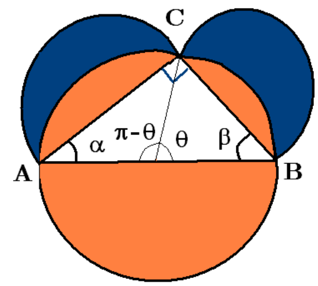
▶ **8-24.**

The triangle $\triangle ABC$ is a right triangle with $\overline{\mathrm{AB}}$ the diameter of a circle with radius r. There are two lunes each associated with a leg of the right triangle.

(a) Find the area of each lune.

(b) Find the area of the triangle $\triangle ABC$

(c) Show the sum of lune areas equals the triangle area.

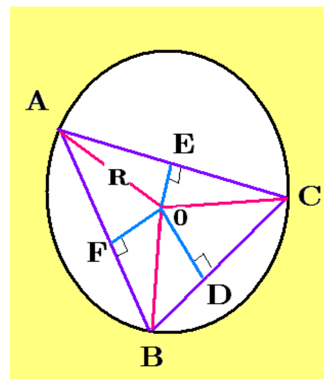


▶ **8-25.**

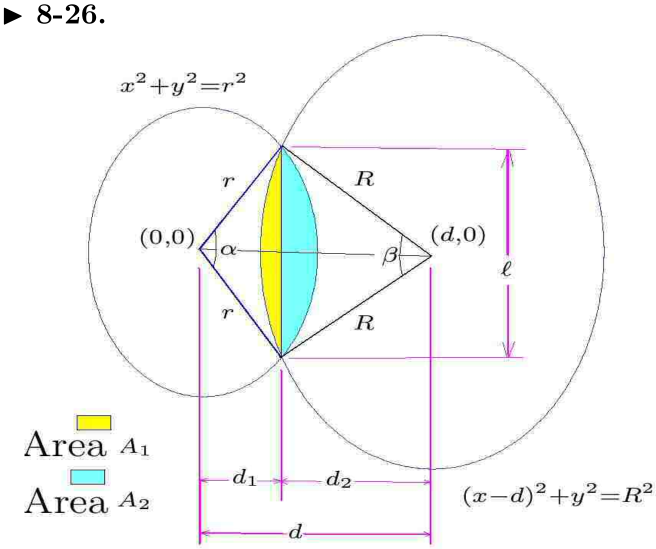
Given triangle $\triangle ABC$ with sides a, b, c circumscribed by circle with center 0. Construct the radii $\overline{0\mathrm{A}}, \overline{0\mathrm{B}}, \overline{0\mathrm{C}}$ and perpendicular lines $\overline{0\mathrm{D}}, \overline{0\mathrm{E}}, \overline{0\mathrm{F}}$.

(a) Show $\angle B0D = \angle A, \quad \angle C0E = \angle B, \quad \angle A0F = \angle C$

(b) Show $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$



▶ **8-26.**



Given a circle centered at the origin $(0, 0)$ having radius r which intersects a circle centered at $(d, 0)$ having a radius $R > r$ as sketched above. Construct the chord having length ℓ and construct line joining the center of the circles having length $d_1 + d_2 = d$. Define the angles α and β as illustrated.

(a) Show $r \sin \alpha/2 = R \sin \beta/2$

(b) Show length of chord is $\ell = 2r \sin \alpha/2 = 2R \sin \beta/2$

(c) Show area of segment $A_2 = \frac{r^2}{2}(\alpha - \sin \alpha)$

(d) Show area of segment $A_1 = \frac{R^2}{2}(\beta - \sin \beta)$

(e) Find the area common to both circles.

(f) Examine the special case $r = R$ with $d = r$.

Geometry

Chapter 9

More properties of geometric shapes

The circle

Let us review some of our knowledge of the circle and also introduce some new terminology. A circle is the set of points equidistant from a fixed point called the **center of the circle.** The **circumference of the circle** is the distance around the circle. The **radius** (plural radii) of the circle is the shortest distance from the center of the circle to a point on the circumference.

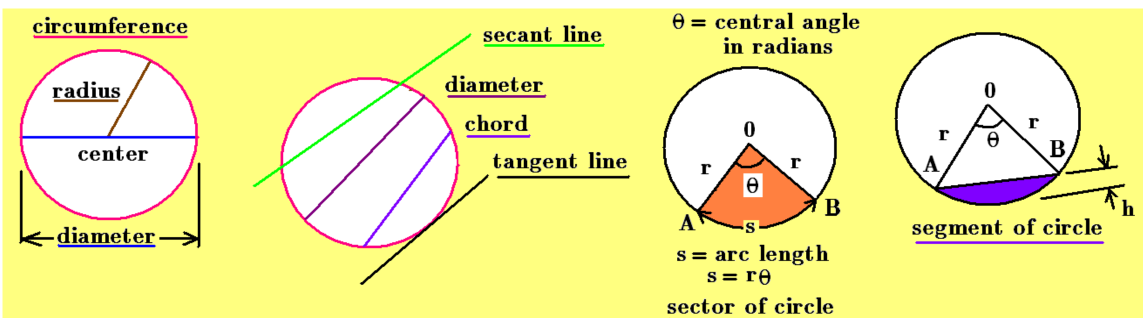


Figure 9-1. Terminology for the circle

We have previously shown that

The **diameter** (d) of the circle is twice the radius (r) or $d = 2r$

The **circumference** (c) of a circle is 2π times the radius (r) or $c = 2\pi r$

The **area** (A) of a circle in π times the radius squared or $A = \pi r^2$

Any line joining two distinct points on the circumference of the circle is called a **chord**. Any line which intersects the circle in two points is called a **secant line.** A **tangent line** to a circle, in the plane of the circle, is any line which intersects with the circle in exactly one point. This one point is called **the point of tangency.**

Any angle θ with vertex at the center of the circle is called a **central angle.**[1] The central angle defines an **arc length**[2] $s = \overset{\frown}{AB}$ on the circumference of the circle,

[1] All angles will be expressed in radians. The conversion factor $1^\circ = 2\pi/360$ radians will convert degrees to radians.

[2] There are two measures for the arc length on a circle (i) the central angle θ in degrees associated with the subtended arc and (ii) the arc length $s = r\theta$ with θ in radians. Remember that the central angle divides the circumference of the circle into a minor arc and a major arc.

with distance $s = r\theta$ (θ in radians), determined by the central angle θ and the radial distance of the circle. The area bounded by two radii making up the sides of the central angle together with the arc $\overset{\frown}{\text{AB}}$ subtended by the central angle defines **a sector of the circle**, where we have previously shown the area of the sector is given by

$$A_{sector} = \frac{1}{2}r^2\theta$$

where the central angle θ is in radians.

The two sides of the central angle intersect the circumference of the circle at two points A and B. The chord joining the points A and B and the arc on the circumference $\overset{\frown}{\text{AB}}$ define an area called a **segment of the circle.** We have shown that the area of a segment is given by

$$\begin{aligned}\text{Area of segment} = A_{segment} =& \text{Area of sector} - \text{Area of triangle } \triangle A0B\\ A_{segment} =& \frac{1}{2}r^2(\theta - \sin\theta)\\ \text{sagitta} = h =& r\left(1 - \cos\frac{\theta}{2}\right)\\ \text{chord length } \overline{AB} =& 2r\sin\frac{\theta}{2}\end{aligned}$$

where the central angle θ is in radians.

The geometric representation associated with the above terminology is illustrated in the figure 9-1.

Equation of circle Cartesian coordinates

In Cartesian coordinates let (x, y) denote a variable point which can move around the x, y-plane and let (x_0, y_0) denote a fixed point as illustrated in the figure 9-2. The circle can be defined in Cartesian coordinates as the locus of points (x, y) in a plane which are at a constant distance from a given fixed point (x_0, y_0).

As the point (x, y) moves, it is to remain a constant distance r from the fixed point (x_0, y_0), which represents the center of the circle. If the the distance r is to remain constant as (x, y) moves, then by the distance formula obtained by using the Pythagorean theorem we have the requirement

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r = \text{a constant}$$

or

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{9.1}$$

This is the equation in Cartesian coordinates for a circle with radius r and center at (x_0, y_0) .

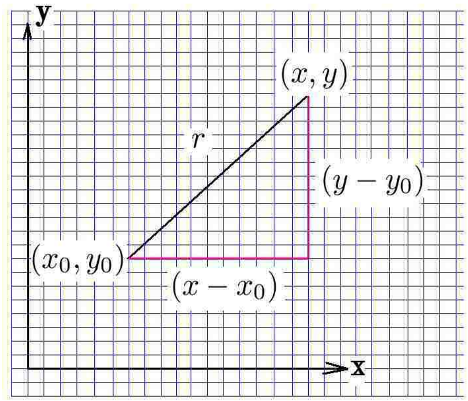


Figure 9-2. Sketch to find set of points (x, y) equidistant from fixed point (x_0, y_0)

In the special case (x_0, y_0) is the origin $(0, 0)$, then the equation (9.1) reduces to

$$x^2 + y^2 = r^2 \tag{9.2}$$

The equation (9.1) results from equation (9.2) by replacing x by $x - x_0$ and y by $y - y_0$ and equation (9.1) is referred to as a **translation** of the circle origin from $(0, 0)$ to the point (x_0, y_0).

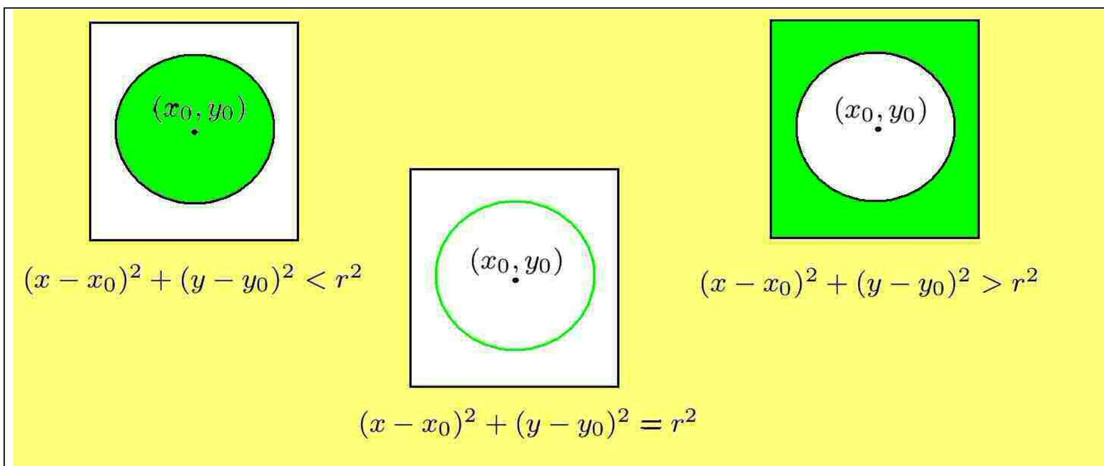


Figure 9-3. Points inside, on and outside of circle centered at (x_0, y_0)

The set of points satisfying the inequality

$$(x - x_0)^2 + (y - y_0)^2 < r^2 \tag{9.3}$$

represents all points inside the circle and the set of points satisfying the inequality

$$(x - x_0)^2 + (y - y_0)^2 > r^2 \tag{9.4}$$

represents all points outside the circle.

General equation for a circle

Equations which can be expanded and rearranged to have the form

$$x^2 + y^2 + ax + by + c = 0, \qquad a^2 + b^2 - 4c > 0 \tag{9.5}$$

are said to represent the **general form for a circle.** One can rearrange equation (9.5) into the form

$$(x^2 + ax \quad) + (y^2 + by \quad) = -c \tag{9.6}$$

and then complete the square on the x-terms and y-terms.

To complete the square on the x-terms take $\frac{1}{2}$ of the x-coefficient, square it, and then add the result to both sides of equation (9.6). This produces

$$\left(x^2 + ax + \frac{a^2}{4}\right) + (y^2 + by \quad) = \ -c + \frac{a^2}{4} \tag{9.7}$$

To complete the square on the y-terms take $\frac{1}{2}$ of the y-coefficient, square it, and then add the result to both sides of equation (9.7). This produces

$$\left(x^2 + ax + \frac{a^2}{4}\right) + \left(y^2 + by + \frac{b^2}{4}\right) = \ -c + \frac{a^2}{4} + \frac{b^2}{4} \tag{9.8}$$

The terms in equation (9.8) which are inside the parenthesis are perfect squares so that equation (9.8) simplifies to

$$(x + \frac{a}{2})^2 + (y + \frac{b}{2})^2 = \ r^2 \tag{9.9}$$

where $r^2 = -c + \frac{a^2}{4} + \frac{b^2}{4} > 0$. The equation (9.9) is the equation of a circle with center $(\frac{-a}{2}, \frac{-b}{2})$ and radius $r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - c}$.

In the cases
(i) $\frac{a^2}{4}+\frac{b^2}{4}-c=0$, the equation (9.9) is a point circle with radius 0.
(ii) $\frac{a^2}{4}+\frac{b^2}{4}-c<0$, the equation (9.9)is called an imaginary circle.
In general, any equation of the form

$$(x-x_0)^2+(y-y_0)^2=r^2 \tag{9.10}$$

represents a circle of radius r centered at the point (x_0, y_0). The equation (9.10) has the expanded form

$$x^2+y^2+ax+by+c=0 \tag{9.11}$$

for appropriate constants a, b, c.

Three conditions determine a circle

The general second degree equation representing a circle has the form

$$x^2+y^2+ax+by+c=0 \tag{9.12}$$

with three constants a, b, c to be determined. These three constants can be determined by imposing any one of the following conditions

(i) Require the circle to pass through three distinct specified points

$$P_1: \ (x_1, y_1) \qquad P_2: \ (x_2, y_2) \qquad P_3: \ (x_3, y_3)$$

This produces the three requirements

$$\begin{aligned}
x_1^2+y_1^2+ax_1+by_1+c=&0\\
x_2^2+y_2^2+ax_2+by_2+c=&0\\
x_3^2+y_3^2+ax_3+by_3+c=&0
\end{aligned} \tag{9.13}$$

which represents three equations in the three unknowns a, b, c to be solved.

(ii) Specify the center (x_0, y_0) of the circle and require the circle to pass through a given point $P_1: \ (x_1, y_1)$. In this case one can use the equation (9.9) in the form

$$(x+\frac{a}{2})^2+(y+\frac{b}{2})^2= \ -c+\frac{a^2}{4}+\frac{b^2}{4}>0 \tag{9.14}$$

because this form determines the values for a and b from the requirement

$$\frac{a}{2}=-x_0 \qquad \text{and} \qquad \frac{b}{2}=-y_0 \tag{9.15}$$

Knowing a and b the equation (9.14) reduces to the form

$$(x - x_0)^2 + (y - y_0)^2 = -c + x_0^2 + y_0^2 \tag{9.16}$$

The constant c is now determined by the requirement that (x_1, y_1) is to satisfy the equation for the circle so that

$$c = x_0^2 + y_0^2 - (x_1 - x_0)^2 - (y_1 - y_0)^2 \tag{9.17}$$

(iii) Specify conditions that the circle is to satisfy
  (a) Circle required to be tangent to a given line.
  (b) Circle center required to lie on some given line.
  (c) Circle required to pass through some specified point.

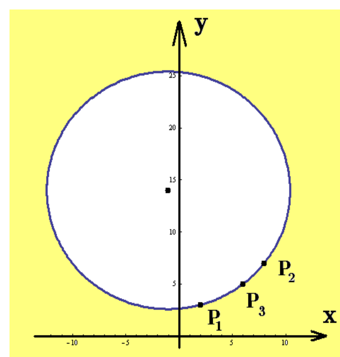
Note that restrictions like those above might produce more than one solution satisfying the given requirements.

Example 9-1. Find the equation of the circle which passes through the points
$P_1: \ (2,3), \ P_2: \ (8,7), \ P_3: \ (6,5)$

Solution

If the equation of the circle is given by

$$x^2 + y^2 + ax + by + c = 0 \tag{9.18}$$



and the coordinates of the points P_1, P_2, P_3 are to each satisfy the equation (9.18), then one obtains the three equations

when $x = 2, y = 3$ the equation (9.18) becomes $2a + 3b + c = -13$
when $x = 8, y = 7$ the equation (9.18) becomes $8a + 7b + c = -113$
when $x = 6, \ y = 5$ the equation (9.18) becomes $6a + 5b + c = -61$

One can solve these simultaneous equations and show

$$a = 2, \qquad b = -28, \qquad c = 67$$

giving the equation of the circle as

$$x^2 + y^2 + 2x - 28y + 67 = 0$$

or after completing the square on the x and y-terms one finds

$$(x+1)^2 + (y-14)^2 = 130$$

as the equation of the circle.

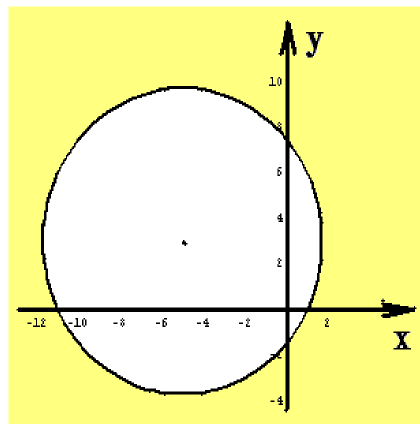
■

Example 9-2.

Find the circle centered at $(-5,3)$ which passes through the point $(1,0)$.

Solution

The general equation of a circle centered at the point (x_0, y_0) is



$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

It is given that $x_0 = -5$ and $y_0 = 3$ so the equation of the circle must have the form

$$(x+5)^2 + (y-3)^2 = r^2$$

If the point $(1,0)$ is to lie on this circle, then from the equation

$$(1+5)^2 + (0-3)^2 = r^2 \quad \text{one finds} \quad 36+9 = r^2 = 45$$

Hence, the equation of the circle satisfying the given conditions is

$$(x+5)^2 + (y-3)^3 = 45$$

■

Example 9-3. Find the equation of the circle (i) tangent to the y-axis (ii) having its center (x_0, y_0) on the line $y = 2x+1$ (iii) passing through the point $(3,4)$.

Solution

The requirement that the circle be tangent to the y-axis, specifies that the radius of the circle be x_0 and so the equation for the circle must have the form

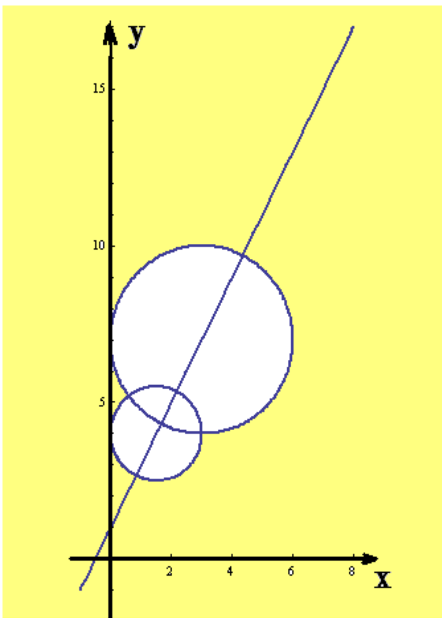
$$(x-x_0)^2 + (y-y_0)^2 = x_0^2 \tag{9.19}$$

The center of the circle is to lie on the line $y = 2x+1$ and so must satisfy the equation

$$y_0 = 2x_0 + 1 \tag{9.20}$$

The point $(3,4)$ is to be on the circle and so must satisfy the equation

$$(3 - x_0)^2 + (4 - y_0)^2 = x_0^2 \tag{9.21}$$



Solve the simultaneous equations (9.20) and (9.21) by substituting y_0 from equation (9.20) into equation (9.21) to obtain

$$(3 - x_0)^2 + (4 - (2x_0 + 1))^2 = x_0^2 \tag{9.22}$$

Expand equation (9.22) to obtain the equation

$$4x_0^2 - 18x_0 + 18 = 0$$

which can be factored into the form

$$2(x_0 - 3)(2x_0 - 3) = 0$$

Thus, there are two solutions for x_0.

$$\begin{aligned} Case1: \quad x_0 &= 3 \\ y_0 &= 2x_0 + 1 = 7 \\ (x-3)^2 + (y-7)^2 &= 3^2 \end{aligned} \qquad \begin{aligned} Case2: \quad x_0 &= 3/2 \\ y_0 &= 2x_0 + 1 = 4 \\ (x-3/2)^2 + (y-4)^2 &= (3/2)^2 \end{aligned}$$

These two solutions are illustrated in the figure above.

■

Parametric equations for representing the circle

Consider a point P with coordinates (x, y) which is moving around a circle with radius r as illustrated in the figure 9-4. Note that the distance x can be interpreted as

$$x = r\cos\theta = \text{projection of } r \text{ onto } x \text{ axis}$$

The distance y can be interpreted as

$$y = r\cos\phi = \text{projection of } r \text{ onto the } y \text{ axis}$$

Note that $\phi = \frac{\pi}{2} - \theta$ with $\cos\phi = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ so that

$$y = r\cos\phi = r\sin\theta = \text{projection of } r \text{ onto } y \text{ axis}$$

The set of equations

$$\begin{aligned} x =& r\cos\theta \\ y =& r\sin\theta \end{aligned} \tag{9.79}$$

for $0 \le \theta \le 2\pi$ are called the parametric equations for representing the circle centered at the origin. These equations describe the motion of the point P as (x, y) moves around the circle as the parameter θ varies from 0 to 2π. Note that when the equations (9.79) are substituted into the equation (9.2) and simplified, then one obtains the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ which holds true for all values of the parameter θ. If the origin of the circle is translated to the point (x_0, y_0), then one only need replace x by $x - x_0$ and y by $y - y_0$ in the equations (9.79) to obtain

$$\begin{aligned} x - x_0 =& r\cos\theta \\ y - y_0 =& r\sin\theta \end{aligned} \tag{9.24}$$

as θ varies from 0 to 2π. The set of equations (9.24) represent the parametric equations for a circle of radius r centered at the point (x_0, y_0).

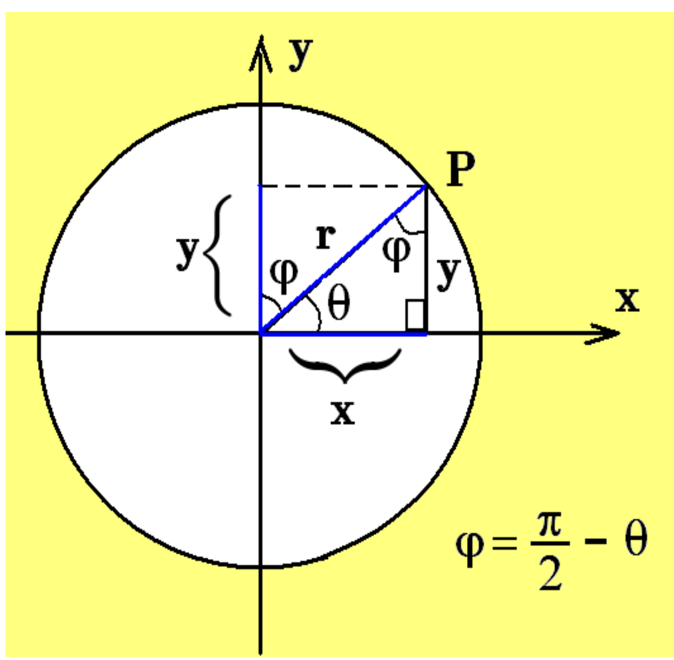


Figure 9-4. Point P with coordinates (x, y) moving around circle with radius r.

Given a set of parametric equations with parameter θ representing a curve in Cartesian coordinates, one can determine the (x, y)-equation for the curve by algebraically eliminating the parameter θ. For example, one can write

$$\begin{array}{lll} & r\cos\theta = x - x_0, & r\sin\theta = y - y_0 \\ \text{square both sides} & r^2\cos^2\theta = (x - x_0)^2, & r^2\sin^2\theta = (y - y_0)^2 \end{array}$$

and then by addition

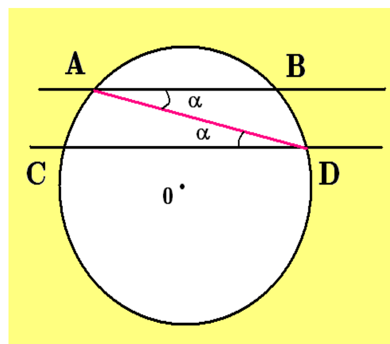
$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta =&(x - x_0)^2 + (y - y_0)^2 \\ r^2(\cos^2\theta + \sin^2\theta) =&(x - x_0)^2 + (y - y_0)^2 \\ r^2 =&(x - x_0)^2 + (y - y_0)^2 \end{aligned} \tag{9.25}$$

The parameter θ is eliminated and one obtains the equation for the circle.

Example 9-4. Show that if the line $\overline{\text{AB}}$ is parallel to line $\overline{\text{CD}}$ ($\overline{\text{AB}} \parallel \overline{\text{CD}}$) and these lines intersect a circle, then arc $\widehat{\text{AC}}$ is congruent to arc $\widehat{\text{BD}}$ ($\widehat{\text{AC}} \cong \widehat{\text{BD}}$)

Proof

Construct the line $\overline{\text{AD}}$ which is a transversal line where the alternate interior angles α are equal to one another. These angles are also inscribed angles equal to $\frac{1}{2}$ of the central angle $\angle B0D$. The arcs $\widehat{\text{BD}}$ and $\widehat{\text{AC}}$ are both associated with the inscribed angle α. Things equal to the same thing are equal to each other so that one can say $\widehat{\text{BD}} \cong \widehat{\text{AC}}$.



■

Tangent-chord theorem

Let the line segment $\overline{\text{AC}}$ denote a chord associated with the circle $x^2 + y^2 = r^2$ centered at the origin of a rectangular coordinate system as illustrated in the figure 9-5. In this figure the point A has the coordinates (x_1, y_1) and point C has the coordinates (x_3, y_3). The endpoints of the chord lie on the circle so that the equations

$$x_1^2 + y_1^2 = r^2 \qquad \text{and} \qquad x_3^2 + y_3^2 = r^2$$

are satisfied. In the following discussions concerning the circle assume that $x_1 \neq 0$ and $y_1 \neq 0$.

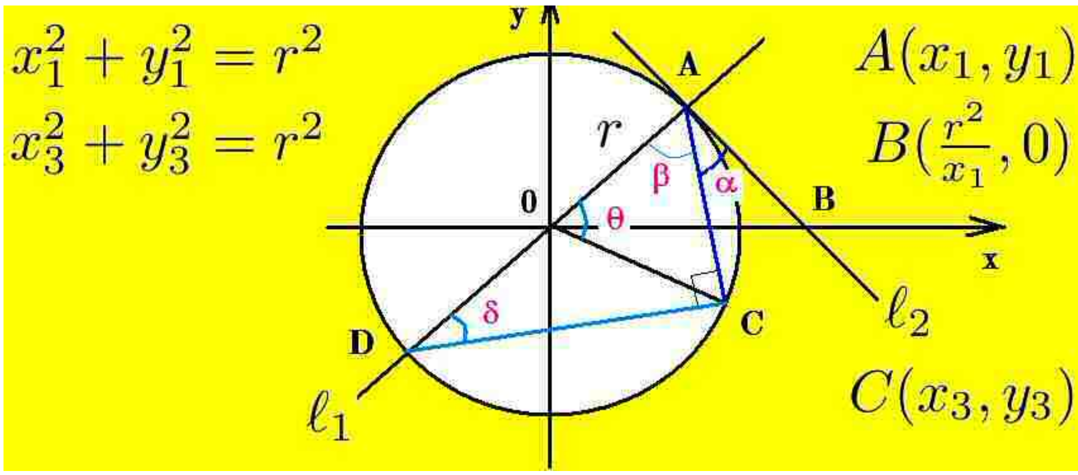


Figure 9-5. Figure for tangent-chord theorem

Construct the secant line ℓ_1 which passes through the center of the circle and the point A having coordinates (x_1, y_1). The slope of the line ℓ_1 is given by

$$m_{0A} = \frac{\text{change in y}}{\text{change in x}} = \frac{y_1}{x_1} \tag{9.26}$$

Using the point-slope formula one can verify that the equation of the secant line ℓ_1 is given by

$$\ell_1 \quad \text{secant line through } (0,0) \text{ and } (x_1, y_1) \text{ is} \qquad y = \frac{y_1}{x_1}x \tag{9.27}$$

Next construct the line ℓ_2 which is tangent to the circle at the point A. The tangent line to the circle will be perpendicular to the line ℓ_1 and so its slope will be the negative reciprocal of the slope m_{0A} or

$$m = \text{slope of tangent line} = \frac{-1}{m_{0A}} = -\frac{x_1}{y_1} \tag{9.28}$$

This tangent line is to pass through the point A with coordinates (x_1, y_1) so using the point-slope formula the equation of the tangent line ℓ_2 is given by

$$y - y_1 = \left(\frac{-x_1}{y_1}\right)(x - x_1) \tag{9.29}$$

The tangent line ℓ_2 intersects the x-axis at the point B illustrated in the figure 9-5. Observe that B has the y-coordinate of 0 and so setting $y = 0$ in equation (9.29) one obtains the x value for the intersection of the tangent line ℓ_2 with the x-axis. One can verify that the point of intersection B has the coordinates $(\frac{r^2}{x_1}, 0)$.

In figure 9-5 the angle $\alpha = \angle BAC$ is called the tangent-chord angle. The angle θ is called the central angle associated with the circular arc $\widehat{\text{AC}}$. Recall by Thales theorem the angle $\angle DCA$ is a right angle. The sum of the angles in any triangle is 180° or π radians, so that in the right triangle $\triangle ACD$ of figure 9-5 on finds

$$\frac{\pi}{2} + \delta + \beta = \pi \qquad \text{or} \qquad \beta + \delta = \frac{\pi}{2} \tag{9.30}$$

The line ℓ_2 is tangent to the circle at point A so that the angle $\angle BAD$ is a right angle or

$$\beta + \alpha = \frac{\pi}{2} \tag{9.31}$$

Subtract equation (9.31) from the equation (9.30) and show

$$\delta - \alpha = 0 \qquad \text{or} \qquad \delta = \alpha \tag{9.32}$$

The tangent-chord angle α is associated with the circular arc $\widehat{\text{AC}}$ and by the inscribed angle theorem one finds

$$\alpha = \frac{1}{2}\theta \quad \text{or} \quad \theta = 2\alpha \tag{9.33}$$

This result is known as the tangent-chord angle theorem.

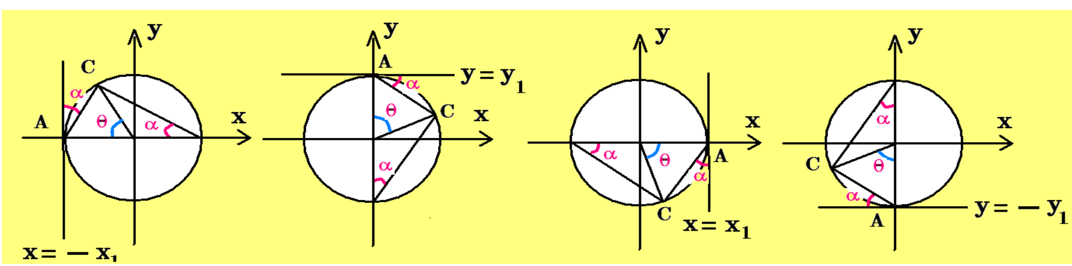


Figure 9-6. Special cases for tangent-chord theorem

The figure 9-6 illustrates four special positions for the point A associated with the tangent-chord theorem. It is left as an exercise to verify that in each of these cases $\alpha = \frac{1}{2}\theta$.

Example 9-5. Use Cartesian coordinates to show the angle $\angle ACD$ in figure 9-5 is a right angle. (Another verification of Thales theorem.)

Solution

Given that point A has the coordinates (x_1, y_1), one can show that the point D has the coordinates $(-x_1, -y_1)$ as these points are antipodal points. We know these points are on the given circle so they must satisfy the equation which defines the circle or $x_1^2 + y_1^2 = r^2$. The point C lies on the circle and has the coordinates (x_3, y_3) which must also satisfy the equation for the circle $x_3^2 + y_3^2 = r^2$. One can then use the formula for calculating the distance between two points to show

$$\begin{aligned} &\text{length of } \overline{\text{DC}} \text{ squared is } \overline{\text{DC}}^2 = (x_3 - (-x_1))^2 + (y_3 - (-y_1))^2 \\ &\text{length of } \overline{\text{AC}} \text{ squared is } \overline{\text{AC}}^2 = (x_3 - x_1)^2 + (y_1 - y_3)^2 \\ &\text{the length of the diameter } \overline{\text{AD}} \text{ squared is } \overline{\text{AD}}^2 = (2r)^2 = 4r^2 \end{aligned} \tag{9.34}$$

Use algebra and expand the equations (9.34) and show

$$\begin{aligned} \overline{\text{DC}}^2 &= x_3^2 + 2x_1x_3 + x_1^2 + y_3^2 + 2y_1y_3 + y_1^2 \\ \overline{\text{AC}}^2 &= x_3^2 - 2x_1x_3 + x_1^2 + y_1^2 - 2y_1y_3 + y_3^2 \end{aligned} \tag{9.35}$$

Addition of the equations (9.35) produces the result

$$\overline{\text{DC}}^2 + \overline{\text{AC}}^2 = 2(x_3^2 + y_3^2) + 2(x_1^2 + y_1^2) = 4r^2 \tag{9.36}$$

since the points (x_1, y_1) and (x_3, y_3) lie on the given circle. Consequently, it can be shown that

$$\overline{\text{DC}}^2 + \overline{\text{AC}}^2 = \overline{\text{AD}}^2 \tag{9.38}$$

so by the converse of the Pythagorean theorem, the angle $\angle ACD$ is a right angle

■

Additional considerations

The figure 9-5 gives us the opportunity to use the converse of the Pythagorean theorem to verify that the line ℓ_2 is indeed perpendicular to line ℓ_1. Examine the figure 9-5 and verify that

$$\begin{aligned} &\text{The line segment } \overline{\text{0B}} \text{ has length} \quad \overline{\text{0B}} = \frac{r^2}{x_1} \\ &\text{The line segment } \overline{\text{0A}} \text{ has length} \quad \overline{\text{0A}} = r = \sqrt{x_1^2 + y_1^2} \\ &\text{The line segment } \overline{\text{AB}} \text{ has length} \quad \overline{\text{AB}} = \sqrt{\left(\frac{r^2}{x_1} - x_1\right)^2 + y_1^2} \end{aligned}$$

Using these lengths one can verify that

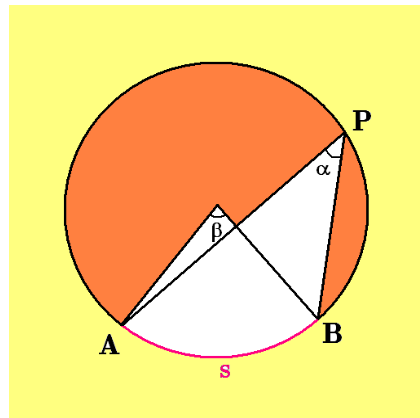
$$\overline{0A}^2 + \overline{AB}^2 = \overline{0B}^2$$

and consequently by the converse Pythagorean theorem the angle $\angle 0AB$ is a right angle.

Note also the result that the tangent line to a circle is always perpendicular to the radius or diameter of the circle meeting at the point of tangency. (Euclid Elements, book 3, proposition 18)

Arc length subtended by inscribed angle

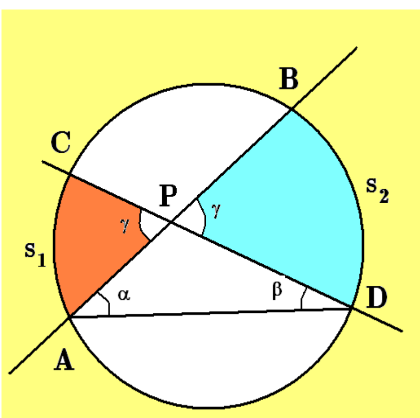
Given an inscribed angle $\alpha = \angle APB$ which subtends an arc $\overset{\frown}{AB} = s$ on a given circle of radius r. Remember that the inscribed angle is always related to the central angle by the relation $\beta = 2\alpha$, so that the arc length along the circumference between the points A and B is



$$s = \overset{\frown}{AB} = r(2\alpha) = r\beta \tag{9.38}$$

where α and β are in radians.

Intersection of secant lines within a circle



If two secant lines have a point of intersection P inside a circle of radius r, then label the points of intersection with the circle circumference as A, B, C and D as illustrated in the figure. Also label the angle of intersection as γ. Construct the line segment $\overline{AD}$ as illustrated and make note that this construction creates the inscribed angles $\alpha = \angle DAB$ and $\beta = \angle ADC$ which subtends the arcs $\overset{\frown}{BD} = s_2$ and $\overset{\frown}{CA} = s_1$.

We know that these arc lengths are given by

$$s_1 = r\theta_1 = r(2\beta) \quad \text{and} \quad s_2 = r\theta_2 = r(2\alpha) \tag{9.39}$$

where θ_1 and θ_2 are the central angles associated with the arcs s_1 and s_2 respectively. Here the inscribed angles α and β and central angles are in radians and r is the radius of the given circle.

In triangle $\triangle APD$ the angle γ is an exterior angle and we know an exterior angle equals the sum of the two opposite interior angles so that

$$\gamma = \alpha + \beta = \frac{1}{2r}(s_1 + s_2) = \frac{1}{2}(\theta_1 + \theta_2) \tag{9.40}$$

where all angles are in radians and r is the radius of the given circle. Here the measure of the angle of intersection γ is the average of the angular measures associated with arcs s_1 and s_2.

Intersection of secant lines outside circle

If the secant lines intersect outside the circle, as illustrated in the figure 9-7, one can again construct the line $\overline{\text{AD}}$ to form the inscribed angles $\angle DAB = \alpha$ and $\angle ADB = \beta$ having respectively the subtended arcs $s_1 = \widehat{\text{BD}}$ and $s_1 = \widehat{\text{BD}}$ with arc lengths

$$s_1 = r(2\beta) = r\theta_1 \quad \text{and} \quad s_2 = r(2\alpha) = r\theta_2$$

where θ_1 and θ_2 are the central angles associated with arcs s_1 and s_2.

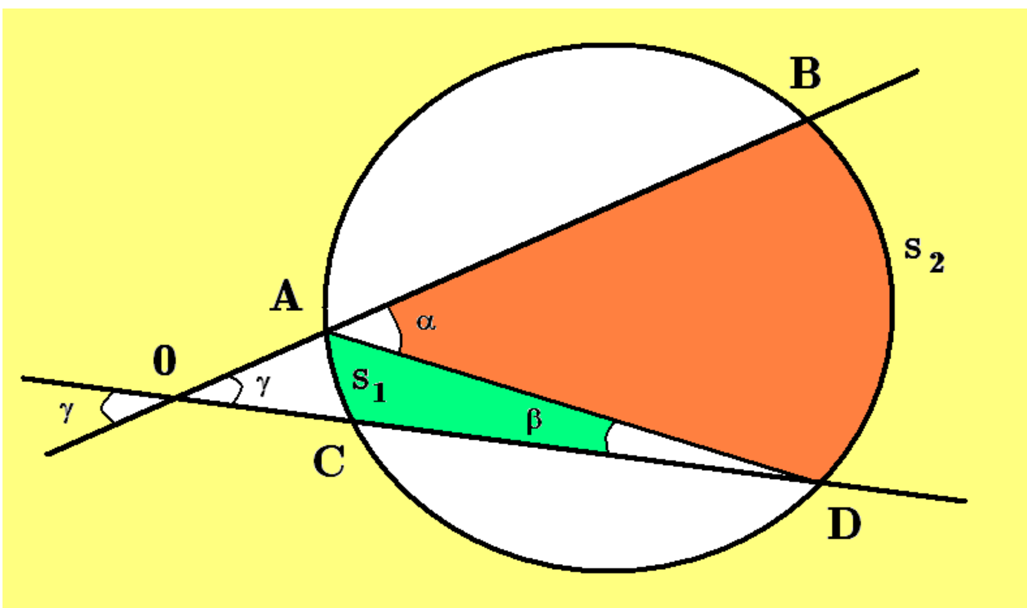


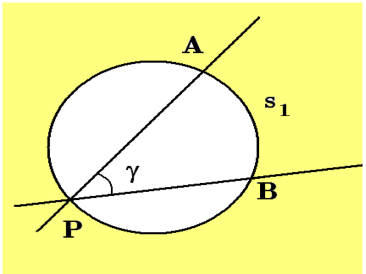
Figure 9-7. Secant lines intersecting outside of circle

The angle α is the exterior angle to the triangle $\triangle 0AD$ so that

$$\alpha = \gamma + \beta \quad \text{or} \quad \gamma = \alpha - \beta \quad \text{or} \quad \gamma = \frac{1}{2r}(s_2 - s_1) = \frac{1}{2}(\theta_2 - \theta_1) \tag{9.41}$$

Here the measure of the angle of intersection γ is half the difference of the central angles defining the arcs s_1 and s_2.

Intersection of secant lines on the circle



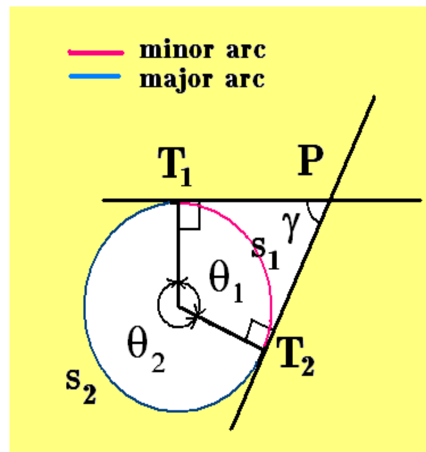
If two secant lines intersect on the circle, then the angle of intersection γ is an inscribed angle and the minor arc $s_1 = \widehat{AB}$ is given by

$$s_1 = r(2\gamma) = r\theta_1$$

where θ_1 is the central angle associated with the arc $\widehat{AB}$.

Intersection of two tangent lines to a circle

If two tangent lines to the same circle intersect at a point P outside the circle, then the angle $\angle T_1PT_2 = \gamma$ is determined by the central angles associated with the minor arc $s_1 = \widehat{T_1T_2}$ and the major arc $s_2 = \widehat{T_2T_1}$ the arcs subtended by the tangent lines.



$$\begin{aligned} \text{The minor arc is} \quad s_1 =& r\theta_1 \\ \text{The major arc is} \quad s_2 =& r\theta_2 = r(2\pi - \theta_1) \end{aligned} \tag{9.42}$$

where θ_1 and θ_2 are the central angles associated with the subtended arcs.

The sum of the angles in the quadrilateral $T_1 0 T_2 P$ is given by

$$\frac{\pi}{2} + \theta_1 + \frac{\pi}{2} + \gamma = 2\pi \quad \Rightarrow \quad \pi - \theta_1 = \gamma \tag{9.43}$$

The major arc s_2 can be expressed

$$s_2 = r\theta_2 = r(2\pi - \theta_1) = r(\pi + (\pi - \theta_1)) = r(\pi + \gamma) \quad \Rightarrow \quad \theta_2 - \pi = \gamma \tag{9.44}$$

Addition of the equations (9.43) and (9.44) produce the result

$$2\gamma = \theta_2 - \theta_1 \quad \text{or} \quad \gamma = \frac{1}{2}(\theta_2 - \theta_1) \tag{9.45}$$

Again we find the measure of the angle of intersection γ is half the difference of the central angles defining the arcs s_2 and s_2.

Tangent-Secant length theorem

Let P denote a point outside a given circle and then construct the tangent line $\overline{\text{PA}}$. Next construct a secant line $\overline{\text{PB}}$ which intersects the circle at point C. The tangent-secant length theorem states that the lengths $\overline{\text{PA}}$, $\overline{\text{PC}}$, $\overline{\text{PB}}$ are related by the formula

$$\overline{\text{PA}}^2 = \overline{\text{PC}} \cdot \overline{\text{PB}} \tag{9.46}$$

That is, the tangent length $\overline{\text{PA}}$ squared must equal the product of the secant line segments $\overline{\text{PC}}$ and $\overline{\text{PB}}$.

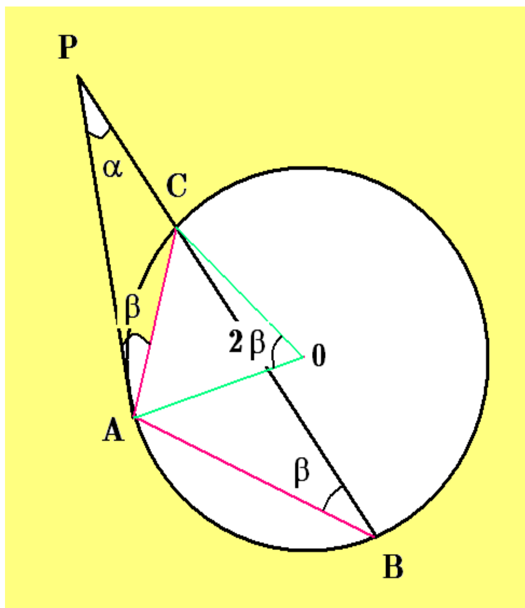


Figure 9-8. Tangent line and secant line to circle

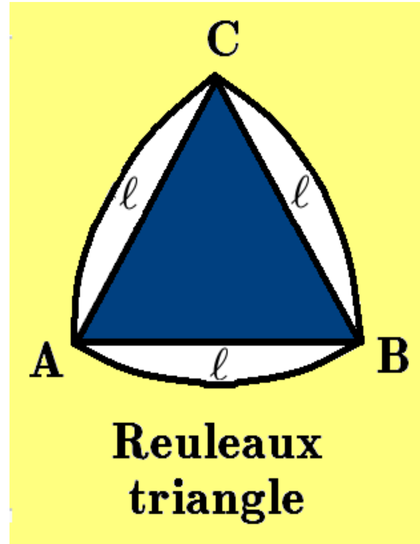
This theorem results from using a combination of previous results. Observe that the angle $\beta = \angle ABP$ is an inscribed angle associated with the arc length $\widehat{\text{AC}} = r(2\beta)$ and central angle 2β. Also the chord $\overline{\text{AC}}$ intersects the tangent line $\overline{\text{PA}}$ forming the tangent-chord angle $\angle CAP$. The angle $\angle CAP = \beta$ by the tangent-chord theorem. The angle α is common to the two triangles $\triangle PAB$ and $\triangle PCA$. The triangles $\triangle PAB$ and $\triangle PCA$ have the same shape and two angles in common. Therefore, the triangles are similar and their sides are proportional so that one can write

$$\frac{\overline{\text{PC}}}{\overline{\text{PA}}} = \frac{\overline{\text{PA}}}{\overline{\text{PB}}} \quad \text{or} \quad \overline{\text{PA}}^2 = \overline{\text{PC}} \cdot \overline{\text{PB}}$$

by using the cross ratio product.

Reuleaux triangle

The circle is not the only figure with a constant width. One can construct an equilateral triangle $\triangle ABC$ with equal sides of length ℓ. Using a drawing compass one can construct circular arcs $\overset{\frown}{\mathrm{AB}}$, $\overset{\frown}{\mathrm{BC}}$, $\overset{\frown}{\mathrm{CA}}$ having centers for the arcs at the vertices C, A and B of the equilateral triangle. The radius associated with each arc is the length ℓ. The resulting triangle $\triangle A\widehat{B}C$ **with curved sides** is called a Reuleaux[3] triangle. One of its properties is that it is a geometric figure with a constant width.



The area of a Reuleaux triangle is the central equilateral triangle area plus the areas associated with the 3 segments of a circle.

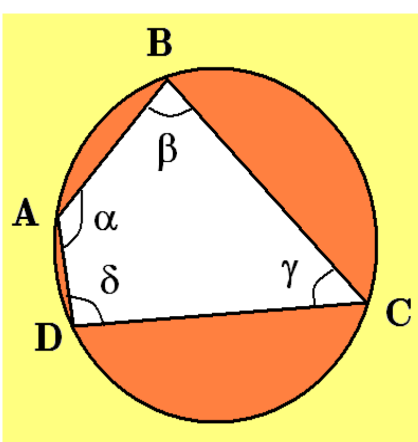
$$\begin{aligned}\text{Area equilateral triangle} &= \frac{\sqrt{3}}{4}\ell^2 \\ \text{Area of one segment} &= \frac{1}{2}\ell^2(\theta - \sin\theta),\ \ \theta = \frac{\pi}{3}\end{aligned} \quad \Rightarrow \quad \begin{aligned}\text{Total Area} &= \frac{\sqrt{3}}{4}\ell^2 + \frac{3}{2}\ell^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \\ \text{Total Area} &= \frac{1}{2}\left(\pi + \sqrt{3}\right)\ell^2\end{aligned}$$

The perimeter of the Reuleaux triangle is three times the arc length of one side or

$$\text{Perimeter} = 3\,(\ell\theta) = 3\left(\ell\frac{\pi}{3}\right) = \pi\ell$$

or π times the diameter (Barbier's theorem).

Quadrilateral inside circle



Let us reexamine the quadrilateral constructed inside a circle. Any quadrilateral whose vertices lie on the circumference of a circle is called a **cyclic quadrilateral**. We have previously demonstrated that the opposite angles (α and γ) as well as (β and δ) in a cyclic quadrilateral will always add to π radians. That is, opposite angles in a cyclic quadrilateral will always add to 180° or π radians.
(Euclid's Elements, Book 3, Proposition 22.)

The opposite angles in a cyclic quadrilateral being supplementary results from an observation of the inscribed angles within the circle. The inscribed angle α subtends the major arc $\mathrm{B\widehat{C}D}$ and the inscribed angle γ subtends the minor arc $\mathrm{D\widehat{A}B}$. These arc lengths can be represented

$$\mathrm{B\widehat{C}D} = r(2\alpha) \qquad \mathrm{D\widehat{A}B} = r(2\gamma) \tag{9.47}$$

But a summation of the arc lengths gives $\mathrm{B\widehat{C}D} + \mathrm{D\widehat{A}B} = 2\pi r$ which represents the circumference of the circle. Consequently,

$$\mathrm{B\widehat{C}D} + \mathrm{D\widehat{A}B} = 2(\alpha+\gamma)r = 2\pi r \quad \Rightarrow \quad (\alpha+\gamma) = \pi \tag{9.48}$$

In a similar fashion we have $\mathrm{A\widehat{B}C} = r(2\delta)$ and $\mathrm{A\widehat{D}C} = r(2\beta)$ with

$$\mathrm{A\widehat{D}C} + \mathrm{A\widehat{B}C} = 2(\beta+\delta)r = 2\pi r \quad \Rightarrow \quad (\beta+\delta) = \pi \tag{9.49}$$

Ptolemy's theorem

Given a cyclic quadrilateral $ABCD$, let

$$\ell_1 = \overline{\mathrm{AB}},\ \ell_2 = \overline{\mathrm{BC}},\ \ell_3 = \overline{\mathrm{CD}},\ \ell_4 = \overline{\mathrm{AD}}$$

denote the length of the sides and let $d_1 = \overline{\mathrm{AC}},\ d_2 = \overline{\mathrm{BD}}$ denote the length of the diagonals of the quadrilateral as illustrated in the figure 9-10.

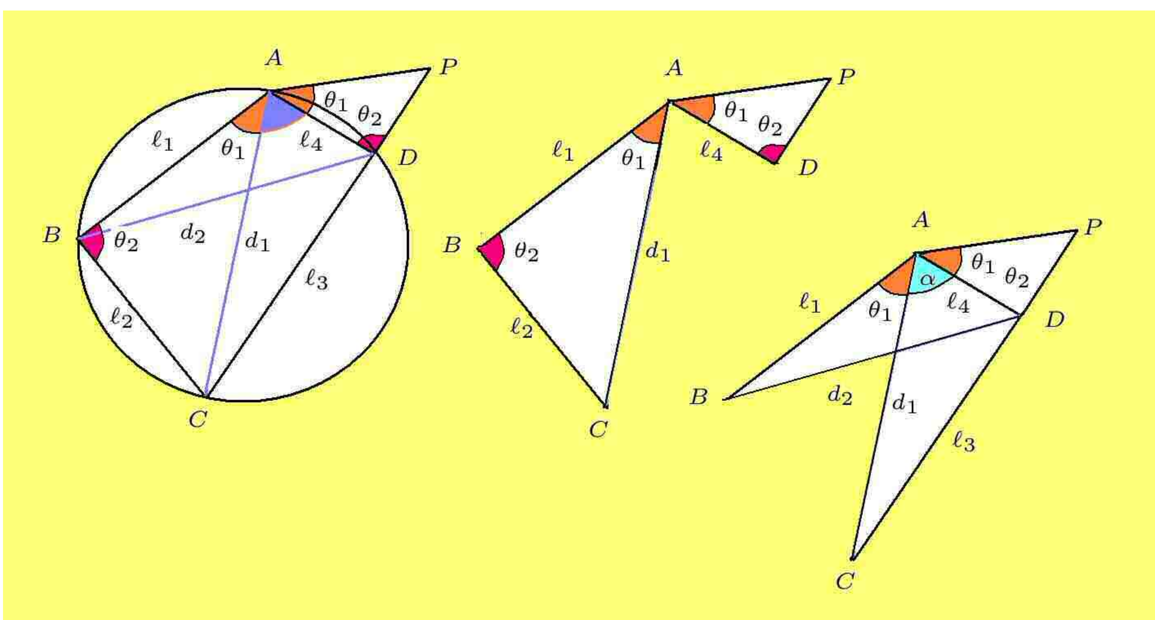


Figure 9-10. Quadrilateral inside circle with extended side $\overline{\mathrm{CD}}$

Multiply the lengths of the opposite sides of the cyclic quadrilateral and add their products together to obtain

$$\ell_1\ell_3 + \ell_2\ell_4 \tag{9.50}$$

Next multiply the diagonals to obtain

$$d_1 d_2 \tag{9.51}$$

Ptolemy's[4] theorem states that the sum of the products given by the representation (9.50) must equal the product of the diagonals given by the representation (9.51) or

$$\ell_1\ell_3 + \ell_2\ell_4 = d_1 d_2 \tag{9.52}$$

Proof of Ptolemy's theorem

Make some constructions in the figure 9-10. (i) Extend the side $\overline{\text{CD}}$ and (ii) construct a line through the vertex A such that $m\angle BAC = m\angle DAP = \theta_1$. Here the quadrilateral is cyclic so that the opposite interior angles sum to π and since $\overline{\text{CDP}}$ is a straight line, then $m\angle ADC + m\angle ADP$ also add to π. Hence one can write

$$\begin{aligned} m\angle ABC + m\angle ADC =& \pi \\ m\angle ADC + m\angle ADP =& \pi \end{aligned}$$

These equations imply that

$$m\angle ABC = m\angle ADP = \theta_2$$

and therefore triangle $\triangle BAC$ is similar to triangle $\triangle ADP$ or $\triangle BAC \sim \triangle ADP$. This results in the triangle sides being proportional so that

$$\frac{\overline{\text{AB}}}{\overline{\text{AD}}} = \frac{\overline{\text{BC}}}{\overline{\text{DP}}} \;\Rightarrow\; \overline{\text{DP}} = \frac{(\overline{\text{AD}})(\overline{\text{BC}})}{\overline{\text{AB}}} = \frac{\ell_3\ell_4}{\ell_1} \tag{9.53}$$

The similar triangles $\triangle BAC \sim \triangle ADP$ also produce the proportion

$$\frac{\overline{\text{AB}}}{\overline{\text{AD}}} = \frac{\overline{\text{AC}}}{\overline{\text{AP}}} \tag{9.54}$$

and the geometry of figure 9-10 one finds $m\angle BAD = m\angle CAP = \theta_1 + \alpha$. Here one can conclude the triangles $\triangle ABD$ and $\triangle ACP$ are similar ($\triangle ABD \sim \triangle ACP$) because

[4] Claudius Ptolemy 85-165 CE Greek astronomer and mathematician

the sides are proportional and the include angles are congruent. This produces the proportion

$$\frac{\overline{\text{BD}}}{\overline{\text{CP}}} = \frac{\overline{\text{AB}}}{\overline{\text{AC}}} \Rightarrow \overline{\text{CP}} = \frac{(\overline{\text{AC}})(\overline{\text{BD}})}{\overline{\text{AB}}} = \frac{d_1 d_2}{\ell_1} \tag{9.55}$$

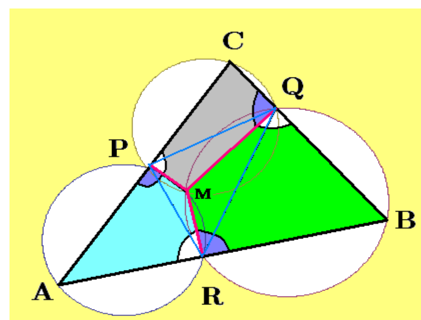
One can then use the results from equations (9.53) and (9.55) to demonstrate that

$$\begin{aligned} \overline{\text{CP}} =& \overline{\text{CD}} + \overline{\text{DP}} \\ \frac{d_1 d_2}{\ell_1} =& \ell_3 + \frac{\ell_3 \ell_4}{\ell_1} \\ \text{or } d_1 d_2 =& \ell_1 \ell_3 + \ell_2 \ell_4 \end{aligned}$$

which is Ptolemy's theorem.

Example 9-6.

Given a triangle $\triangle ABC$ select three arbitrary points P, Q, R on sides $\overline{\text{AC}}, \overline{\text{BC}}, \overline{\text{AB}}$ respectively. This creates the three triangles $\triangle APR$, $\triangle RQB$, $\triangle PCQ$ within the original triangle. Find the circumcenter of each of these interior triangles and construct the circumcircles for each of these interior triangles. These triangles intersect at a point M called the **Miquel point**[5]



This can be demonstrated as follows. Let the circles PCQ and RBQ intersect at point M and then construct the lines $\overline{\text{MQ}}$ and $\overline{\text{MR}}$. This creates two cyclic quadrilaterals PCQM and MQBR. If angle $\angle CQM = \theta$, then the opposite angle $\angle MPC = \pi - \theta$. In quadrilateral MQBR the angle $\angle MQB = \pi - \theta$ and the angle opposite is θ. Hence, $\angle MRA = \pi - \theta$ and the angle opposite is $\angle APM = \theta$, which demonstrates the quadrilateral APMR is also cyclic. This shows the circles APR, PCQ and RQB meet at a point of concurrency.

■

Area of cyclic quadrilateral

Given the **cyclic quadrilateral** ABCD, one can extend two opposite sides of the quadrilateral to meet at a point E as illustrated in the figure 9-9. Denote the length of the extended sides by $c_1 = \overline{\text{ED}}$ and $a_1 = \overline{\text{EA}}$. We know the opposite interior angles of a cyclic quadrilateral must sum to π radians. Therefore, if angle $\angle ABC = \beta$, then

[5] Named after Auguste Miquel (1816-1851) a French mathematician.

the angle $\angle ADC = \pi - \beta$. Similarly,if angle $\angle DCB = \gamma$, then the opposite interior angle is $\angle DAB = \pi - \gamma$. Observe that the angles $\angle EDC$ and $\angle EAB$ are straight angles so by the definition of supplementary angles one finds $\angle EDA = \beta$ and $\angle EAD = \gamma$. The angle α is common to both triangles $\triangle EDA$ and $\triangle EBC$ so the these triangles have three angles in common and consequently are similar triangles ($\triangle EDA \sim \triangle EBC$). The sides of similar triangles are proportional and so one can write the similarity ratios

$$\frac{d}{b} = \frac{a_1}{c_1 + c} = \frac{c_1}{a_1 + a} \tag{9.56}$$

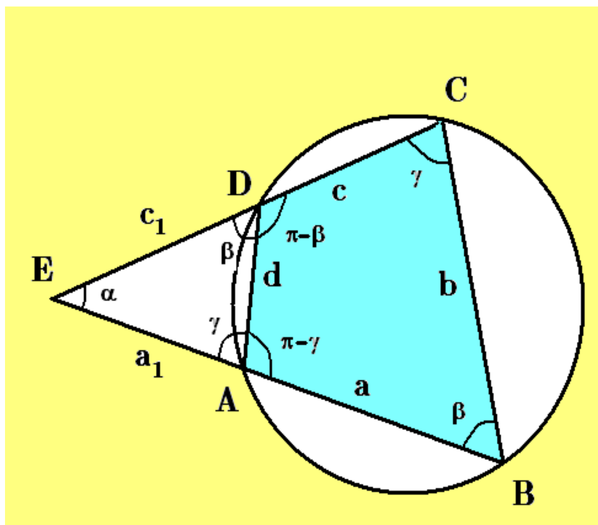


Figure 9-9. Cycle quadrilateral ABCD with extended sides

Recall that when dealing with similar triangles the ratio of the areas is proportional to the similarity ratio squared. Thus, if

$$[EDA] = \text{Area } \triangle EDA \qquad \text{and} \qquad [EBC] = \text{Area } \triangle EBC$$

then one can write

$$\frac{[EDA]}{[EBC]} = \left(\frac{d}{b}\right)^2 \tag{9.57}$$

Let A denote the area of the **cyclic quadrilateral** and note that this area can be expressed as the difference in area of the two similar triangles or

$$A = [EBC] - [EDA] = [EBC]\left(1 - \left(\frac{d}{b}\right)^2\right) = \left(\frac{b^2 - d^2}{b^2}\right)[EBC] \tag{9.58}$$

Using Heron's formula for the area of the triangle $\triangle EBC$ one finds

$$[EBC] = \sqrt{s(s-(a_1+a))(s-b)(s-(c_1+c)} \tag{9.59}$$

where

$$s = \frac{1}{2}\left((a_1+a)+b+(c_1+c)\right) \tag{9.60}$$

is the semiperimeter of triangle $\triangle EBC$.

Using the cross product property of the similarity ratios given in the equations (9.56) one can write

$$\begin{aligned} a_1 b =& d(c_1+c) \\ \text{and} \qquad d(a_1+a) =& c_1 b \end{aligned} \tag{9.61}$$

The equations (9.61) represents two equations in two unknowns a_1 and c_1. Solving this system of equations one finds

$$a_1 = \frac{ad^2+bcd}{b^2-d^2} \qquad \text{and} \qquad c_1 = \frac{cd^2+abd}{b^2-d^2} \tag{9.62}$$

Substituting these values into the terms associated with the equations (9.59) and (9.60) one can verify that

$$\begin{aligned} s =& \frac{1}{2}((a_1+a)+b+(c_1+c)) = \frac{b(a+b+c-d)}{2(b-d)} \\ s-(a_1+a) =& \frac{1}{2}(b+(c_1+c)-(a_1+a)) = \frac{b(b+c+d-a)}{2(b+d)} \\ (s-b) =& \frac{1}{2}((a_1+a)+(c_1+c)-b) = \frac{b(a+c+d-b)}{2(b-d)} \\ s-(c_1+c) =& \frac{1}{2}((a_1+a)+b-(c_1+c)) = \frac{a+b+d-c}{2(b+d)} \end{aligned} \tag{9.63}$$

The equation (9.58) representing the area of the cyclic quadrilateral then takes on the form

$$A = \frac{b^2-d^2}{b^2}\sqrt{b^4\frac{(a+b+c-d)}{2(b-d)}\frac{(b+c+d-a)}{2(b+d)}\frac{(a+c+d-b)}{2(b-d)}\frac{(a+b+d-c)}{2(b+d)}} \tag{9.64}$$

Define

$$\begin{aligned} \bar{s} =& \frac{1}{2}(a+b+c+d) \\ \text{then show} \qquad \bar{s}-a =& \frac{1}{2}(b+c+d-a) \\ \bar{s}-b =& \frac{1}{2}(a+c+d-b) \\ \bar{s}-c =& \frac{1}{2}(a+b+d-c) \\ \bar{s}-d =& \frac{1}{2}(a+b+c-d) \end{aligned} \tag{9.65}$$

and simplify the equation (9.64) to obtain the following result. The area of a cyclic quadrilateral with sides a, b, c, d can be represented in the form

$$A = \sqrt{(\bar{s}-a)(\bar{s}-b)(\bar{s}-c)(\bar{s}-d)} \quad \text{where } \bar{s} = \frac{1}{2}(a+b+c+d) \tag{9.67}$$

The above area representation for a cyclic quadrilateral is known as Brahmagupta's[6] formula. Note the form of the equation representing the area of a cyclic quadrilateral is very similar in form to Heron's formula for the area of a triangle.

Area of a general quadrilateral

A general quadrilateral is any polygon with four sides and four vertices. Quadrilaterals are called simple if the sides are not self intersecting. Consider the convex quadrilateral illustrated. Its area can be thought of as the sum of the two triangles $\triangle ABD$ and $\triangle CBD$.

$$\text{Area } \triangle ABD = \frac{1}{2}\ell_2 h_1 \quad \text{and} \quad \text{Area } \triangle CBD = \frac{1}{2}\ell_2 h_2$$

where h_1, h_2 are the triangle altitudes and the diagonal $\ell_2 = \overline{\text{BD}}$ is the base of both triangles

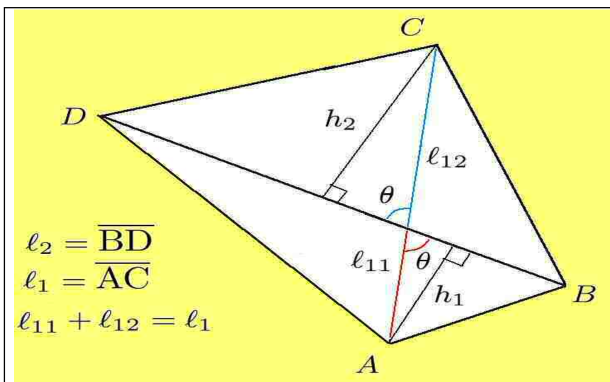


Figure 9-11. Quadrilateral ABCD

By addition one finds

$$\text{Area of quadrilateral} = \frac{1}{2}\ell_2(h_1 + h_2) \tag{9.67}$$

The intersection of the quadrilateral diagonals defines an angle θ. By definition of the sine function of the angle θ, the sine of θ can be represented

$$\sin\theta = \frac{h_2}{\ell_{12}} \quad \text{and} \quad \sin\theta = \frac{h_1}{\ell_{11}} \tag{9.68}$$

[6] Brahmagupta (598-670)CE an Indian mathematician.

where by addition the diagonal is $\ell_1 = \ell_{11} + \ell_{12}$ Using the substitutions from the equations (9.68) one can write

$$\ell_{11} + \ell_{12} = \ell_1 = \frac{h_1}{\sin\theta} + \frac{h_2}{\sin\theta} = \frac{h_1 + h_2}{\sin\theta} \tag{9.69}$$

Replacing the term $(h_1 + h_2)$ in equation (9.67) with the same term from equation (9.69), the area of the quadrilateral can be expressed

$$\text{Area of quadrilateral} = \frac{1}{2}\ell_1\ell_2\sin\theta$$

The power of a point

In 1826 a mathematician named Jakob Steiner[7] defined the concept of **the power of a point P with respect to the center 0 of a circle having radius r.** The point P can be outside the circle, on the circumference of the circle or inside the circle.

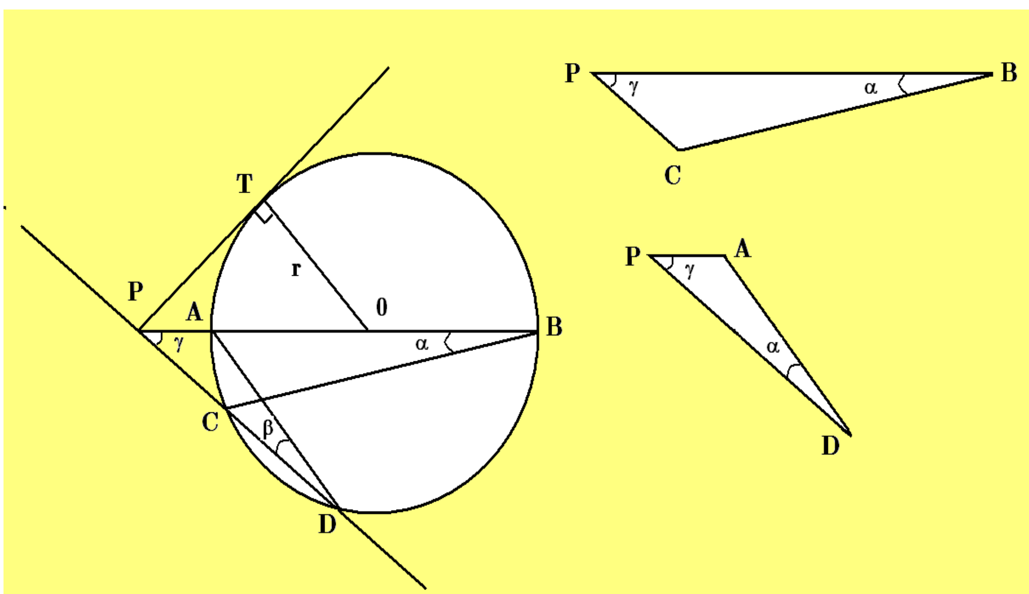


Figure 9-12. Power of a point sketch.

Assume the point P is outside a given circle with known radius r. Construct the straight line passing through the center of the circle and the point P. This line intersects the circle circumference at the points labeled A and B as illustrated

[7] Jakob Steiner (1796-1863) a Swiss mathematician.

in the figure 9-12. Construct the tangent line $\overline{\text{PT}}$ to the circle having the point of tangency T. Construct also any line through the point P which intersects the circle in two points C and D and produces the angle $\gamma = \angle CPA$. These constructions are illustrated in the figure 9-12.

The power of point P is defined

$$power = \overline{\text{P0}}^2 - r^2 \tag{9.70}$$

Observe that if the point P is

(i) outside the circle, then the power > 0

(ii) on the circle, then the power $= 0$

(iii) inside the circle, then the power < 0

There are equivalent definitions for the power of a point. For example, the triangle $\triangle PT0$ is a right triangle with right angle $\angle PT0$. By the Pythagorean theorem

$$\overline{\text{PT}}^2 + r^2 = \overline{\text{P0}}^2 \quad \text{or} \quad \overline{\text{PT}}^2 = \overline{\text{P0}}^2 - r^2 = power \tag{9.71}$$

In figure 9-12 construct the line segments $\overline{\text{CB}}$ and $\overline{\text{AD}}$ forming the inscribed angles α and β. The inscribed angles α and β both subtend the arc $\widehat{\text{AC}}$ so that

$$\widehat{\text{AC}} = (2\alpha)\, r \,=\, (2\beta)\, r \quad \Rightarrow \quad \alpha = \beta \tag{9.72}$$

by the inscribed angle theorem. The triangles $\triangle PCB$ and $\triangle PAD$ are similar triangles ($\triangle PCB \sim \triangle PAD$) because the angle γ is common to both triangles, the inscribed angles $\alpha = \beta$ and $\pi - (\alpha + \gamma)$ is the remaining angle in both triangles. In the two triangles cited one finds all the angles are equal and so the triangles are similar. Therefore, their sides are proportional so that

$$\frac{\overline{\text{PC}}}{\overline{\text{PA}}} = \frac{\overline{\text{PB}}}{\overline{\text{PD}}} \quad \Rightarrow \quad \overline{\text{PC}} \cdot \overline{\text{PD}} = \overline{\text{PA}} \cdot \overline{\text{PB}} \tag{9.73}$$

This ratio of the sides associated with similar triangles can be written

$$\overline{\text{PC}} \cdot \overline{\text{PD}} = \overline{\text{PA}} \cdot \overline{\text{PB}} = (\overline{\text{P0}} - r)(\overline{\text{P0}} + r) = \overline{\text{P0}}^2 - r^2 = power \tag{9.74}$$

This result is known as the power of a point theorem.

Power of a point theorem

Given a circle with a point P inside, on or outside the circle. Construct any two lines through the point P which intersects the circle in two points.

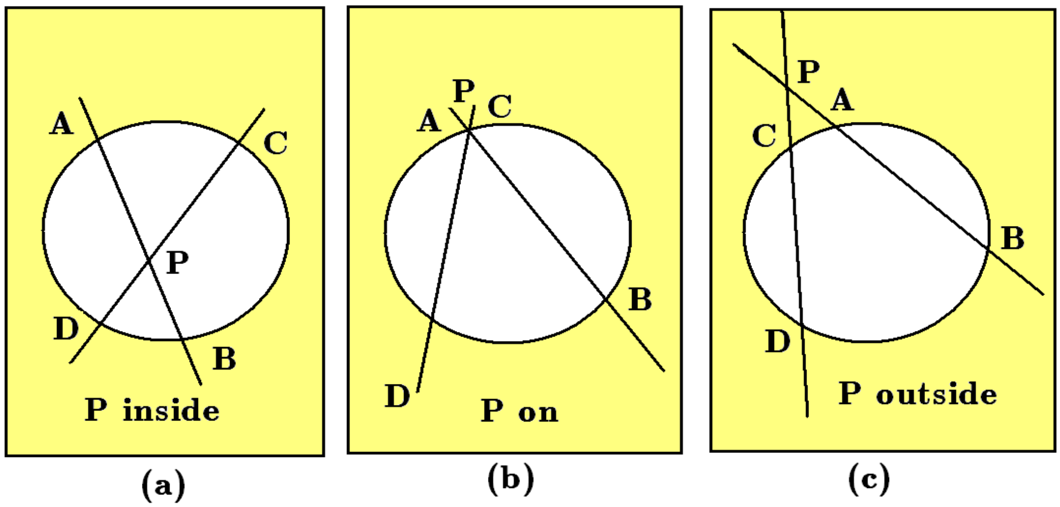


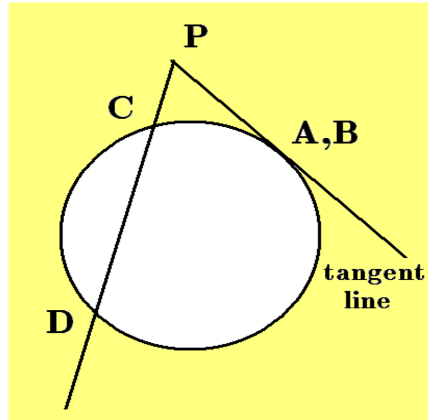
Figure 9-13. Figures for power of a point theorem

As you construct the two lines through point P, be sure to label the first point of intersection with the circle as point A and label the second point of intersection as B. The power of a point theorem states

> The product of the distance P to the first point of intersection A or $\overline{\text{PA}}$ times the distance P to the second point of intersection B or $\overline{\text{PB}}$ is always a constant. The constant is known as the power of point P.
>
> $$\overline{\text{PA}} \cdot \overline{\text{PB}} = \text{a constant} = \overline{\text{PC}} \cdot \overline{\text{PD}}$$

In the figure 9-13 (a) and (c) $\overline{\text{PA}} \cdot \overline{\text{PB}} = \text{a constant} = \overline{\text{PC}} \cdot \overline{\text{PD}}$ Note the middle figure 9-13(b) gives $\overline{\text{PA}} = 0$ since the power of the point P is zero.

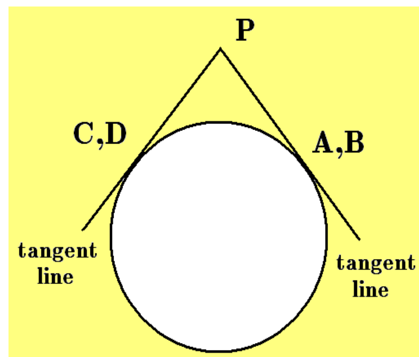
Special cases



1) In figure 9-13(a) the result $\overline{\text{PA}}\cdot\overline{\text{PB}} = \overline{\text{PC}}\cdot\overline{\text{PD}}$ is also known as the **intersecting chord theorem**. (Euclid's Elements, book 3, proposition 35)

2) If one of the lines through the point P is a tangent line to the circle, then the distance from P to the first point of intersection and the distance P to the second point of intersection are treated as being the same distance giving

$$\overline{\text{PA}}\cdot\overline{\text{PB}} = \overline{\text{PC}}\cdot\overline{\text{PD}}$$
$$\overline{\text{PA}}^2 = \overline{\text{PC}}\cdot\overline{\text{PD}}$$



3) If both lines are tangent from P to the circle, then one has

$$\overline{\text{PA}}\cdot\overline{\text{PB}} = \overline{\text{PC}}\cdot\overline{\text{PD}}$$
$$\overline{\text{PA}}\cdot\overline{\text{PA}} = \overline{\text{PC}}\cdot\overline{\text{PC}}$$
$$\overline{\text{PA}}^2 = \overline{\text{PC}}^2$$

The triangle

Given a general triangle $\triangle ABC$ with vertices A, B, C having Cartesian coordinates $A : (x_1, y_1), B : (x_2, y_2), C : (x_3, y_3)$ as illustrated in the figure 9-14.

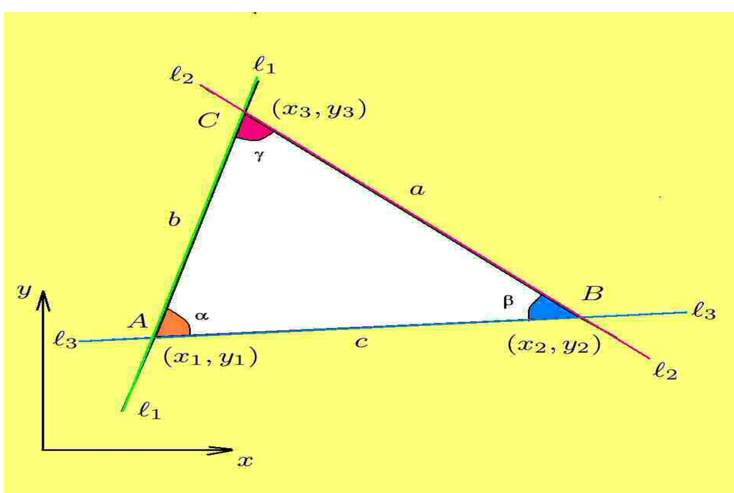


Figure 9-14. General triangle $\triangle ABC$

Midpoint of sides

The midpoint of the sides of the triangle illustrated are represented as

$$\begin{aligned} M_{AB}: &\ (\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)) \\ M_{AC}: &\ (\frac{1}{2}(x_1+x_3), \frac{1}{2}(y_1+y_3)) \\ M_{BC}: &\ (\frac{1}{2}(x_2+x_3), \frac{1}{2}(y_2+y_3)) \end{aligned} \tag{9.75}$$

Length of sides

The Pythagorean theorem can be used to calculate the sides of the triangle illustrated

$$\begin{aligned} a =&\sqrt{(x_2-x_3)^2+(y_2-y_3)^2} \\ b =&\sqrt{(x_3-x_1)^2+(y_3-y_1)^2} \\ c =&\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \end{aligned} \tag{9.76}$$

Slope of sides

The lines ℓ_1, ℓ_2, ℓ_3 are extensions of the sides of the triangle in figure 9-14. The slopes of these lines are calculated using the relation $m = \text{slope} = \dfrac{\text{change in y}}{\text{change in x}}$. One can verify that

$$m_3 = m_{AB} = \frac{y_2-y_1}{x_2-x_1}, \qquad m_2 = m_{BC} = \frac{y_3-y_2}{x_3-x_2}, \qquad m_1 = m_{AC} = \frac{y_3-y_1}{x_3-x_1} \tag{9.77}$$

Angles of triangle

The extended sides of the triangle produce intersecting lines. The formula for the angles associated with intersecting lines can be used to determine the angles associated with a given triangle. Using the slopes associated with the intersecting lines one finds

$$\tan\alpha = \frac{\overleftarrow{m_1 - m_3}}{1+m_1m_3}, \quad \tan\beta = \frac{\overleftarrow{m_3 - m_2}}{1+m_2m_3}, \quad \tan\gamma = \frac{\overleftarrow{m_2 - m_1}}{1+m_1m_2} \tag{9.78}$$

where the arrow is to remind you which line is rotating counterclockwise to produce the angle. One can then express the angles of the triangle as

$$\alpha = \tan^{-1}\left(\frac{m_1-m_3}{1+m_1m_3}\right), \quad \beta = \tan^{-1}\left(\frac{m_3-m_2}{1+m_2m_3}\right), \quad \gamma = \tan^{-1}\left(\frac{m_2-m_1}{1+m_1m_2}\right) \tag{9.79}$$

Equation of lines ℓ_1, ℓ_2 and ℓ_3

Using the point-slope formula one finds the equations representing the lines ℓ_1, ℓ_2 and ℓ_3 of figure 9-14 are given by

$$\begin{aligned}\ell_1 : &\quad y - y_3 = m_1(x - x_3)\\ \ell_2 : &\quad y - y_2 = m_2(x - x_2)\\ \ell_3 : &\quad y - y_1 = m_3(x - x_1)\end{aligned} \tag{9.80}$$

Inscribed circle

The inscribed circle associated with the triangle of figure 9-14 has its center where the **angle bisectors meet**. This point of concurrency is called the incenter of the inscribed circle. We have previously demonstrated (Chapter 4) that the incenter has the coordinates (x_I, y_I) given by

$$x_I = \frac{ax_1 + bx_2 + cx_3}{a + b + c} \qquad \text{and} \qquad y_I = \frac{ay_1 + by_2 + cy_3}{a + b + c} \tag{9.81}$$

where a, b, c represent the sides of the triangle $\triangle ABC$.

The area of triangle $\triangle ABC$ can be thought of as the area of three triangles as illustrated in the figure 9-15. Each triangle has a height r which represents the radius of the inscribed circle. We know that half the base times height is one method to obtain the area of a given triangle. Applying this method to each of the three triangles of figure 9-15 one finds

$$[ABC] = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = rs, \quad \text{where} \quad s = \frac{1}{2}(a + b + c) \tag{9.82}$$

is called the semiperimeter of the triangle. This equation can be used to express the radius r of the inscribed circle as

$$r = \frac{[ABC]}{s} \tag{9.83}$$

Using Heron's formula, the area of the triangle $\triangle ABC$ is given by

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{where} \quad s = \frac{1}{2}(a + b + c) \tag{9.84}$$

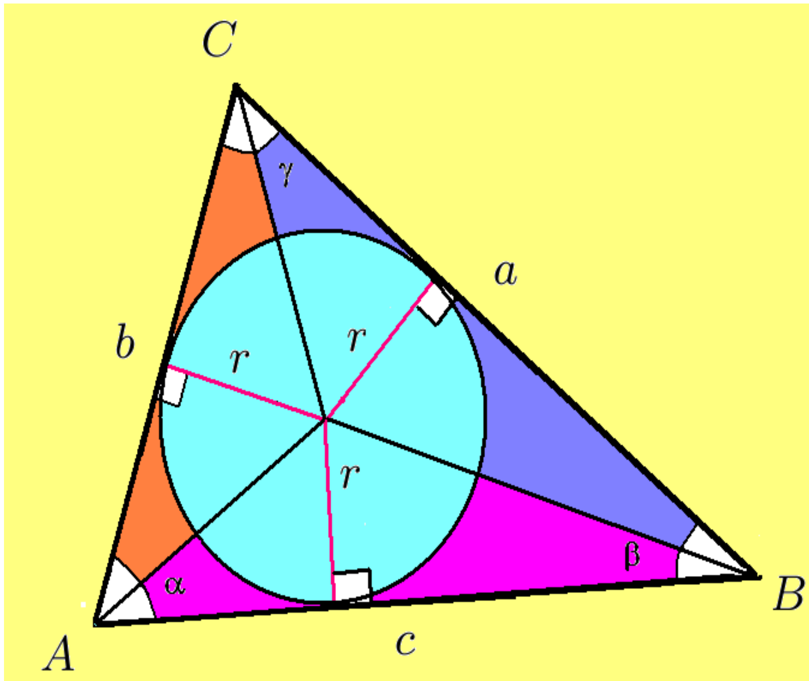


Figure 9-15. Triangle $\triangle ABC$ divided into three parts

Substitute Heron's formula for the area in equation (9.83) and express the **radius of the inscribed circle** by the formula

$$r = \frac{[ABC]}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \tag{9.85}$$

The equation for the inscribed circle can then be expressed

$$(x - x_I)^2 + (y - y_I)^2 = r^2 \tag{9.86}$$

Circumscribed circle

The center of the circumscribed circle resides at the point of concurrency of the lines which are perpendicular bisectors of each side of the given triangle $\triangle ABC$. This circumscribed circle has center at (x_c, y_c) and radius R. The circumference of the circle passes through each vertex of the triangle as illustrated in the figure 9-16.

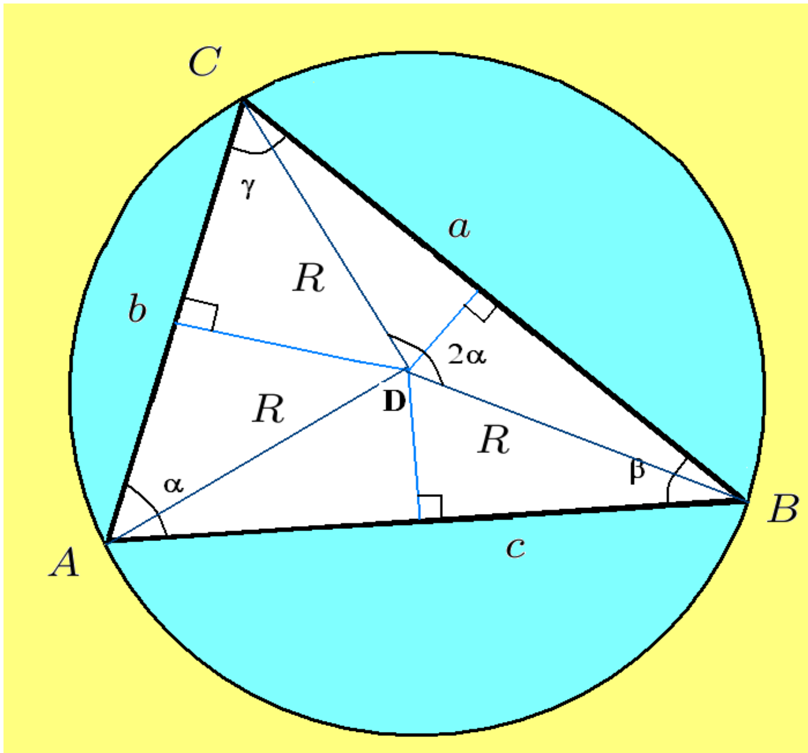


Figure 9-16. Triangle $\triangle ABC$ with circumscribed circle.

The center (x_c, y_c) of the circumscribed circle passing through the triangle vertices is called the circumcenter and can be calculated as follows. We know the slope of the side $\overline{\mathrm{BC}}$ is given by $m_2 = m_{BC} = \dfrac{y_3 - y_2}{x_3 - x_2}$ and the slope of any line perpendicular to $\overline{\mathrm{BC}}$ must have the slope

$$m_{\perp BC} = -\frac{x_3 - x_2}{y_3 - y_2} \tag{9.87}$$

which is the negative reciprocal of the slope m_2. The line which represents the perpendicular bisector of $\overline{\mathrm{BC}}$ must pass through the midpoint $(\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3))$ and by the point-slope formula for the equation of a line, we find its equation is

$$y - \frac{1}{2}(y_2 + y_3) = m_{\perp BC}(x - \frac{1}{2}(x_2 + x_3)) \tag{9.88}$$

One can verify the slope of the side $\overline{\mathrm{AB}}$ is $m_3 = m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$ and any line perpendicular to $\overline{\mathrm{AB}}$ must have the slope

$$m_{\perp AB} = -\frac{x_2 - x_1}{y_2 - y_1} \tag{9.89}$$

which is the negative reciprocal of m_3. The perpendicular bisector of line $\overline{\mathrm{AB}}$ must pass through the midpoint $(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2))$. Using the point-slope formula the perpendicular bisector line associate with side $\overline{\mathrm{AB}}$ has the form

$$y - \frac{1}{2}(y_1+y_2) = m_{\perp AB}(x - \frac{1}{2}(x_1+x_2)) \tag{9.90}$$

The intersection of the lines (9.88) and (9.90) produces the circumcenter (x_c, y_c). Solving the system of equations (9.88) and (9.90) one obtains the circumcenter coordinates

$$\begin{aligned} x_c =& \frac{(x_1^2+y_1^2)(y_3-y_2)+(x_2^2+y_2^2)(y_1-y_3)+(x_3^2+y_3^2)(y_2-y_1)}{2\,(\,x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1))} \\ y_c =& \frac{(x_1^2+y_1^2)(x_2-x_3)+(x_2^2+y_2^2)(x_3-x_1)+(x_3^2+y_3^2)(x_1-x_2)}{2\,(\,x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1))} \end{aligned} \tag{9.91}$$

Note the cyclic rotation of the subscripts associated with the first terms in both the numerator and denominator of the above expressions representing the coordinates of the circumcenter.

In the figure 9-16 observe that angle α is an inscribed angle and the angle 2α is a central angle. Applying the law of cosines to triangle $\triangle DCB$ one can write

$$\begin{aligned} a^2 =& R^2 + R^2 - 2R^2\cos(2\alpha) \\ a^2 =& 2R^2(1-\cos(2\alpha)) \qquad \text{Now use identity } \cos(2\alpha) = 1 - 2\sin^2\alpha \\ \text{to show} \quad a^2 =& 2R^2(2\sin^2\alpha) \\ \text{or} \quad a =& 2R\sin\alpha \end{aligned} \tag{9.92}$$

Recall that the area of triangle $\triangle ABC$ can be expressed

$$[ABC] = \frac{1}{2}bc\sin\alpha \tag{9.93}$$

and substituting for $\sin\alpha$ from equation (9.92) one finds the area of the triangle is related to the circumradius R by the relation

$$[ABC] = \frac{abc}{4R} \tag{9.94}$$

Again use the Heron's formula for the area of the triangle and show

$$R = \frac{abc}{4[ABC]} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \tag{9.95}$$

The equation for the circumscribed circle having circumcenter (x_c, y_c) and radius R is given by

$$(x - x_c)^2 + (y - y_c)^2 = R^2 \tag{9.96}$$

Altitudes

The altitudes of triangle $\triangle ABC$ are line segments h_a, h_b, h_c through each vertex of the triangle which are perpendicular to the sides opposite the vertex as illustrated in the figures 9-17.

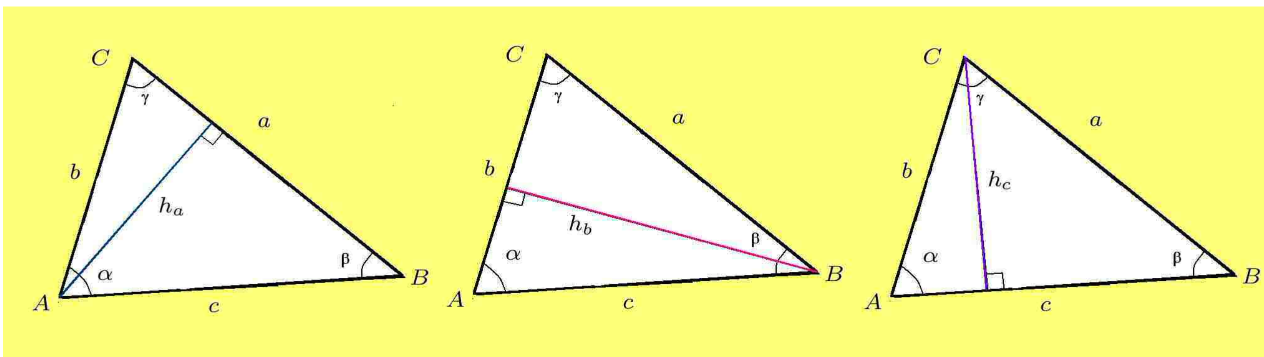


Figure 9-17. Altitudes h_a, h_b, h_c of triangle $\triangle ABC$

Using trigonometry one can verify that

$$h_a = b \sin\gamma = c \sin\beta \tag{9.97}$$

and by a cyclic rotation of the symbols one finds the relations

$$h_b = c \sin\alpha = a \sin\gamma \qquad \text{and} \qquad h_c = a \sin\beta = b \sin\alpha \tag{9.98}$$

which can also be verified using the trigonometric definition of the sine function. Using $[ABC]$ as a shorthand notation for the area of triangle $\triangle ABC$. The area of triangle $\triangle ABC$ can be expressed

$$[ABC] = \frac{1}{2} a\, h_a = \frac{1}{2} b\, h_b = \frac{1}{2} c\, h_c \tag{9.99}$$

where the sides of the triangle are treated as bases and the altitudes represent the corresponding heights. If one uses the Heron's formula for the area of the triangle, then the altitudes can be calculated using the equations

$$\begin{aligned} h_a =& \frac{2[ABC]}{a} = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} \\ h_b =& \frac{2[ABC]}{b} = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)} \\ h_c =& \frac{2[ABC]}{c} = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)} \end{aligned} \tag{9.101}$$

where

$$s = \frac{1}{2}(a+b+c) \tag{9.101}$$

is the semiperimeter of the triangle $\triangle ABC$.

Observe that the equation (9.83) gives the radius r the inscribed circle of triangle $\triangle ABC$ as $r = \dfrac{[ABC]}{s}$. One can use this result and show

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a}{2[ABC]} + \frac{b}{2[ABC]} + \frac{c}{2[ABC]} = \frac{s}{[ABC]} = \frac{1}{r} \tag{9.102}$$

where s is the semiperimeter given by equation (9.101).

The altitudes of the triangle $\triangle ABC$ intersect at a point of concurrency called **the orthocenter**. The orthocenter coordinates (x_H, y_H) can be calculated as follows. We know the slope of the line $\overline{\text{BC}}$ is given by $m_2 = \dfrac{y_2 - y_3}{x_2 - x_3}$ and the slope of any line perpendicular to the segment $\overline{\text{BC}}$ will have a slope $m_{\perp AB} = -\dfrac{1}{m_2} = -\dfrac{x_2 - x_3}{y_2 - y_3}$. Using the point-slope formula, the equation for the altitude passing through the vertex A with coordinates (x_1, y_1) is

$$y - y_1 = -\frac{x_2 - x_3}{y_2 - y_3}(x - x_1) \tag{9.103}$$

The slope of the side $\overline{\text{AB}}$ is $m_3 = \dfrac{y_2 - y_1}{x_2 - x_1}$ and the slope of any line perpendicular to $\overline{\text{AB}}$ must have the slope $m_{\perp AB} = -\dfrac{1}{m_3} = -\dfrac{x_2 - x_1}{y_2 - y_1}$. Using the point-slope formula for the equation of a line, the altitude passing through the vertex C with coordinates (x_3, y_3) is

$$y - y_3 = -\frac{x_2 - x_1}{y_2 - y_1}(x - x_3) \tag{9.104}$$

Solving the simultaneous equations (9.103) and (9.104) gives the coordinates of the orthocenter (x_H, y_H) as

$$\begin{aligned} x_H =& \frac{\lambda(y_1-y_3)+\mu(y_3-y_2)+\nu(y_2-y_1)}{y_2(x_1-x_3)+y_1(x_3-x_2)+y_3(x_2-x_1)} \\ y_H =&-\frac{\lambda(x_1-x_3)+\mu(x_3-x_2)+\nu(x_2-x_1)}{y_2(x_1-x_3)+y_1(x_3-x_2)+y_3(x_2-x_1)} \end{aligned} \quad \text{where} \quad \begin{aligned} \lambda =& x_3x_1+y_3y_1 \\ \mu =& x_2x_3+y_2y_3 \\ \nu =& x_1x_2+y_1y_2 \end{aligned} \tag{9.105}$$

Medians

The medians of triangle $\triangle ABC$ are line segments joining a vertex of the triangle to the midpoint of the opposite side. The three medians of triangle $\triangle ABC$ are illustrated in the figure 9-18. The three medians intersect at a point of concurrency (x_c, y_c) called the centroid[8] of the triangle.

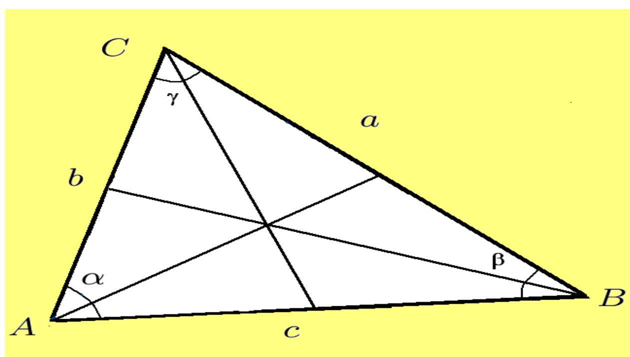


Figure 9-18. Medians of triangle $\triangle ABC$

The median connecting vertex A to the opposite side $\overline{\text{BC}}$ passes through the two points (x_1, y_1) and $(\frac{1}{2}(x_2+x_3), \frac{1}{2}(y_2+y_3))$. The slope of this line is

$$m_A = \frac{\frac{1}{2}(y_2+y_3)-y_1}{\frac{1}{2}(x_2+x_3)-x_1}$$

and the point-slope formula for the equation of the median is

$$y-y_1 = \left[\frac{\frac{1}{2}(y_2+y_3)-y_1}{\frac{1}{2}(x_2+x_3)-x_1}\right](x-x_1) \tag{9.106}$$

[8] If the triangle is made of a homogeneous material of uniform density, then the centroid of the triangle is called the center of gravity.

The median connecting the vertex C to the opposite side $\overline{\text{AB}}$ passes through the points (x_3, y_3) and $(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2))$. The slope of this line is

$$m_C = \frac{\frac{1}{2}(y_1+y_2) - y3}{\frac{1}{2}(x_1+x_2) - x_3}$$

and the point-slope formula for the equation of the median is

$$y - y_2 = \left[\frac{\frac{1}{2}(y_1+y_2) - y3}{\frac{1}{2}(x_1+x_2) - x_3}\right](x - x_3) \tag{9.107}$$

Solving the simultaneous equations (9.106) and (9.107) one finds the centroid of the triangle is given by

$$x_G = \frac{1}{3}(x_1+x_2+x_3) \qquad\qquad y_G = \frac{1}{3}(y_1+y_2+y_3) \tag{9.108}$$

Here the centroid (x_G, y_G) of any triangle is such that the abscissa x_G is the average of the x-values of the vertices and the ordinate value y_G is the average of the y-values of the vertices.

Example 9-7.

Given the triangle $\triangle ABC$ with vertices $A:(-2,8),\ B:(3,-4),\ C:(12,8)$ as illustrate in the figure 9-19.

Midpoints

The midpoints of the sides are given by

$$\begin{aligned} M_{AB} =& (\frac{1}{2}(-2+3), \frac{1}{2}(8+-4)) = (\frac{1}{2}, 2)\\ M_{BC} =& (\frac{1}{2}(3+12), \frac{1}{2}(-4+8)) = (\frac{15}{2}, 2)\\ M_{AC} =& (\frac{1}{2}(-2+12), \frac{1}{2}(8+8)) = (5,8) \end{aligned}$$

These midpoints are illustrated in the figure 9-19.

Length of sides

The sides a, b, c opposite the vertices A, B, C have the following lengths

$$\begin{aligned} b =& 12 - (-2) = 14\\ c =& \sqrt{(3-(-2))^2 + ((8-(-4))^2} = \sqrt{169} = 13\\ a =& \sqrt{(12-3)^2 + (8-(-4))^2} = \sqrt{225} = 15 \end{aligned}$$

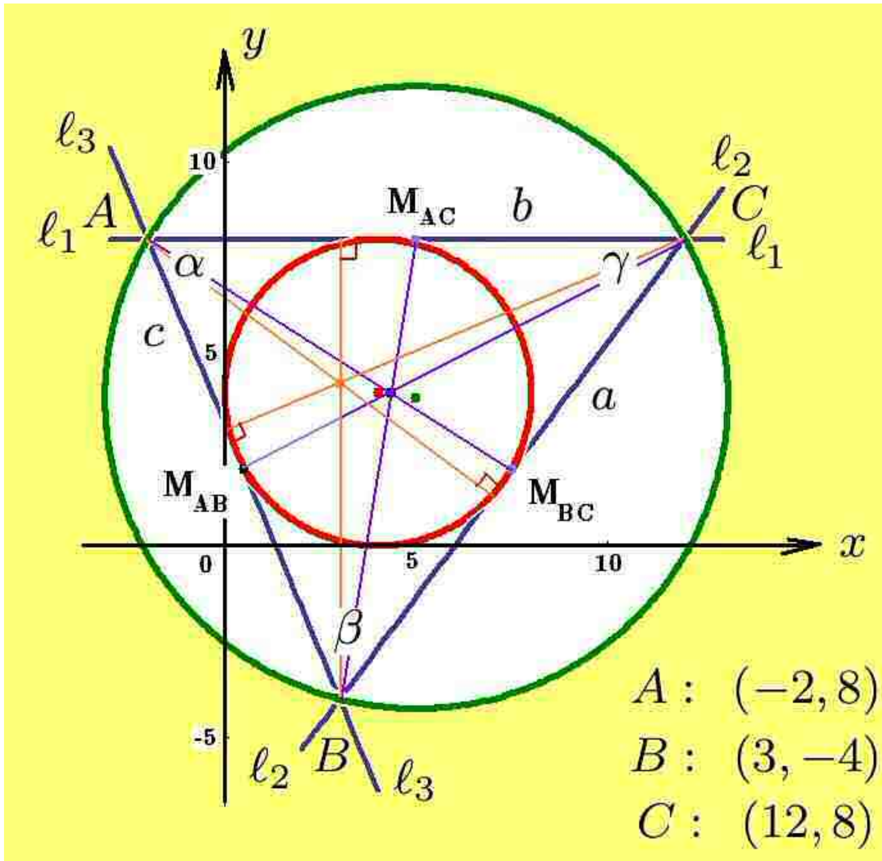


Figure 9-19. Triangle $\triangle ABC$

Slope of sides

The given triangle has sides consisting of line segments on the lines ℓ_1, ℓ_2, ℓ_3 with slopes $m = \dfrac{\text{change in y}}{\text{change in x}}$ given by

$$\begin{aligned} m_{\ell_3} =& \frac{-4-8}{3-(-2)} = -\frac{12}{5} \\ m_{\ell_2} =& \frac{8-(-4)}{12-3} = \frac{4}{3} \\ m_{\ell_1} =& \frac{8-8}{12-(-2)} = 0 \end{aligned}$$

Angles of triangle

Knowing the slopes of the lines ℓ_1, ℓ_2, ℓ_3 one can use the formula $\tan\gamma = \dfrac{m_{\ell_2} - m_{\ell_1}}{1 + m_{\ell_1} m_{\ell_2}}$ to determine the angle γ between intersecting lines. In a similar fashion one can determine the other interior angles of the given triangle. One finds

$$
\begin{aligned}
\tan\gamma =& \frac{m_{\ell_2} - m_{\ell_1}}{1 + m_{\ell_1} m_{\ell_2}} = \frac{4}{3}, \qquad \gamma = \tan^{-1}\left(\frac{4}{3}\right) = 53.1301^\circ \\
\tan\beta =& \frac{m_{\ell_3} - m_{\ell_2}}{1 + m_{\ell_2} m_{\ell_3}} = \frac{56}{33}, \qquad \beta = \tan^{-1}\left(\frac{56}{33}\right) = 59.4818^\circ \\
\tan\alpha =& \frac{m_{\ell_1} - m_{\ell_3}}{1 + m_{\ell_1} m_{\ell_3}} = \frac{12}{5}, \qquad \alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.3801^\circ
\end{aligned}
$$

As a check, observe that a summation of the angles α, β and γ gives 180°.

Equations of lines ℓ_1, ℓ_2, ℓ_3

The lines ℓ_1, ℓ_2, ℓ_3 which define the given triangle have equations determined by use of the point-slope formula to have the form

$$
\begin{aligned}
\ell_1 :& \quad y = 8 \\
\ell_2 :& \quad y + 4 = \frac{4}{3}(x - 3) \\
\ell_3 :& \quad y - 8 = \frac{-12}{5}(x + 2)
\end{aligned}
$$

Inscribed circle

The inscribed circle or incircle of the triangle touches all three sides of the triangle and has a center or incenter at the coordinates (x_I, y_I) given by equation (9.81) as

$$
x_I = \frac{15(-2) + 14(3) + 13(12)}{15 + 14 + 13} = 4, \qquad \text{and} \qquad y_I = \frac{15(8) + 14(-4) + 13(8)}{15 + 14 + 13} = 4
$$

In order to find the radius of the incircle, the equation (9.82) requires knowledge of both the area of the triangle and the semiperimeter of the triangle. One can calculate the semiperimeter as

$$
s = \frac{1}{2}(15 + 14 + 13) = 31
$$

The area of the triangle can then be calculate using Heron's formula

$$
[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = 84
$$

or one can use the base of the triangle as side $c = 14$ and the height $h_b = 12$ of the triangle as length of altitude through vertex B. The area of the triangle is then half the base times the height. The radius of the incircle is given by equation (9.83) as

$$
r = \frac{[ABC]}{s} = \frac{84}{21} = 4
$$

The equation describing the incircle is given by

$$(x-4)^2+(y-4)^{=}4^2$$

or the parametric form for the incircle is given by

$$x=4+4\cos t, \qquad y=4+4\sin t, \qquad 0\le t\le 2\pi$$

The incircle and incenter have the color red in figure 9-19.

Circumscribed circle

The easy way to calculate the center of the circumscribed circle is to construct the lines perpendicular to the sides of the triangle which pass through the midpoints of the sides or one can use the equations (9.91). One can verify that the line perpendicular to side $\overline{\text{BC}}$ and passing through the midpoint M_{BC} is given by $y-2=(-3/4)(x-15/2)$ and the line perpendicular to side $\overline{\text{AB}}$ and passing through the midpoint M_{AB} is given by $y-2=(5/12)(x-1/2)$. The intersection of these lines produces the center of the circumscribed circle at $x_O=5$, $y_O=\frac{31}{8}$. These same results can be obtained using the equations (9.91).

The radius R of the circumscribed circle is obtained from equation (9.94). One finds

$$R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}=\frac{65}{8}$$

The equation for the circumscribed circle is therefore

$$(x-5)^2+(y-\frac{31}{8})^2=\left(\frac{65}{8}\right)^2$$

or one can write the parametric form for the circumscribed circle as

$$x=5+\frac{65}{8}\cos t, \qquad y=\frac{31}{8}+\frac{65}{8}\sin t, \qquad 0\le t\le 2\pi$$

The circumscribed circle and circumcenter are illustrated in green in the figure 9-19.

Altitudes

The length of the altitudes for triangle $\triangle ABC$ are given by the equations (9.101).

$$\begin{aligned} h_a =& \frac{2[ABC]}{a} = 2\frac{84}{15} = \frac{56}{5} \\ h_b =& \frac{2[ABC]}{b} = 2\frac{84}{14} = 12 \\ h_c =& \frac{2[ABC]}{c} = 2\frac{84}{13} = \frac{168}{13} \end{aligned}$$

The equations for the lines determined by the altitudes are

$$\begin{array}{ll} \text{Line } \perp \text{ to } \overline{\text{AC}} \text{ through vertex } B & x = 3 \\ \text{Line } \perp \text{ to } \overline{\text{AB}} \text{ through vertex } C & y - 8 - \dfrac{5}{12}(x - 12) \\ \text{Line } \perp \text{ to } \overline{\text{BC}} \text{ through vertex } A & y - 8 = \dfrac{-3}{4}(x + 2) \end{array}$$

The point of concurrency of these lines produces the orthocenter $(0_x, 0_y) = (3, \frac{17}{4}$. These lines are in orange in the figure 9-19.

Medians

One can verify that the median line

$$\begin{array}{ll} \text{through } M_{BC} \text{ and vertex } A \text{ is} & y - 8 = \dfrac{-12}{19}(x + 2) \\ \text{through } M_{AB} \text{ and vertex } C \text{ is} & y - 8 = \dfrac{12}{22}(x - 12) \\ \text{through } M_{AC} \text{ and vertex } B \text{ is} & y - 8 = 6(x - 5) \end{array}$$

The point of concurrency of these three lines is the centroid $(x_c, y_c) = (\frac{13}{3}, 4)$. One can also obtain this same centroid using the equations (9.108). The median lines and centroid are in purple when examining the figure 9-19.

The circumcenter, orthocenter and centroid of a triangle will all lie on the same line which is called the Euler[9] line. One can verify that the Euler line associated with triangle $\triangle ABC$ is given by $y - \frac{17}{4} = \frac{-3}{16}(x - 3)$

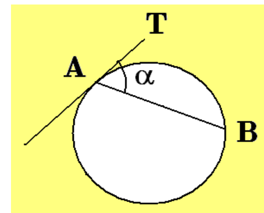
■

[9] Leonhard Euler (1707-1783) A famous Swiss mathematician.

Exercises

▶ **9-1.**

Given the tangent line T intersecting a unit circle at point A. Through the point A construct a chord $\overline{\text{AB}}$. Show that the angle $\alpha = \angle BAT$ equals $\frac{1}{2}\,\widehat{\text{AB}}$.



▶ **9-2.**

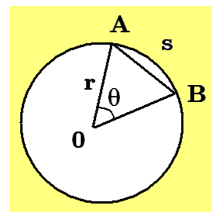
For the circle illustrated $\theta = \frac{\pi}{6}$, $r = 2$

(a) Find the arc length s.

(b) Find the area of the sector.

(c) Find the area of the segment.

(d) Find the area of the triangle $\triangle 0AB$.



▶ **9-3.** Find the equation of the circle

(a) centered at $(0,0)$ with radius 1.

(b) centered at $(5,-7)$ with radius 2.

(c) centered at $(6,8)$ with radius 3.

▶ **9-4.** Complete the square to find the center and radius of the given circle.

(a) $x^2 + y^2 - 6x + 4y - 23 = 0$

(b) $x^2 + y^2 + 10x - 4y - 20 = 0$

(c) $x^2 + y^2 - 8y - 48 = 0$

▶ **9-5.** Find the equation of the circle passing through the given points.

(a) $(3,2),\ (-2,4),\ (-1,-4)$

(b) $(1,\sqrt{15}),\ (2,\sqrt{12}),\ (3,\sqrt{7})$

(c) $(1,8),\ (5,6),\ (6,3)$

▶ **9-6.** Find the equation of the circle centered at

(a) $(2,2)$ which passes through the point $(3,4)$

(b) $(5,7)$ which passes through the point $(0,0)$

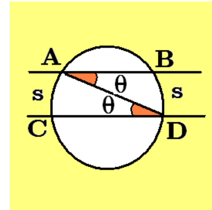
(c) $(-2,-4)$ which passes through the point $(3,0$

▶ **9-7.** Find the equation of the circle tangent to the x-axis having its center on the line $y = \frac{1}{2}x + 1$ which passes through the point $(3,5)$

► **9-8.**

Given a circle with radius $r = 6$ with lines $\overline{\text{AB}} \parallel \overline{\text{CD}}$.

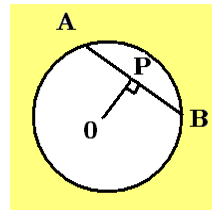
Find the arc length s if $\theta = \dfrac{\pi}{6}$



► **9-9.** Find the equation of the tangent line to the circle $(x-3)^2+(y-4)^2=25$ which touches the circle at the point $(6,8)$.

► **9-10.**

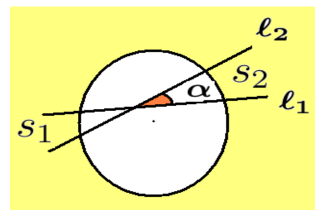
Let $\overline{\text{AB}}$ denote the chord of a circle and then construct the line from the center of the circle perpendicular to the chord and intersecting it at point P. Show that $\overline{\text{AP}} = \overline{\text{PB}}$.



► **9-11.**

Given the circle $(x-3)^2+(y-4)^2=25$ and the secant lines

$$\begin{aligned} \ell_1 :\ y =& 5 + \frac{1}{2}(x-2) \\ \ell_2 :\ y =& 5 + 3(x-2) \end{aligned}$$

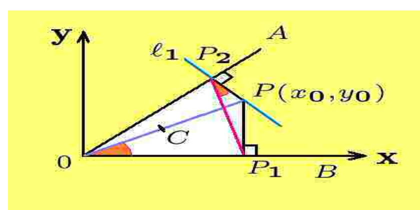


(a) Show the angle of intersection of the given lines is $\alpha = \frac{\pi}{6}$

(b) Find the average of the arc lengths s_1 and s_2.

► **9-12.**

Given the acute angle $\angle B0A$ select an arbitrary point P with coordinates (x_0, y_0) within the angle and make the following constructions.



(i) The line $\overline{\text{0P}}$

(ii) The perpendicular line from P to side $\overline{\text{0A}}$ intersecting at P_2

(iii) The perpendicular line from P to side $\overline{\text{0B}}$ intersecting at P_1

(iv) The line $\overline{P_1P_2}$

(a) Find coordinates of the midpoint C of the line segment $\overline{\text{0P}}$.

(b) Find the coordinates of the point P_1.

(c) Find the distance $r_1^2 = \overline{CP_1}^2$.

(d) Find the distance $r_2^2 = \overline{\text{CP}}^2$.

(e) Find the distance $r_3^2 = \overline{\text{0C}}^2$.

(f) Find the equation of the line $\overline{\text{0A}}$ if the slope is $m = \tan\theta$ where $\theta = \angle B0A$.

(g) Find the equation of the line ℓ_1 which is perpendicular to line $\overline{0A}$ and passing through point P.

(h) Find the coordinates of the point P_2 where lines $\overline{0A}$ and ℓ_1 intersect.

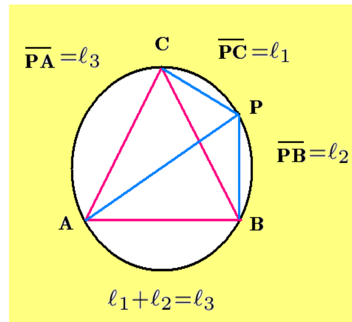
(i) Find the distance $r_4^2 = \overline{CP_2}^2$

(j) Prove the points $0, P_1, P, P_2$ all lie on a circle with radius $r_1 = r_2 = r_3 = r_4$.

(k) Prove angle $\angle B0P$ equals angle $\angle P_1P_2P$

▶ **9-13.**

An equilateral triangle is inscribed within an given circle. Select any point P on the circumference of the circle and then make the following constructions. (i) Lines to the two nearest vertices and (ii) a line to the furthest vertex. These are the lines $\overline{PC} = \ell_1, \overline{PB} = \ell_2$ and $\overline{PA} = \ell_3$ as illustrated in the accompanying figure. Use Ptolemy's theorem to show $\ell_1 + \ell_2 = \ell_3$.



▶ **9-14.** Investigate Ptolemy's theorem in the following special cases

(a) The quadrilateral is a rectangle inscribed within a circle. Construct a diagram and write out in words the conclusion of your findings.

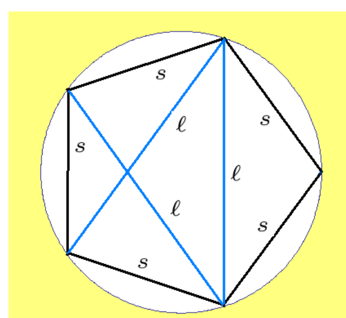
(b) The quadrilateral is a square inscribed within a circle. Construct a diagram and write out in words the conclusion of your findings.

▶ **9-15.**

Construct a pentagon inscribed within a given circle and then construct the diagonals of the pentagon.

(a) If the sides of the pentagon are of length s and the diagonals are of length ℓ show that $\ell^2 = s\ell + s^2$

(b) Show $\phi = \frac{\ell}{s}$ is a golden ratio.



▶ **9-16.**

Given the cyclic quadrilateral $ABCD$ within a circle with diameter $\overline{\mathrm{BE}} = 1$ and sides s_1, s_2, s_3, s_4.

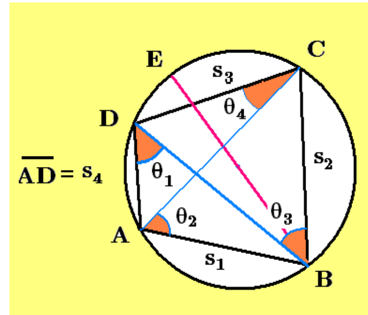
(a) Show that $s_1 = \sin\theta_1$

(b) Show that $s_2 = \sin\theta_2$

(c) Show that $s_3 = \sin\theta_3$

(d) Show that $s_4 = \sin\theta_4$

(e) In this special case express Ptolemy's theorem in terms of trigonometric functions.



▶ **9-17.** Given the triangle $\triangle ABC$ with vertices $A: (0,0),\ B: (6,0),\ C: (3,4)$

(a) Find the length of each side.

(b) Find the midpoints of each side.

(c) Find the Cartesian equation for the medians.

(d) Find the slopes of each side.

(e) Find the Cartesian equation for each side.

(f) Find the slopes of the altitudes.

(g) Find the Cartesian equation for each altitude.

(h) Find the area of triangle $\triangle ABC$.

▶ **9-18.** Given triangle $\triangle ABC$ with vertices $A: (0,0),\ B: (2+\sqrt{3}, 1),\ C: (4,0)$

(a) Find the vertex angles $\angle A = \alpha, \angle B = \beta, \angle C = \gamma$

(b) Find the length of the triangle sides.

(c) Find the area of triangle $\triangle ABC$.

▶ **9-19.** Given the triangle $\triangle ABC$ with vertices $A: (0,0),\ B: (42,0),\ C: (12,16)$

(a) Find the Cartesian equation for the medians.

(b) Find where the medians intersect.

(c) Check your answer with equation (9.108)

▶ **9-20.** Given the triangle $\triangle ABC$ with vertices $A: (0,0),\ B: (40,0),\ C: (35,12)$

(a) Find the Cartesian equation for the altitudes.

(b) Find where the altitudes intersect.

(c) Check your answers with equation (9.105).

▶ **9-21.** Given the triangle $\triangle ABC$ with vertices $A: \ (0,0), \ B: \ (120,0), \ C: \ (8,15)$

(a) Find the incenter.

(b) Find the radius of the inscribed circle.

(c) Find the area of triangle $\triangle ABC$.

(d) Find the circumcenter.

(e) Find the circumradius.

▶ **9-22.** Given the triangle $\triangle ABC$ with vertices $A: \ (0,0), \ B: \ (39,0), \ C: \ (32,24)$

(a) Find the area of this triangle.

(b) Find the altitudes h_a, h_b, h_c.

(c) Find the radius r of the inscribed circle.

(d) Find the radius R of the circumscribed circle.

(e) Find the incenter.

(f) Find the cirumcenter.

▶ **9-23.** Given the triangle $\triangle ABC$ with vertices $A: \ (0,0), \ B: \ (51,0), \ C: \ (9,41)$

(a) Find the area of this triangle.

(b) Find the altitudes h_a, h_b, h_c.

(c) Find the radius r of the inscribed circle.

(d) Find the radius R of the circumscribed circle.

(e) Find the incenter.

(f) Find the circumcenter.

(g) Find the orthocenter.

(h) Find the centroid.

Geometry

Chapter 10

Mathematics and Geometry

Vectors used in geometry

A **vector** is a quantity that has **both** a magnitude and a direction while a **scalar** quantity has only a magnitude. Some examples of vector and scalar quantities are

Vectors	**Scalars**
Force	temperature
velocity	speed
acceleration	time
momentum	size of angle
torque	energy
angular velocity	length

A vector can be represented by a line segment with an arrowhead attached at one end to indicate the **direction** of the vector, while the length of the line segment represents the magnitude of the vector. In this text vectors are represented by capital letters with an arrow on top. For example, the vectors $\vec{A}$, $\frac{1}{2}\vec{A}$, $2\vec{A}$, $-\vec{A}$ are illustrated in the figure 10-1.

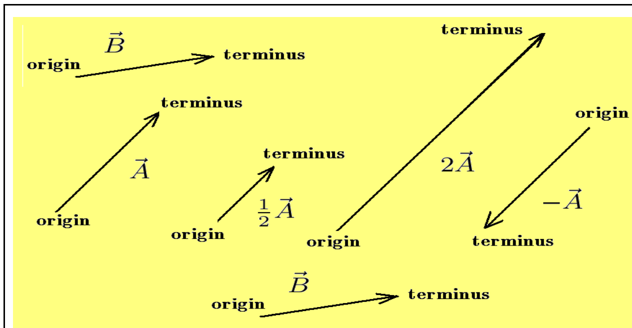


Figure 10-1.
Scalar multiples of vector $\vec{A}$

The magnitude of the vector $\vec{A}$ is denoted A or $|\vec{A}|$ and represents the length of the vector $\vec{A}$. The tail end of the vector is called the vector origin and the arrowhead of the vector is called the terminus.

Properties of vectors

Some important properties of vectors are

1. Two vectors $\vec{A}$ and $\vec{B}$ are equal if they have the **same magnitude (length) and direction**. Equality is denoted by $\vec{A}=\vec{B}$.
2. The magnitude of a vector is **a nonnegative scalar quantity**. The magnitude of a vector $\vec{B}$ is denoted by the symbols B or $|\vec{B}|$ and in some textbooks the length of the vector is denoted $\parallel \vec{B} \parallel$.
3. A vector $\vec{B}$ is equal to zero only if its magnitude is zero. A vector whose magnitude is zero is called the **zero or null vector** and denoted by the symbol $\vec{0}$.
4. Multiplication of a nonzero vector $\vec{A}$ by a positive scalar m is denoted by $m\vec{A}$ and produces a new vector whose direction is the same as $\vec{A}$ but whose magnitude is m times the magnitude of $\vec{A}$. Symbolically, $|m\vec{A}| = m|\vec{A}|$. If m is a negative scalar the direction of $m\vec{A}$ is opposite to that of the direction of $\vec{A}$. In figure 10-1 several vectors obtained from $\vec{A}$ by scalar multiplication are exhibited.
5. Vectors are considered as **"free vectors".** The term "**free vector**" is used to mean the following. Any vector may be moved to a new position in space provided that in the new position it is **parallel to and has the same direction as its original position.** See for example the vector $\vec{B}$ in the figure 10-1. In many of the examples that follow, there are times when a given vector is moved to a convenient point in space in order to emphasize a special geometrical or physical concept.

Vector Addition and Subtraction

Let $\vec{C} = \vec{A} + \vec{B}$ denote **the sum of two vectors** $\vec{A}$ and $\vec{B}$. To find the vector sum $\vec{A}+\vec{B}$, slide **the origin of the vector** $\vec{B}$ **to the terminus point of the vector** $\vec{A}$, then draw the line **from the origin of** $\vec{A}$ **to the terminus of** $\vec{B}$ to represent $\vec{C}$. Alternatively, start with the vector $\vec{B}$ and place the origin of the vector $\vec{A}$ at the terminus point of $\vec{B}$ to construct the vector $\vec{B}+\vec{A}$. Adding vectors in this way employs the **parallelogram law for vector addition** which is illustrated in the figure 10-2. Note that vector addition is commutative. That is, using the shifted vectors $\vec{A}$ and $\vec{B}$, as illustrated in the figure 10-2, the commutative law for vector addition $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, is illustrated using the parallelogram illustrated. The addition of vectors can be thought of as connecting the origin and terminus of directed line segments.

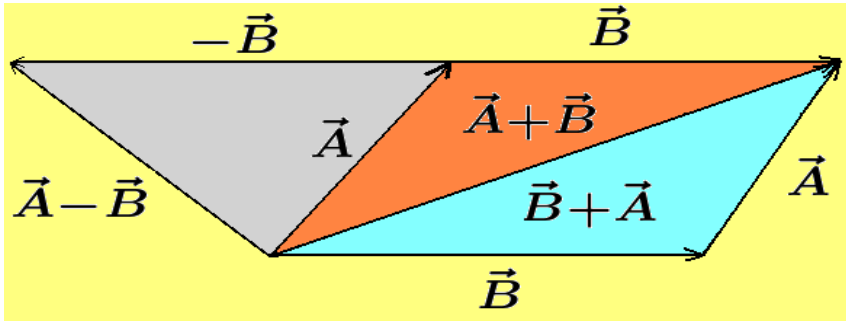


Figure 10-2. Parallelogram law for vector addition

If $\vec{F} = \vec{A} - \vec{B}$ denotes **the difference of two vectors** $\vec{A}$ and $\vec{B}$, then $\vec{F}$ is determined by the above rule for vector addition by writing $\vec{F} = \vec{A} + (-\vec{B})$. Thus, subtraction of the vector $\vec{B}$ from the vector $\vec{A}$ is represented by the addition of the vector $-\vec{B}$ to $\vec{A}$. In figure 10-2 observe that the vectors $\vec{A}$ and $\vec{B}$ are free vectors and have been translated to appropriate positions to illustrate the concepts of addition and subtraction. The sum of two or more force vectors is sometimes referred to as **the resultant force**. In general, the **resultant force** acting on an object is calculated by using **a vector addition** of all the forces acting on the object.

Vector addition and notation

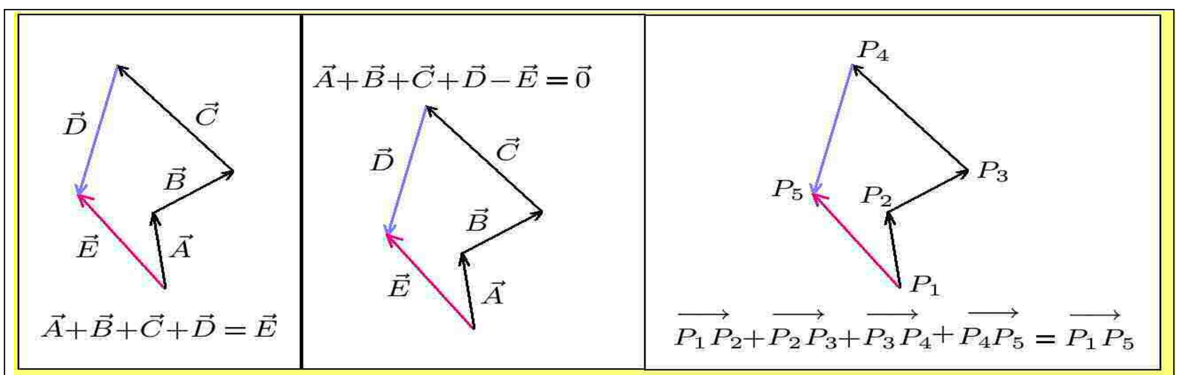


Figure 10-3. Vector addition

The figure 10-3 illustrates two notations for representing vector addition. In the first two notations note that the representation of the vector $\vec{E}$ does not change the orientation of the arrow head. Only the sign of the vector $\vec{E}$ changes. The last notation illustrates that the vectors can be represented by directed line segments if the endpoints of the vectors are known. Also observe there is a difference in the notations $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ because $\overrightarrow{P_1P_2} = -\overrightarrow{P_2P_1}$.

Linear combination

If there exists scaler constants $c_1, c_2, \ldots, c_n$, not all zero, together with a set of vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$, such that

$$\vec{A} = c_1\vec{A}_1 + c_2\vec{A}_2 + \cdots + c_n\vec{A}_n,$$

then the vector $\vec{A}$ is said to be **a linear combination** of the vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$.

Two nonzero vectors $\vec{A}$ and $\vec{B}$ are said to be **linearly dependent** if it is possible to find scalars $k_1,\ k_2$ not both zero, such that the equation

$$k_1\vec{A} + k_2\vec{B} = \vec{0} \tag{10.1}$$

is satisfied. If $k_1 = 0$ and $k_2 = 0$ are the only scalars for which the above equation is satisfied, then the vectors $\vec{A}$ and $\vec{B}$ are said to be **linearly independent**.

The above definitions can be interpreted geometrically. If $k_1 \neq 0$, then equation (10.1) implies that $\vec{A} = -\frac{k_2}{k_1}\vec{B} = m\vec{B}$ showing that $\vec{A}$ is a scalar multiple of $\vec{B}$. That is, $\vec{A}$ and $\vec{B}$ have the same direction and therefore, they are called **collinear vectors**. If $\vec{A}$ and $\vec{B}$ are not collinear, then they are linearly independent **(noncollinear)**. If two nonzero vectors $\vec{A}$ and $\vec{B}$ are linearly independent, then any vector $\vec{C}$ lying in the plane of $\vec{A}$ and $\vec{B}$ can be expressed as a linear combination of the these vectors. Construct as in figure 10-4 a parallelogram with diagonal $\vec{C}$ and sides parallel to the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide.

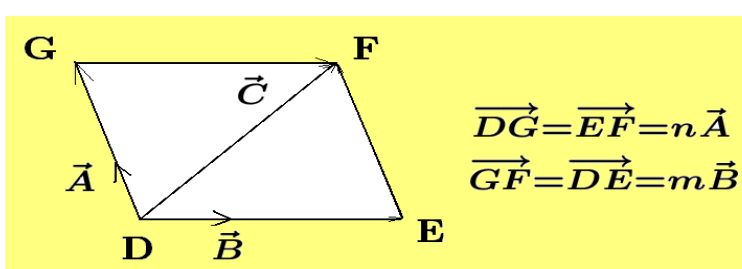


Figure 10-4. Vector $\vec{C}$ is a linear combination of vectors $\vec{A}$ and $\vec{B}$.

Since the vector side $\overrightarrow{DE}$ is parallel to $\vec{B}$ and the vector side $\overrightarrow{EF}$ is parallel to $\vec{A}$, then there exists scalars m and n such that $\overrightarrow{DE} = m\vec{B}$ and $\overrightarrow{EF} = n\vec{A}$. With vector addition,

$$\vec{C} = \overrightarrow{DE} + \overrightarrow{EF} = m\vec{B} + n\vec{A} \tag{10.2}$$

which shows that $\vec{C}$ is a linear combination of the vectors $\vec{A}$ and $\vec{B}$.

Example 10-1. Show that the medians of a triangle meet at a trisection point.

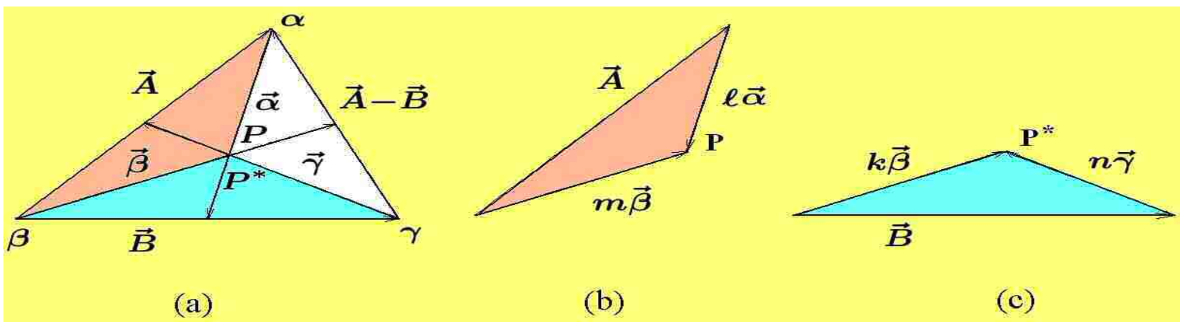


Figure 10-5. Constructing medians of a triangle

Solution: Let the sides of a triangle with vertices α, β, γ be denoted by the vectors $\vec{A}$, $\vec{B}$, and $\vec{A} - \vec{B}$ as illustrated in the figure 10-5. Further, let $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ denote the vectors from the respective vertices of α, β, γ to the **midpoints of the opposite sides.** By using vector addition one can construct the following vector equations

$$\vec{A} + \vec{\alpha} = \frac{1}{2}\vec{B} \qquad \vec{B} + \frac{1}{2}(\vec{A} - \vec{B}) = \vec{\beta} \qquad \vec{B} + \vec{\gamma} = \frac{1}{2}\vec{A}. \tag{10.3}$$

Let the vectors $\vec{\alpha}$ and $\vec{\beta}$ intersect at a point designated by P, Similarly, let the vectors $\vec{\beta}$ and $\vec{\gamma}$ intersect at the point designated P^*. The problem is to show that the points P and P^* are the same. Figures 10-5(b) and 10-5(c) illustrate that for suitable scalars k, ℓ, m, n, the points P and P^* determine the vectors equations

$$\vec{A} + \ell\vec{\alpha} = m\vec{\beta} \qquad \text{and} \qquad \vec{B} + n\vec{\gamma} = k\vec{\beta}. \tag{10.4}$$

In these equations the scalars k, ℓ, m, n are unknowns to be determined. Use the set of equations (10.3), to solve for the vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ in terms of the vectors $\vec{A}$ and $\vec{B}$ and show

$$\vec{\alpha} = \frac{1}{2}\vec{B} - \vec{A} \qquad \vec{\beta} = \frac{1}{2}(\vec{A} + \vec{B}) \qquad \vec{\gamma} = \frac{1}{2}\vec{A} - \vec{B}. \tag{10.5}$$

These equations can now be substituted into the equations (10.4) to yield, after some simplification, the equations

$$(1-\ell-\frac{m}{2})\vec{A}=(\frac{m}{2}-\frac{\ell}{2})\vec{B} \quad \text{and} \quad (\frac{k}{2}-\frac{n}{2})\vec{A}=(1-n-\frac{k}{2})\vec{B}.$$

Since the vectors $\vec{A}$ and $\vec{B}$ are linearly independent (noncollinear), the scalar coefficients in the above equation must equal zero, because if these scalar coefficients were not zero, then the vectors $\vec{A}$ and $\vec{B}$ would be linearly dependent (collinear) and a triangle would not exist. By equating to zero the scalar coefficients in these equations, there results the simultaneous scalar equations

$$(1-\ell-\frac{m}{2})=0, \qquad (\frac{m}{2}-\frac{\ell}{2})=0, \qquad (\frac{k}{2}-\frac{n}{2})=0, \qquad (1-n-\frac{k}{2})=0$$

The solution of these equations produces the fact that $k=\ell=m=n=\frac{2}{3}$ and hence the conclusion $P=P^*$ is a trisection point.

■

Unit Vectors

A vector having length or magnitude of one is called **a unit vector**. If $\vec{A}$ is a nonzero vector of length $|\vec{A}|$, a unit vector in the direction of $\vec{A}$ is obtained by multiplying the vector $\vec{A}$ by the scalar $m=\frac{1}{|\vec{A}|}$. The unit vector so constructed is denoted

$$\hat{\mathbf{e}}_A=\frac{\vec{A}}{|\vec{A}|} \qquad \text{and satisfies} \qquad |\,\hat{\mathbf{e}}_A|=1.$$

The symbol $\hat{\mathbf{e}}$ is reserved for unit vectors and the notation $\hat{\mathbf{e}}_A$ is to be read **"a unit vector in the direction of $\vec{A}$."** The hat or carat ($\hat{}$) notation is used to represent a **unit vector or normalized vector.**

The figure 10-6 illustrates unit base vectors $\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$ in the directions of the positive x,y,z-coordinate axes in a rectangular three dimensional Cartesian coordinate system. These unit base vectors in the direction of the x,y,z axes have historically been represented by a variety of notations. Some of the more common notations employed in various textbooks to denote **rectangular unit base vectors** are

$$\hat{\imath},\ \hat{\jmath},\ \hat{k}, \quad \hat{\mathbf{e}}_x,\ \hat{\mathbf{e}}_y,\ \hat{\mathbf{e}}_z, \quad \hat{\imath}_1,\ \hat{\imath}_2,\ \hat{\imath}_3, \quad \overline{1}_x,\ \overline{1}_y,\ \overline{1}_z, \quad \hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$$

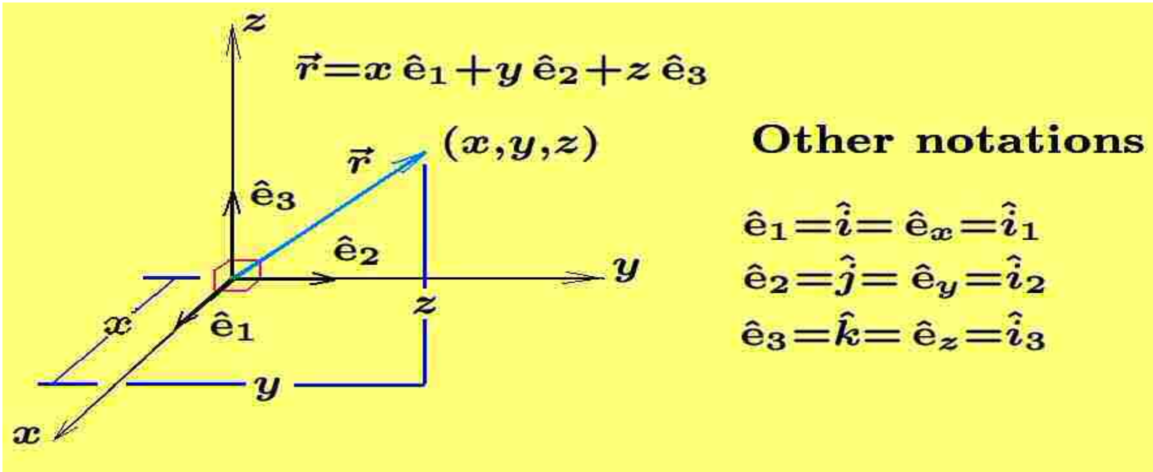


Figure 10-6. Cartesian axes.

The notation $\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$ to represent the unit base vectors in the direction of the x, y, z axes will be used in the discussions that follow as this notation makes it easier to generalize vector concepts to n-dimensional spaces.

Observe in figure 10-6 the position vector $\vec{r}$ from the origin to the point (x, y, z) is obtained by vector addition of the vectors $x\,\hat{\mathbf{e}}_1,\ y\,\hat{\mathbf{e}}_2,\ z\,\hat{\mathbf{e}}_3$.

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

Also make note of the fact that the base vectors are all perpendicular to one another.

Scalar or Dot Product (inner product)

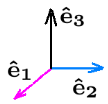
The **scalar or dot product of two vectors** is sometimes referred to as **an inner product of vectors**.

Definition (Dot product) The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is denoted

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\,\cos\theta, \tag{10.6}$$

and represents the magnitude of $\vec{A}$ times the magnitude $\vec{B}$ times the cosine of θ, where θ is the angle between the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide.

The angle between any two of the orthogonal unit base vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in Cartesian coordinates is 90° or $\frac{\pi}{2}$ radians. Using the results $\cos\frac{\pi}{2} = 0$ and $\cos 0 = 1$, there results the following dot product relations for these unit vectors



$$\begin{array}{lll} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = 1 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 = 0 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 = 0 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = 1 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_2 = 0 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3 = 0 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = 0 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 = 1 \end{array} \tag{10.7}$$

The dot product satisfies the following properties

Commutative law $\quad \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

Distributive law $\quad \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

Magnitude squared $\quad \vec{A} \cdot \vec{A} = A^2 = |\vec{A}|^2$

which are proved using the definition of a dot product.

The physical interpretation of **projection** can be assigned to the dot product as is illustrated in figure 10-7. In this figure $\vec{A}$ and $\vec{B}$ are nonzero vectors with $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_B$ **unit vectors in the directions of** $\vec{A}$ **and** $\vec{B}$, respectively. The figure 10-7 illustrates the physical interpretation of the following equations:

$$\hat{\mathbf{e}}_B \cdot \vec{A} = \frac{\vec{B}}{\left|\vec{B}\right|} \cdot \vec{A} = |\vec{A}| \cos\theta = \text{Projection of } \vec{A} \text{ onto direction of } \hat{\mathbf{e}}_B$$

$$\hat{\mathbf{e}}_A \cdot \vec{B} = \frac{\vec{A}}{\left|\vec{A}\right|} \cdot \vec{B} = |\vec{B}| \cos\theta = \text{Projection of } \vec{B} \text{ onto direction of } \hat{\mathbf{e}}_A.$$

In general, the dot product of a nonzero vector $\vec{A}$ with a unit vector $\hat{\mathbf{e}}$ is given by $\vec{A} \cdot \hat{\mathbf{e}} = \hat{\mathbf{e}} \cdot \vec{A} = |\vec{A}||\hat{\mathbf{e}}| \cos\theta$ and represents the projection of the given vector onto the direction of the unit vector. The **dot product of a vector with a unit vector** is a **basic fundamental concept** which arises in a variety of science and engineering applications.

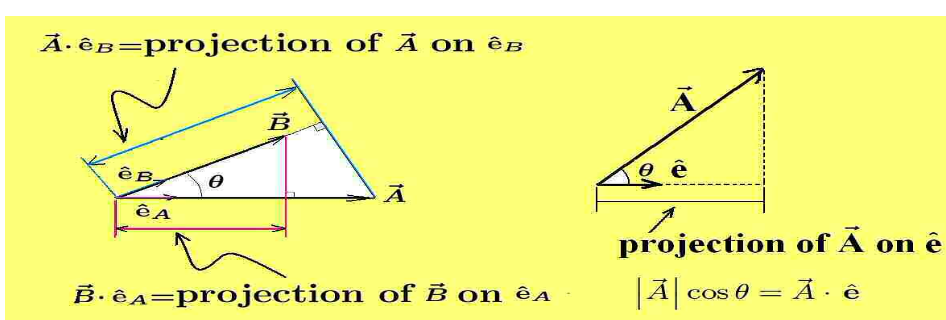


Figure 10-7. Projection of one vector onto another.

Observe that if the dot product of two vectors is zero, $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = 0$, then this implies that either $\vec{A} = \vec{0}$, $\vec{B} = \vec{0}$, or $\theta = \frac{\pi}{2}$. If $\vec{A}$ and $\vec{B}$ are **both nonzero vectors** and their **dot product is zero**, then the angle between these vectors, when their origins coincide, must be $\theta = \frac{\pi}{2}$. One can then say **the vector $\vec{A}$ is perpendicular to the vector $\vec{B}$** or one can state that **the projection of $\vec{B}$ on $\vec{A}$ is zero.** If $\vec{A}$ and $\vec{B}$ are nonzero vectors and $\vec{A}\cdot\vec{B} = 0$, then the vectors $\vec{A}$ and $\vec{B}$ are said to be **orthogonal vectors**.

Direction Cosines Associated With Vectors

Let $\vec{A}$ be a nonzero vector having its origin at the origin of a rectangular Cartesian coordinate system. The dot products

$$\vec{A}\cdot\hat{\mathbf{e}}_1 = A_1 \qquad \vec{A}\cdot\hat{\mathbf{e}}_2 = A_2 \qquad \vec{A}\cdot\hat{\mathbf{e}}_3 = A_3 \tag{10.8}$$

represent respectively, **the components or projections of the vector $\vec{A}$** onto the x, y and z-axes. The projections A_1, A_2, A_3 of the vector $\vec{A}$ onto the coordinate axes are scalars which are called **the components of the vector $\vec{A}$**. From the definition of the dot product of two vectors, the scalar components of the vector $\vec{A}$ satisfy the equations

$$A_1 = \vec{A}\cdot\hat{\mathbf{e}}_1 = |\vec{A}|\cos\alpha, \qquad A_2 = \vec{A}\cdot\hat{\mathbf{e}}_2 = |\vec{A}|\cos\beta, \qquad A_3 = \vec{A}\cdot\hat{\mathbf{e}}_3 = |\vec{A}|\cos\gamma, \tag{10.9}$$

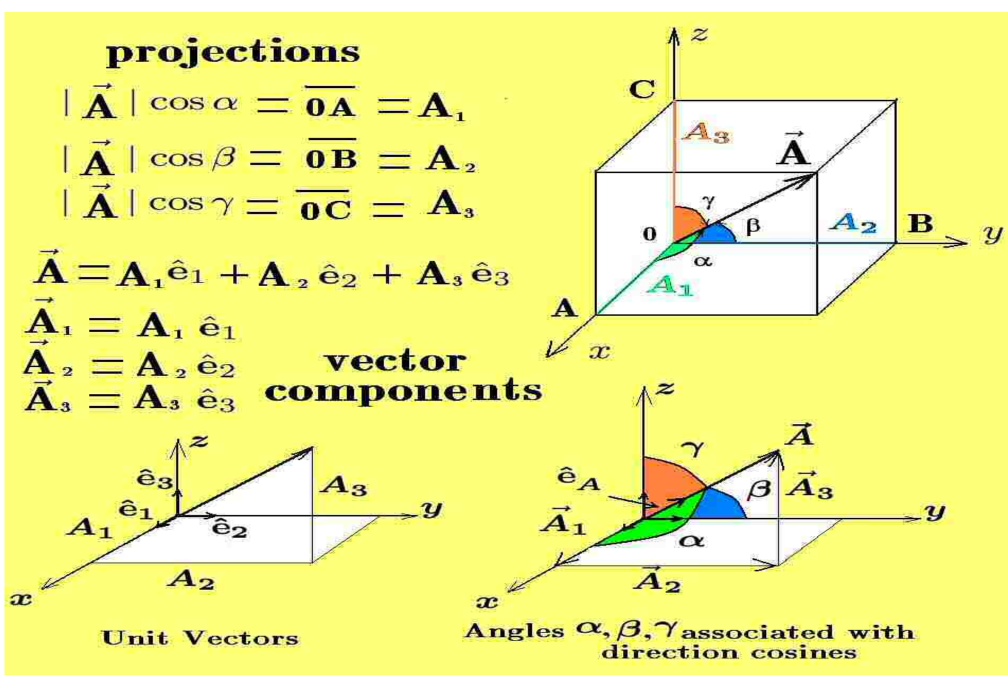


Figure 10-8. Projections, vector components and direction cosines

where $\alpha,\ \beta,\ \gamma$ are respectively, the smaller angles between the vector $\vec{A}$ and the $x,\ y,\ z$ coordinate axes. The cosine of these angles are referred to as the **direction cosines** of the vector $\vec{A}$. These angles are illustrated in figure 10-8. The vector quantities

$$\vec{A}_1 = A_1\,\hat{\mathbf{e}}_1, \qquad \vec{A}_2 = A_2\,\hat{\mathbf{e}}_2, \qquad \vec{A}_3 = A_3\,\hat{\mathbf{e}}_3 \tag{10.10}$$

are called **the vector components of the vector $\vec{A}$.** From the **addition property of vectors,** the vector components of $\vec{A}$ may be added to obtain

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3 = |\vec{A}|(\cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3) = |\vec{A}|\ \hat{\mathbf{e}}_A \tag{10.11}$$

This vector representation $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ is called the **component form of the vector $\vec{A}$** and the unit vector $\hat{\mathbf{e}}_A = \cos\alpha\ \hat{\mathbf{e}}_1 + \cos\beta\ \hat{\mathbf{e}}_2 + \cos\gamma\ \hat{\mathbf{e}}_3$ is **a unit vector in the direction of $\vec{A}$.**

Any numbers proportional to the direction cosines of a line are called **the direction numbers of the line.** Show for $a:b:c$ the direction numbers of a line which are not all zero, then the direction cosines are given by

$$\cos\alpha = \frac{a}{r} \qquad \cos\beta = \frac{b}{r} \qquad \cos\gamma = \frac{c}{r},$$

where $r = \sqrt{a^2+b^2+c^2}$.

Cauchy-Schwarz inequality

Let $\vec{A}$ and $\vec{B}$ denote any two nonzero vectors and then examine the quantity

$$y = y(t) = |t\vec{A} - \vec{B}|^2 = (t\vec{A} - \vec{B})\cdot(t\vec{A} - \vec{B}) = t^2\,|\vec{A}|^2 - 2t\,(\vec{A}\cdot\vec{B}) + |\vec{B}|^2 \tag{10.12}$$

where $|\vec{A}|^2 = \vec{A}\cdot\vec{A}$ and $|\vec{B}|^2 = \vec{B}\cdot\vec{B}$. The quantity y is a parabola

$$y = \alpha\,t^2 - \beta\,t + \gamma \tag{10.13}$$

where $\alpha = |\vec{A}|^2$, $\beta = 2\,\vec{A}\cdot\vec{B}$ and $\gamma = |\vec{B}|^2$. Observe the quantity is always positive because a vector dotted with itself always gives the length of the vector squared. The quantity y will always be positive if the discriminant is negative which requires that

$$\beta^2 - 4\alpha\,\gamma \le 0 \quad \text{or} \quad 4(\vec{A}\cdot\vec{B})^2 \le 4\,|\vec{A}|^2\,|\vec{B}|^2 \tag{10.14}$$

Taking the square root of both sides of equation (10.14) and simplifying gives the Cauchy-Schwarz inequality

$$|\vec{A}\cdot\vec{B}| \le |\vec{A}|\,|\vec{B}| \tag{10.15}$$

The Cauchy-Schwarz inequality tells us that the absolute value of the dot product of two nonzero vectors is always less than or equal to the product of the vector lengths. As an exercise, show the equality sign holds only when the two vectors are collinear.

Triangle inequality

If $\vec{A}$ and $\vec{B}$ are nonzero vectors, then one can write

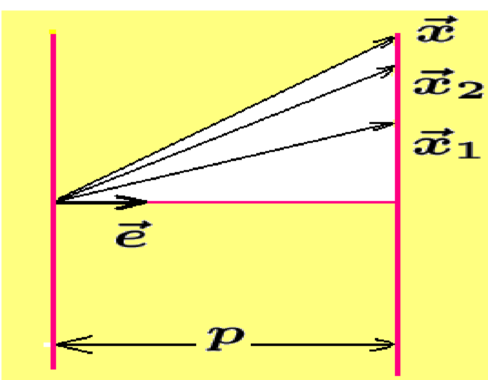
$$\begin{aligned} |\vec{A}+\vec{B}|^2 = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) =& \vec{A}\cdot\vec{A} + 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B} \\ |\vec{A}+\vec{B}|^2 =& |\vec{A}|^2 + 2\,\vec{A}\cdot\vec{B} + |\vec{B}|^2 \\ |\vec{A}+\vec{B}|^2 \le& |\vec{A}|^2 + 2|\vec{A}||\vec{B}| + |\vec{B}|^2 \qquad (10.16) \\ & \text{by using the Cauchy-Schwarz inequality} \\ |\vec{A}+\vec{B}|^2 \le& \left(|\vec{A}| + |\vec{B}|\right)^2 \end{aligned}$$

Taking the square root of both sides of the equation (10.16) one can obtain the triangle inequality

$$|\vec{A}+\vec{B}| \le |\vec{A}| + |\vec{B}| \qquad (10.17)$$

which tells use the absolute value associated with a sum of two vectors will always be less than or equal to a summation of the vector lengths.

Example 10-2.



Sketch a large version of the letter H. Consider the sides of the letter H as parallel lines a distance of p units apart. Place a unit vector $\hat{\mathbf{e}}$ perpendicular to the left side of H and pointing toward the right side of H. Construct a vector $\vec{x}_1$ which runs from the origin of $\hat{\mathbf{e}}$ to a point on the right side of the H. Observe that $\hat{\mathbf{e}}\cdot\vec{x}_1 = p$ is a projection of $\vec{x}_1$ on $\hat{\mathbf{e}}$. Now construct another vector $\vec{x}_2$, different from $\vec{x}_1$, again from the origin of $\hat{e}$ to the right side of the H. Note also that $\hat{e}\cdot\vec{x}_2 = p$ is a projection of $\vec{x}_2$ on the vector $\hat{\mathbf{e}}$. Draw still another vector $\vec{x}$, from the origin of $\hat{\mathbf{e}}$ to the right side of H which is different from $\vec{x}_1$ and $\vec{x}_2$. Observe that the dot product $\hat{\mathbf{e}}\cdot\vec{x} = p$ representing the projection of $\vec{x}$ on $\hat{\mathbf{e}}$ still produces the value p.

Assume you are given $\hat{\mathbf{e}}$ and p and are asked to solve the vector equation $\hat{\mathbf{e}}\cdot\vec{x} = p$ for the unknown quantity $\vec{x}$. You might think that there is some operation like vector division, for example $\vec{x} = p/\hat{\mathbf{e}}$, whereby $\vec{x}$ can be determined. However, if you look at the equation $\hat{\mathbf{e}} \cdot \vec{x} = p$ as a projection, one can observe that there would be an infinite number of solutions to this equation and for this reason there is **no division of vector quantities.**

■

Component Form for Dot Product

Let $\vec{A}$, $\vec{B}$ be two nonzero vectors represented in the component form

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3, \qquad \vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_1\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$$

The **dot product** of these two vectors is

$$\vec{A}\cdot\vec{B} = (A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3)\cdot(B_1\,\hat{\mathbf{e}}_1 + B_1\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3) \tag{10.18}$$

and this product can be expanded utilizing the distributive and commutative laws to obtain

$$\begin{aligned}\vec{A}\cdot\vec{B} =\ & A_1B_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 + A_1B_2\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 + A_1B_3\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3\\ &+ A_2B_1\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 + A_2B_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 + A_2B_3\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_3\\ &+ A_3B_1\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_1 + A_3B_2\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_2 + A_3B_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_3.\end{aligned} \tag{10.19}$$

From the previous properties of the dot product of unit vectors, given by equations (10.7), the dot product reduces to the form

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3. \tag{10.20}$$

Thus, the dot product of two vectors produces a scalar quantity which is **the sum of the products of like components.**

From the definition of the dot product the following useful relationship results:

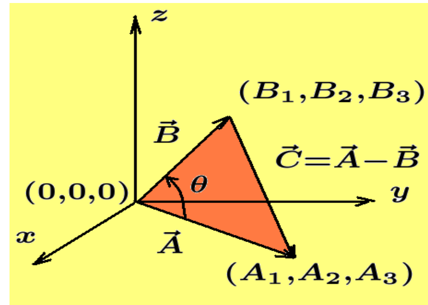
$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = |\vec{A}||\vec{B}|\cos\theta. \tag{10.21}$$

This relation may be used to find the angle between two vectors when their origins are made to coincide and their components are known. If in equation (10.21) one makes the substitution $\vec{B} = \vec{A}$, there results the special formula

$$\vec{A}\cdot\vec{A} = A_1^2 + A_2^2 + A_3^2 = A\cdot A\cos 0 = A^2 = |\vec{A}|^2. \tag{10.22}$$

Consequently, the magnitude of a vector $\vec{A}$ is given by the square root of the sum of the squares of its components or $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2}$

The previous dot product definition is motivated by the **law of cosines** as the following arguments demonstrate. Consider three points having the coordinates $(0,0,0)$, (A_1, A_2, A_3), and (B_1, B_2, B_3) and plot these points in a Cartesian coordinate system as illustrated. Denote by $\vec{A}$ the directed line segment from $(0,0,0)$ to (A_1, A_2, A_3) and denote by $\vec{B}$ the directed straight-line segment from $(0,0,0)$ to (B_1, B_2, B_3).



One can now apply the distance formula from analytic geometry to represent the lengths of these line segments. We find these lengths can be represented by

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \quad \text{and} \quad |\vec{B}| = \sqrt{B_1^2 + B_2^2 + B_3^2}.$$

Let $\vec{C} = \vec{A} - \vec{B}$ denote the directed line segment from (B_1, B_2, B_3) to (A_1, A_2, A_3). The length of this vector is found to be

$$|\vec{C}| = \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2}.$$

If θ is the angle between the vectors $\vec{A}$ and $\vec{B}$, the law of cosines is employed to write

$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\theta.$$

Substitute into this relation the distances of the directed line segments for the magnitudes of $\vec{A}$, $\vec{B}$ and $\vec{C}$. Expanding the resulting equation shows that the law of cosines takes on the form

$$(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2 = A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 - 2|\vec{A}||\vec{B}| \cos\theta.$$

With elementary algebra, this relation simplifies to the form

$$A_1 B_1 + A_2 B_2 + A_3 B_3 = |\vec{A}||\vec{B}| \cos\theta$$

which suggests the definition of a dot product as $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$.

Example 10-3. If $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ is a given vector in component form, then

$$\vec{A}\cdot\vec{A} = A_1^2 + A_2^2 + A_3^3 \quad \text{and} \quad |\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

The vector

$$\widehat{e}_A = \frac{1}{|\vec{A}|}\vec{A} = \frac{A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3}{\sqrt{A_1^2 + A_2^2 + A_3^2}} = \cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3$$

is **a unit vector in the direction of $\vec{A}$**, where

$$\cos\alpha = \frac{A_1}{|\vec{A}|}, \qquad \cos\beta = \frac{A_2}{|\vec{A}|}, \qquad \cos\gamma = \frac{A_3}{|\vec{A}|}$$

are the direction cosines of the vector $\vec{A}$. The dot product

$$\widehat{e}_A\cdot\widehat{e}_A = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

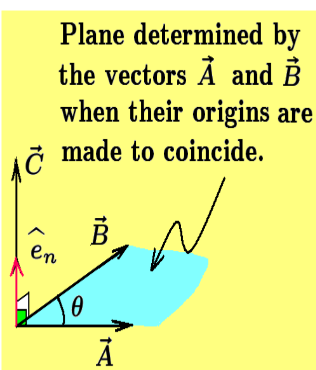
shows that **the sum of squares of the direction cosines is unity.**

■

The Cross Product or Outer Product

The **cross or outer product of two nonzero vectors** $\vec{A}$ and $\vec{B}$ is denoted using the notation $\vec{A}\times\vec{B}$ and represents the construction of a new vector $\vec{C}$ defined as

$$\vec{C} = \vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{\mathbf{e}}_n, \tag{10.23}$$



where θ is the smaller angle between the two nonzero vectors $\vec{A}$ and $\vec{B}$ when their origins coincide, and $\hat{\mathbf{e}}_n$ is a unit vector perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide. The direction of $\hat{\mathbf{e}}_n$ is determined by the **right-hand rule**. Place the fingers of your right-hand in the direction of $\vec{A}$ and rotate the fingers toward the vector $\vec{B}$, then the thumb of the right-hand points in the direction $\vec{C}$.

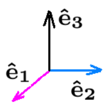
The vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ then form a **right-handed system.**[1] Note that the cross product $\vec{A}\times\vec{B}$ is a vector which will always be perpendicular to the vectors $\vec{A}$ and $\vec{B}$, whenever $\vec{A}$ and $\vec{B}$ are linearly independent nonzero vectors.

[1] Note many European technical books use left-handed coordinate systems which produces results different from using a right-handed coordinate system.

A special case of the above definition occurs when $\vec{A}\times\vec{B}=\vec{0}$ and in this case one can state that **either** $\theta=0$, which implies the vectors $\vec{A}$ and $\vec{B}$ **are parallel** or $\vec{A}=\vec{0}$ or $\vec{B}=\vec{0}$. Further note that the equation $\vec{A}\times\vec{B}=\vec{A}\times\vec{C}$ **does not** imply that $\vec{B}=\vec{C}$.

Unit vectors

Use the above definition of a cross product and show that the orthogonal unit vectors $\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$ satisfy the relations



$$\begin{array}{lll}
\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1=\vec{\mathbf{0}} & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1=-\hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1=\hat{\mathbf{e}}_2\\
\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2=\hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_2=\vec{\mathbf{0}} & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_2=-\hat{\mathbf{e}}_1\\
\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3=-\hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3=\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_3=\vec{\mathbf{0}}
\end{array}\qquad\textbf{(10.24)}$$

Properties of the Cross Product

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}\quad\text{(noncommutative)}$$
$$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}\quad\text{(distributive law)}$$
$$m(\vec{A}\times\vec{B})=(m\vec{A})\times\vec{B}=\vec{A}\times(m\vec{B})\quad m\text{ a scalar}$$
$$\vec{A}\times\vec{A}=\vec{0}\quad\text{since }\vec{A}\text{ is parallel to itself.}$$

Let $\vec{A}=A_1\,\hat{\mathbf{e}}_1+A_2\,\hat{\mathbf{e}}_2+A_3\,\hat{\mathbf{e}}_3$ and $\vec{B}=B_1\,\hat{\mathbf{e}}_1+B_2\,\hat{\mathbf{e}}_2+B_3\,\hat{\mathbf{e}}_3$ be two nonzero vectors in component form and form the cross product $\vec{A}\times\vec{B}$ to obtain

$$\vec{A}\times\vec{B}=(A_1\,\hat{\mathbf{e}}_1+A_2\,\hat{\mathbf{e}}_2+A_3\,\hat{\mathbf{e}}_3)\times(B_1\,\hat{\mathbf{e}}_1+B_2\,\hat{\mathbf{e}}_2+B_3\,\hat{\mathbf{e}}_3).\qquad(10.25)$$

The cross product can be expanded by using the distributive law to obtain

$$\begin{aligned}\vec{A}\times\vec{B}=\ &A_1B_1\,\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1+A_1B_2\,\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2+A_1B_3\,\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3\\
&+A_2B_1\,\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1+A_2B_2\,\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_2+A_2B_3\,\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3\\
&+A_3B_1\,\hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1+A_3B_2\,\hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_2+A_3B_3\,\hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_3.\end{aligned}\qquad(10.26)$$

Simplification by using the previous results from equation (10.24) produces the important cross product formula

$$\vec{A}\times\vec{B}=(A_2B_3-A_3B_2)\,\hat{\mathbf{e}}_1+(A_3B_1-A_1B_3)\,\hat{\mathbf{e}}_2+(A_1B_2-A_2B_1)\,\hat{\mathbf{e}}_3,\qquad(10.27)$$

This result that can be expressed in **the determinant form**[2]

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{\mathbf{e}}_1&\hat{\mathbf{e}}_2&\hat{\mathbf{e}}_3\\A_1&A_2&A_3\\B_1&B_2&B_3\end{vmatrix}=\begin{vmatrix}A_2&A_3\\B_2&B_3\end{vmatrix}\hat{\mathbf{e}}_1-\begin{vmatrix}A_1&A_3\\B_1&B_3\end{vmatrix}\hat{\mathbf{e}}_2+\begin{vmatrix}A_1&A_2\\B_1&B_2\end{vmatrix}\hat{\mathbf{e}}_3.\qquad(10.90)$$

[2] For more information on determinants see chapter 5.

In summary, the cross product of two vectors $\vec{A}$ and $\vec{B}$ is a new vector $\vec{C}$, where

$$\vec{C} = \vec{A} \times \vec{B} = C_1\,\hat{\mathbf{e}}_1 + C_2\,\hat{\mathbf{e}}_2 + C_3\,\hat{\mathbf{e}}_3 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

with components

$$C_1 = A_2B_3 - A_3B_2, \qquad C_2 = A_3B_1 - A_1B_3, \qquad C_3 = A_1B_2 - A_2B_1 \tag{10.91}$$

Geometric Interpretation

A geometric interpretation that can be assigned to the magnitude of the cross product of two vectors is illustrated in figure 10-9.

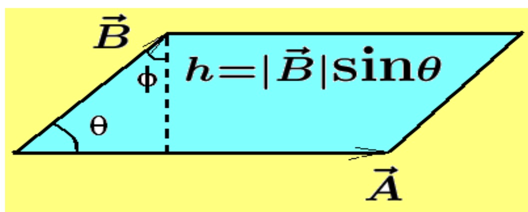


Figure 10-9. Parallelogram with sides $\vec{A}$ and $\vec{B}$.

The area of the parallelogram having the vectors $\vec{A}$ and $\vec{B}$ for its sides is given by

$$\text{Area} = |\vec{A}| \cdot h = |\vec{A}||\vec{B}|\sin\theta = |\vec{A} \times \vec{B}|. \tag{10.30}$$

Here $|\vec{B}|\cos\phi = |\vec{B}|\sin(\frac{\pi}{2} - \theta) = |\vec{B}|\sin\theta$ is the projection of $\vec{B}$ onto a line perpendicular to the vector $\vec{A}$ in the plane of the vectors $\vec{A}$ and $\vec{B}$. Therefore, the magnitude of the cross product of two vectors represents the **area of the parallelogram formed from these vectors when their origins are made to coincide.**

Example 10-4. (Area of parallelogram) Consider the parallelogram illustrated in the figure 10-10 with vertices $(0,0)$, (a,b), (a,c), $(a+c, b+d)$ Define the vectors

$$\begin{aligned} \vec{A} =& a\,\hat{\mathbf{e}}_1 + b\,\hat{\mathbf{e}}_2, \quad |\vec{A}| = A = \sqrt{a^2+b^2} \\ \vec{B} =& c\,\hat{\mathbf{e}}_1 + d\,\hat{\mathbf{e}}_2, \quad |\vec{B}| = B = \sqrt{c^2+d^2} \\ \vec{A}\cdot\vec{B} =& ac + bd, \end{aligned}$$

then the altitude of the parallelogram is given by $h = B\sin\theta$, where $B = |\vec{B}|$. The area of the parallelogram is the base times the altitude or

$$\text{Area} = A\,B\sin\theta = A\,B\sqrt{1-\cos^2\theta} \tag{10.31}$$

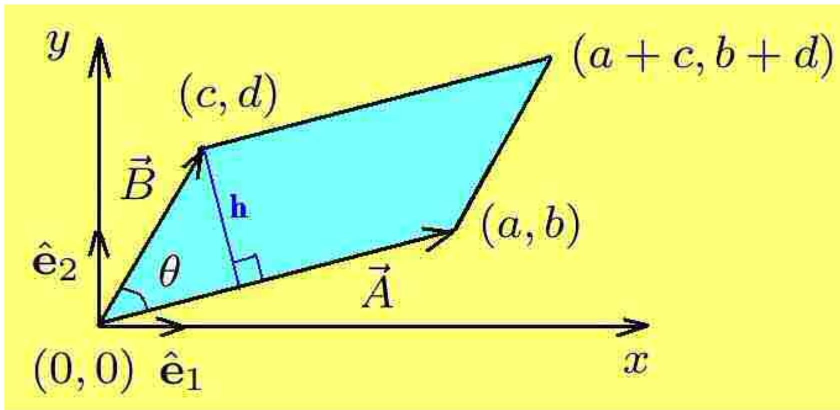


Figure 10-10. Parallelogram

Use the definition of the dot product of vectors

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta \quad \text{to write} \qquad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{A\,B} \tag{10.32}$$

to express the equation (10.31) in the form

$$\text{Area} = A\,B\sqrt{1-\left[\frac{\vec{A}\cdot\vec{B}}{A\,B}\right]^2} = \sqrt{A^2\,B^2 - (\vec{A}\cdot\vec{B})^2}$$

Using some algebra this expression can be written in a different form.

$$\begin{aligned}\text{Area}^2 =& A^2B^2 - (\vec{A}\cdot\vec{B})^2 = (a^2+b^2)(c^2+d^2) - (ac+bd)^2\\ =& a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 - [a^2c^2 + acbd + acbd + b^2d^2]\\ =& ad(ad-bc) + bc(bc-ad)\\ =& (ad-bc)^2\end{aligned} \tag{10.33}$$

Observe that the cross product $\vec{A}\times\vec{B}$ can be expressed

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \hat{\mathbf{e}}_3(ad-bc) \tag{10.34}$$

so that the area of the parallelogram can be expressed

$$\text{Area} = \left|\vec{A}\times\vec{B}\right| = |ad-bc|$$

■

Example 10-5. (Area of a triangle)

Find the area of the triangle having the vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$.

Solution

Define the position vectors

$$\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3, \qquad \vec{r}_2 = x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2 + z_2\,\hat{\mathbf{e}}_3, \qquad \vec{r}_3 = x_3\,\hat{\mathbf{e}}_1 + y_3\,\hat{\mathbf{e}}_2 + z_3\,\hat{\mathbf{e}}_3$$

to the given points defining the vertices and observe that the sides of the triangle in figure 10-11 are defined by the vectors

$$\vec{A} = \vec{r}_2 - \vec{r}_1 \qquad \text{and} \qquad \vec{B} = \vec{r}_3 - \vec{r}_1$$

The situation is illustrated in the figure 10-11. We know the magnitude of the cross product $\vec{A} \times \vec{B}$ gives the area of a parallelogram having the vector sides $\vec{A}$ and $\vec{B}$. Hence, the area of the triangle is one-half the area of the parallelogram or

$$\text{Area} = \frac{1}{2}\left|\vec{A} \times \underline{\mathrm{B}}\right|$$

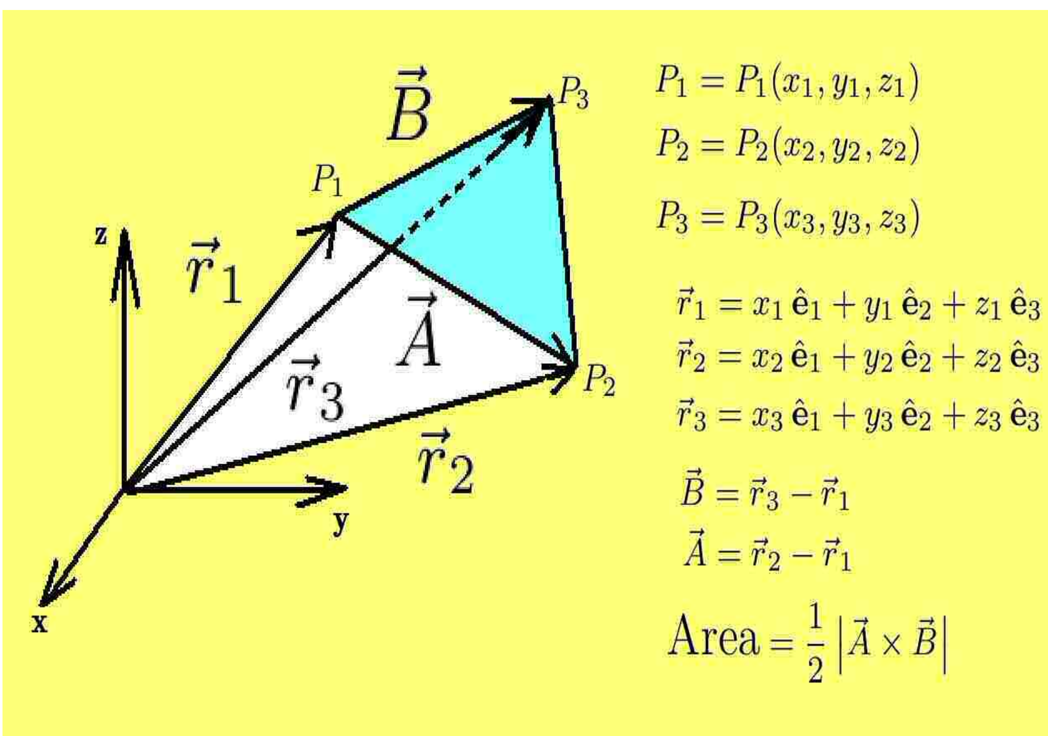


Figure 10-11. Triangle formed from points P_1, P_2 and P_3

∎

Vector Identities

The following vector identities are often needed to simplify various equations in science and engineering.

1. $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (10.35)
2. $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ (10.36)

   An identity known as the **triple scalar product**.
3. $$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \vec{C}\left[\vec{D}\cdot(\vec{A}\times\vec{B})\right] - \vec{D}\left[\vec{C}\cdot(\vec{A}\times\vec{B})\right] = \vec{B}\left[\vec{A}\cdot(\vec{C}\times\vec{D})\right] - \vec{A}\left[\vec{B}\cdot(\vec{C}\times\vec{D})\right] \tag{10.37}$$
4. $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ (10.38)

   The quantity $\vec{A}\times(\vec{B}\times\vec{C})$ is called a **triple vector product**.
5. $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$ (10.39)
6. The triple vector product satisfies

   $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = \vec{0}$ (10.40)

Note that in the triple scalar product $\vec{A}\cdot(\vec{B}\times\vec{C})$ the parenthesis is sometimes omitted because $(\vec{A}\cdot\vec{B})\times\vec{C}$ is meaningless and so $\vec{A}\cdot\vec{B}\times\vec{C}$ can have only one meaning. The parenthesis just emphasizes this one meaning.

A physical interpretation can be assigned to the triple scalar product $\vec{A}\cdot(\vec{B}\times\vec{C})$ is that **its absolute value represents the volume of the parallelepiped[3] formed by the three noncoplaner vectors $\vec{A}, \vec{B}, \vec{C}$ when their origins are made to coincide.** The absolute value is needed because sometimes the triple scalar product is negative. This physical interpretation can be obtained from the following analysis.

In figure 10-12 note the following.

(a) The magnitude $|\vec{B}\times\vec{C}|$ represents the area of the parallelogram $PQRS$.

(b) The unit vector $\hat{\mathbf{e}}_n = \dfrac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}$ is normal to the plane containing the vectors $\vec{B}$ and $\vec{C}$.

[3] A parallelepiped is a solid figure which has parallelograms for each of its faces.

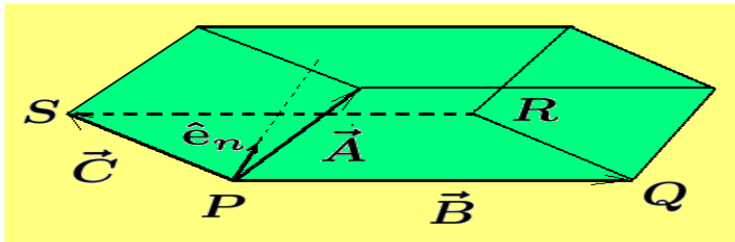


Figure 10-12. Triple scalar product and volume.

(c) The dot product $\vec{A}\cdot\hat{\mathbf{e}}_n = \vec{A}\cdot\dfrac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|} = h$ represents the projection of $\vec{A}$ on $\hat{\mathbf{e}}_n$ and produces the height of the parallelepiped. These results demonstrate that the triple scalar product gives the volume of the parallelepiped

$$\left|\vec{A}\cdot(\vec{B}\times\vec{C})\right| = |\vec{B}\times\vec{C}|\,h = (\text{Area of base})(\text{Height}) = \text{Volume}.$$

Therefore, one can state that the magnitude of the triple scalar product is the volume of the parallelepiped formed when **the origins** of the three vectors forming the sides are made to coincide.

Example 10-6. Show that the triple scalar product satisfies the relations

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Note **the cyclic rotation of the symbols** in the above relations where the first symbol is moved to the last position and the second and third symbols are each moved to the left. This is called **a cyclic permutation** of the symbols.

Solution Use the determinant form for the cross product and express the triple scalar product as a determinant as follows.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) =(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3)\cdot\begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) =(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3)\cdot[(B_2C_3 - B_3C_2)\,\hat{\mathbf{e}}_1 - (B_1C_3 - B_3C_1)\,\hat{\mathbf{e}}_2 + (B_1C_2 - B_2C_1)\,\hat{\mathbf{e}}_3]$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) =A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1)$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) =A_1\begin{vmatrix}B_2 & B_3\\ C_2 & C_3\end{vmatrix} - A_2\begin{vmatrix}B_1 & B_3\\ C_1 & C_3\end{vmatrix} + A_3\begin{vmatrix}B_1 & B_2\\ C_1 & C_2\end{vmatrix} = \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

Determinants have the property[4] that the interchange of two rows of a determinant changes its sign. One can then show

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_2 \end{vmatrix}$$

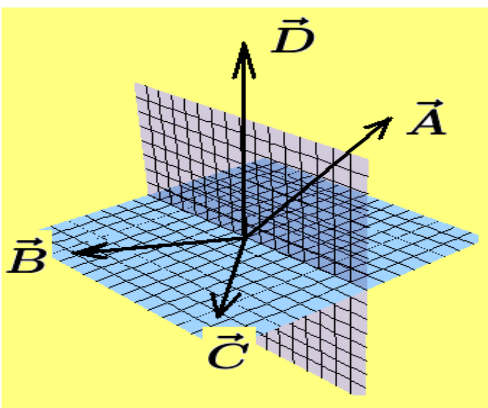
or

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

■

Example 10-7. For nonzero vectors $\vec{A}, \vec{B}, \vec{C}$ show that the triple vector product satisfies $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

That is, the triple vector product is not associative and the order of execution of the cross product is important.



Solution Let $\vec{B} \times \vec{C} = \vec{D}$ denote the vector perpendicular to the plane determined by the vectors $\vec{B}$ and $\vec{C}$. The vector $\vec{A} \times \vec{D} = \vec{E}$ is a vector perpendicular to the plane determined by the vectors $\vec{A}$ and $\vec{D}$ and therefore must lie in the plane of the vectors $\vec{B}$ and $\vec{C}$. One can then say the vectors $\vec{B}, \vec{C}$ and $\vec{A} \times (\vec{B} \times \vec{C})$ are coplanar and consequently there must exist scalars α and β such that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \alpha \vec{B} + \beta \vec{C} \tag{10.41}$$

In a similar fashion one can show that the vectors $(\vec{A} \times \vec{B}) \times \vec{C}$, $\vec{A}$ and $\vec{B}$ are coplanar so that there exists constants γ and δ such that

$$(\vec{A} \times \vec{B}) \times \vec{C} = \gamma \vec{A} + \delta \vec{B} \tag{10.42}$$

The equations (10.41) and (10.42) show that in general

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

■

[4] See chapter 6 for properties of determinants.

Example 10-8. Show that the triple vector product satisfies

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\,\vec{B} - (\vec{A} \cdot \vec{B})\,\vec{C}$$

Solution Use the results from the previous example showing there exists scalars α and β such that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \alpha\vec{B} + \beta\vec{C} \tag{10.43}$$

Let $\vec{B} \times \vec{C} = \vec{D}$ and write

$$\vec{A} \times \vec{D} = \alpha\vec{B} + \beta\vec{C} \tag{10.44}$$

Take the dot product of both sides of equation (10.44) with the vector $\vec{A}$ to obtain the triple scalar product

$$\vec{A} \cdot (\vec{A} \times \vec{D}) = \alpha(\vec{A} \cdot \vec{B}) + \beta(\vec{A} \cdot \vec{C})$$

By the permutation properties of the triple scalar product one can write

$$\vec{A} \cdot (\vec{A} \times \vec{D}) = \vec{A} \cdot (\vec{D} \times \vec{A}) = \vec{D} \cdot (\vec{A} \times \vec{A}) = 0 = \alpha(\vec{A} \cdot \vec{B}) + \beta(\vec{A} \cdot \vec{C}) \tag{10.45}$$

The above result holds because $\vec{A} \times \vec{A} = \vec{0}$ and implies

$$\alpha(\vec{A} \cdot \vec{B}) = -\beta(\vec{A} \cdot \vec{C}) \quad \text{or} \quad \frac{\alpha}{\vec{A} \cdot \vec{C}} = \frac{-\beta}{\vec{A} \cdot \vec{B}} = \lambda$$

where λ is a scalar. This shows that the equation (10.43) can be expressed in the form

$$\vec{A} \times (\vec{B} \times \vec{C}) = \lambda(\vec{A} \cdot \vec{C})\vec{B} - \lambda(\vec{A} \cdot \vec{B})\vec{C} \tag{10.46}$$

which shows that the vectors $\vec{A} \times (\vec{B} \times \vec{C})$ and $(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ are collinear. The equation (10.46) must hold for all vectors $\vec{A}, \vec{B}, \vec{C}$ and so it must be true in the special case $\vec{A} = \hat{\mathbf{e}}_2,\ \vec{B} = \hat{\mathbf{e}}_1,\ \vec{C} = \hat{\mathbf{e}}_2$ where equation (10.46) reduces to

$$\hat{\mathbf{e}}_2 \times (\,\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2) = \hat{\mathbf{e}}_1 = \lambda\,\hat{\mathbf{e}}_1 \quad \text{which implies} \quad \lambda = 1$$

■

Example 10-9. Use vectors to derive the **law of sines** for the triangle illustrated in the figure 10-13.

Solution The sides of the given triangle are formed from the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and since these vectors are free vectors they can be moved to the positions illustrated in figure 10-13.

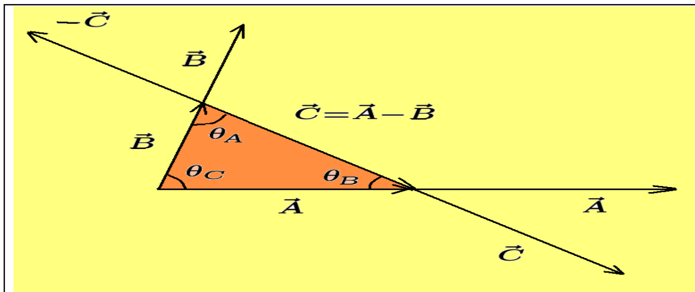


Figure 10-13. Triangle for law of sines.

Also sketch the vector $-\vec{C}$ as illustrated. The new positions for the vectors $\vec{A}, \vec{B}, \vec{C}$ and $-\vec{C}$ are constructed to better visualize certain vector cross products associated with the law of sines.

Examine figure 10-13 and note the following cross products

$$\vec{C} \times \vec{A} = (\vec{A} - \vec{B}) \times \vec{A} = \vec{A} \times \vec{A} - \vec{B} \times \vec{A} = -\vec{B} \times \vec{A} = \vec{A} \times \vec{B}$$

and $\quad \vec{B} \times (-\vec{C}) = \vec{B} \times (-\vec{A} + \vec{B}) = \vec{B} \times (-\vec{A}) + \vec{B} \times \vec{B} = \vec{A} \times \vec{B}.$

Taking the magnitude of the above cross products gives

$$|\vec{C} \times \vec{A}| = |\vec{A} \times \vec{B}| = |\vec{B} \times (-\vec{C})|$$

or

$$AC \sin \theta_B = AB \sin \theta_C = BC \sin \theta_A.$$

Dividing by the product of the vector magnitudes ABC produces the law of sines

$$\frac{\sin \theta_A}{A} = \frac{\sin \theta_B}{B} = \frac{\sin \theta_C}{C}.$$

■

Example 10-10. Use vectors to derive the **law of cosines** for the triangle illustrated.

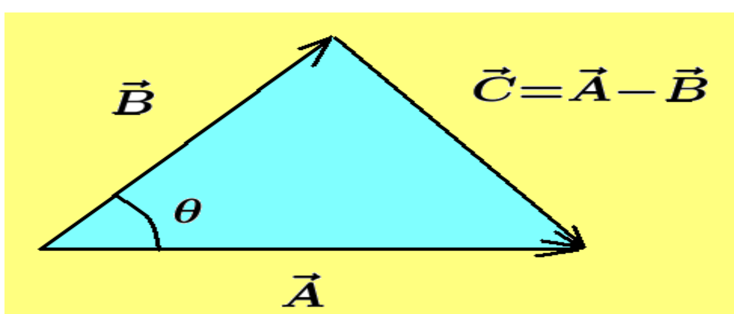


Figure 10-14. Triangle for law of cosines.

Solution Let $\vec{C} = \vec{A} - \vec{B}$ so that the dot product of $\vec{C}$ with itself gives

$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\vec{A} \cdot \vec{B}$$

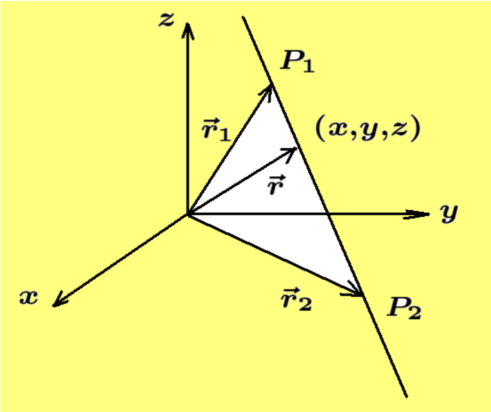
or

$$C^2 = A^2 + B^2 - 2AB\cos\theta,$$

where $A = |\vec{A}|, \quad B = |\vec{B}|, \quad C = |\vec{C}|$ represent the magnitudes of the vector sides.

■

Example 10-11. Find the **vector equation of a line which passes through the two given points** $P_1(x_1, y_1, z_1)$ **and** $P_2(x_2, y_2, z_2)$.



Solution Let

$$\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3$$
$$\text{and} \quad \vec{r}_2 = x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2 + z_2\,\hat{\mathbf{e}}_3$$

denote position vectors to the points P_1 and P_2 respectively and let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector of any other variable point on the line. Observe that the vector $\overrightarrow{P_1P_2} = \vec{r}_\ell = \vec{r}_2 - \vec{r}_1$ is parallel to the line through the points P_1 and P_2 and represents the direction vector of the line. By vector addition the (x, y, z) position on the line is given by

$$\vec{r} = \vec{r}(\lambda) = \vec{r}_1 + \lambda\vec{r}_\ell \qquad -\infty < \lambda < \infty \tag{10.47}$$

where λ is a scalar parameter[5]. Note that as λ varies from 0 to 1 the position vector $\vec{r}$ moves from $\vec{r}_1$ to $\vec{r}_2$. An alternative form for the equation of the line is given by

$$\vec{r} = \vec{r}_2 + \lambda^*(\vec{r}_1 - \vec{r}_2) \qquad -\infty < \lambda^* < \infty$$

where λ^* is some other scalar parameter and the vector $\vec{r}_2 - \vec{r}_1$ is a direction vector the negative of $\vec{r}_\ell$. This second form for the line has the position vector $\vec{r}$ moving from $\vec{r}_2$ to $\vec{r}_1$ as λ^* varies from 0 to 1. The vector $\pm(\vec{r}_2 - \vec{r}_1)$ is called **the direction vector of the line.**

[5] The parameter λ can be replaced by some other parameter such as t, s, u, v etc.

Define the quantities $a = x_2 - x_1$, $b = y_2 - y_1$ and $c = z_2 - z_1$ as a set of **direction numbers for the line.** The direction vector for the line is then $\vec{r}_\ell = a\,\hat{\mathbf{e}}_1 + b\,\hat{\mathbf{e}}_2 + c\,\hat{\mathbf{e}}_3$ then the vector equation for the line is

$$\vec{r} = \vec{r}_1 + \vec{r}_\ell\,\lambda \tag{10.55}$$

where λ is a parameter. Expanding the equation (10.55) one obtains

$$x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3 + a\lambda\,\hat{\mathbf{e}}_1 + b\lambda\,\hat{\mathbf{e}}_2 + c\lambda\,\hat{\mathbf{e}}_3$$

The parametric form for the line is obtained by **equating like components** to obtain

$$x = x_1 + a\lambda, \quad y = y_1 + b\lambda, \quad z = z_1 + c\lambda$$

The line can also be represented in **the symmetric form**

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda \tag{10.49}$$

where a, b, c are direction numbers different from zero.

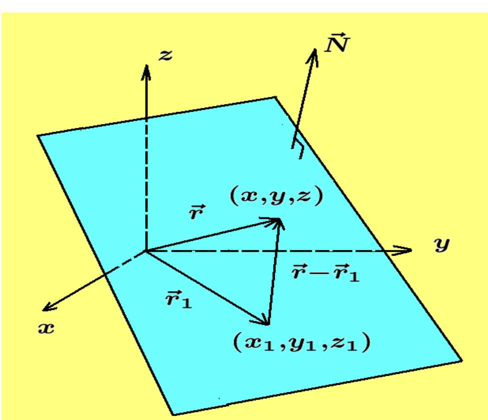
Note that if one of the direction numbers is zero, this states the line lies in a plane. For example, if $a = 0$, the line lies in the plane $x = x_1 = \mathit{constant}$. That is, the line is defined by by

$$y = y_1 + b\lambda, \quad z = z_1 + c\lambda \text{ or } \Rightarrow \; z - z_1 = m(y - y_1), \quad m = c/b$$

in the plane $x = x_1$ a constant.

■

Example 10-12. Find the **equation of the plane which passes through the point** $P_1(x_1, y_1, z_1)$ **and is perpendicular to the given vector** $\vec{N} = N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3$.



Solution Let $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ denote the position vector to the point P_1 and let the vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to any variable point (x, y, z) in the plane. If the vector $\vec{r} - \vec{r}_1$ lies in the plane, then it must be perpendicular to the given vector $\vec{N}$ and consequently the dot product of $(\vec{r} - \vec{r}_1)$ with $\vec{N}$ must be zero and so one can write

$$(\vec{r} - \vec{r}_1) \cdot \vec{N} = 0 \tag{10.50}$$

as the equation representing the plane. In scalar form, the equation of the plane is given as

$$\begin{aligned}(x-x_1)N_1+(y-y_1)N_2+(z-z_1)N_3 =&0\\ \text{or}\qquad N_1x+N_2y+N_3z=&\text{ a constant}\end{aligned} \tag{10.51}$$

where N_1, N_2, N_3 are the direction numbers for the **normal line to the plane** and the constant term is $x_1N_1+y_1N_2+z_1N_3$.

■

A plane can be defined knowing

(a) A point and normal vector.
(b) Three noncollinear points.
(c) A line and point not on the line.
(d) By two intersecting lines.
(e) By two parallel lines.

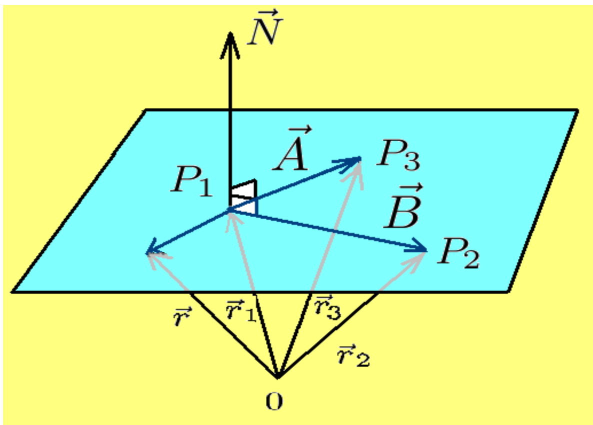
Example 10-13.

Find the equation of the plane defined by three noncollinear points.

Solution:

Define the vectors $\overrightarrow{P_1P_3}=\vec{A}=\vec{r}_3-\vec{r}_1$ and $\overrightarrow{P_1P_2}=\vec{B}=\vec{r}_2-\vec{r}_1$ and let

$$\vec{r}=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3$$



denote a position vector to a point in the plane. The vectors $\vec{A}$ and $\vec{B}$ lie in the plane and the cross product $\vec{N}=\vec{B}\times\vec{A}$ defines a normal vector to the plane.
Knowing a point on the plane and a normal vector to the plane, the equation of the plane is

$$(\vec{r}-\vec{r}_1)\cdot\vec{N}=0$$

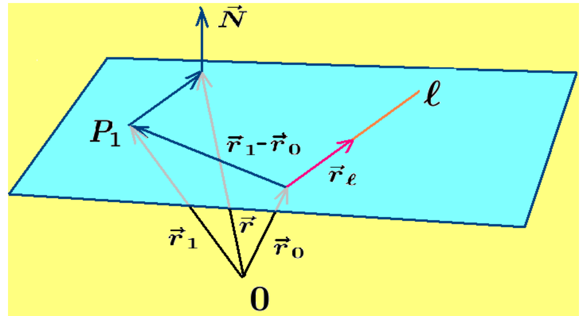
■

Example 10-14.

Find the equation of the plane defined by a line and a point not on the line.

Solution:

If the given line has the vector equation $\vec{r} = \vec{r}_0 + \lambda \vec{r}_\ell$, then when $\lambda = 0$ the vector $\vec{r}_0$ points to a point on the plane. The vectors $\vec{r}_\ell$ and $\vec{r}_1 - \vec{r}_0$ lie in the plane and their cross product $\vec{r}_\ell \times (\vec{r}_1 - \vec{r}_0) = \vec{N}$ produces a normal vector to the plane.



Knowing a point on the plane and a normal vector to the plane, the vector equation of the plane is $(\vec{r} - \vec{r}_1) \cdot \vec{N} = 0$.

■

Example 10-15.

Find the equation of the plane defined by two intersecting lines.

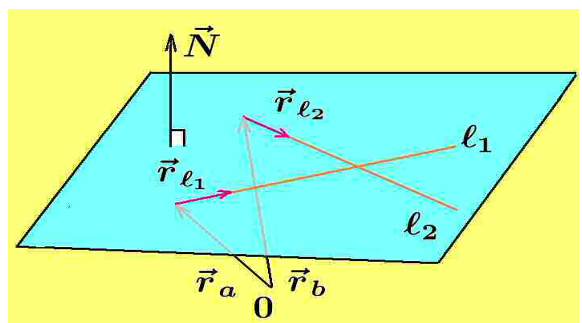
solution:

Let the intersecting lines have the equations

$$\ell_1 : \vec{r} = \vec{r}_a + \lambda_1 \vec{r}_{\ell_1}$$

$$\ell_2 : \vec{r} = \vec{r}_b + \lambda_2 \vec{r}_{\ell_2}$$

When the parameters λ_1 and λ_2 are zero the vectors produced are $\vec{r}_a$ and $\vec{r}_b$ which are vectors to two points in the plane.



Therefore, either of the equations

$$(\vec{r} - \vec{r}_a) \cdot \vec{N} = 0 \qquad \text{or} \qquad (\vec{r} - \vec{r}_b) \cdot \vec{N} = 0$$

represents the equation of the plane where $\vec{N} = \vec{r}_{\ell_2} \times \vec{r}_{\ell_1}$ is a normal vector to the given plane.

■

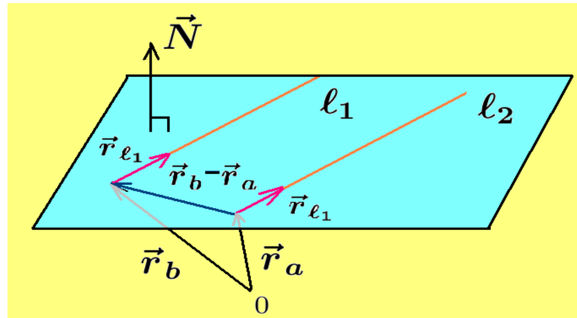
Example 10-16. Find the equation of the plane defined by two parallel lines.

Solution:

Let the parallel lines be defined by the equations

$$\vec{r} = \vec{r}_a + \lambda_1 \vec{r}_{\ell_1}$$
$$\vec{r} = \vec{r}_b + \lambda_2 \vec{r}_{\ell_1}$$

where both lines have the same direction vector. When the parameters $\lambda_1 = \lambda_2 = 0$ there results the position vectors $\vec{r}_a$ and $\vec{r}_b$ representing two points in the plane.



The vector difference $\vec{r}_b - \vec{r}_a$ also lies in the plane and so a normal vector to the plane is given the cross product $\vec{r}_{\ell_1} \times (\vec{r}_b - \vec{r}_a) = \vec{N}$. Therefore, either of the equations

$$(\vec{r} - \vec{r}_b) \cdot \vec{N} = 0 \quad \text{or} \quad (\vec{r} - \vec{r}_a) \cdot \vec{N} = 0$$

can be used to represent the equation of the plane.

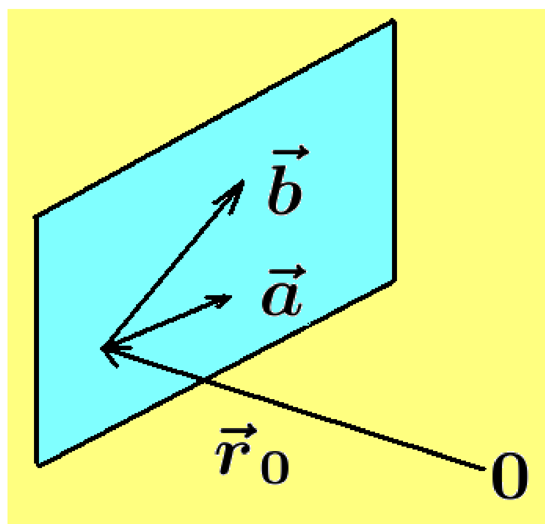
■

Parametric equation of a plane

Select two linearly independent vectors $\vec{a}$ and $\vec{b}$ from within the plane or parallel to the plane desired. Let $\vec{r}_0$ denote the position vector to a point in the plane. The position of a general point $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ on the plane can then be represented

$$\vec{r} = \vec{r}_0 + u\vec{a} + v\vec{b}, \quad -\infty < u < \infty, \quad -\infty < v < \infty$$

where u and v are called parameters of the plane determined by the vectors $\vec{r}_0, \vec{a}, \vec{b}$.



Example 10-17. Three points determine a plane. Find the parametric equation representing the plane passing through the three points

$$P_1: \ (4,5,6) \quad P_2: \ (-1,4,3) \quad P_3: \ (7,0,4)$$

Solution:

One can define the position vectors

$$\vec{r}_1 = 4\,\hat{\mathbf{e}}_1 + 5\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3$$
$$\vec{r}_2 = -1\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + 3\,\hat{\mathbf{e}}_3$$
$$\vec{r}_3 = 7\,\hat{\mathbf{e}}_1 + 0\,\hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3$$

to each of the given points. The vectors

$$\overrightarrow{P_1P_3} = \vec{a} = \vec{r}_3 - \vec{r}_1 = 3\,\hat{\mathbf{e}}_1 - 5\,\hat{\mathbf{e}}_2 - 2\,\hat{\mathbf{e}}_3$$
$$\overrightarrow{P_1P_2} = \vec{b} = \vec{r}_2 - \vec{r}_1 = -5\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - 3\,\hat{\mathbf{e}}_3$$

are independent vectors lying in the plane and so one form for the parametric equation for the plane is

$$\vec{r} = \vec{r}_1 + u\vec{a} + v\vec{b}$$

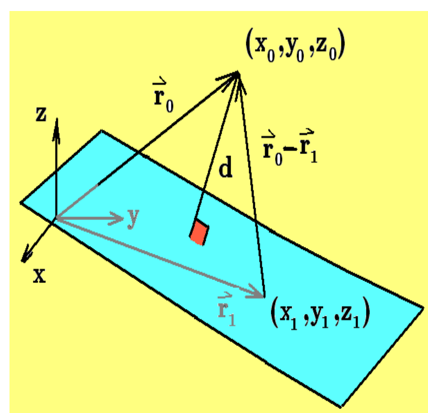
■

Example 10-18. Find **the perpendicular distance d from a given plane**

$$(x - x_1)N_1 + (y - y_1)N_2 + (z - z_1)N_3 = 0 \tag{10.52}$$

to a given point (x_0, y_0, z_0) not on the plane.

Solution



Let the vector $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ point to the given point (x_0, y_0, z_0) and the vector $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ point the point (x_1, y_1, z_1) lying in the plane. Construct the vector $\vec{r}_0 - \vec{r}_1$ which points from the terminus of $\vec{r}_1$ to the terminus of $\vec{r}_0$ and construct the unit normal to the plane which is given by

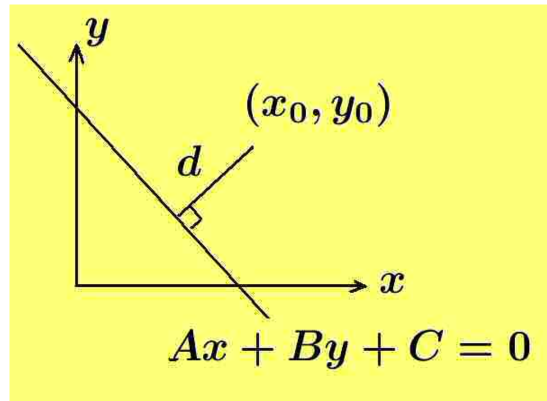
$$\hat{\mathbf{e}}_N = \frac{N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3}{\sqrt{N_1^2 + N_2^2 + N_3^2}}$$

Observe that the dot product $\hat{\mathbf{e}}_N \cdot (\vec{r}_0 - \vec{r}_1)$ equals the projection of $\vec{r}_0 - \vec{r}_1$ onto the normal vector $\hat{\mathbf{e}}_N$. This gives the perpendicular distance

$$d = |\,\hat{\mathbf{e}}_N \cdot (\vec{r}_0 - \vec{r}_1)|$$
$$d = \left| \frac{(x_0 - x_1)N_1 + (y_0 - y_1)N_2 + (z_0 - z)N_3}{\sqrt{N_1^2 + N_2^2 + N_3^2}} \right| \tag{10.53}$$

where the absolute value signs guarantee that the sign of d is always positive and does not depend upon the direction selected for the unit normal $\hat{\mathbf{e}}_N$ to the plane.

As a special case of equations (10.52) and (10.53) set the N_3 and $\hat{\mathbf{e}}_3$ components to zero to produce the two dimensional problem of finding the perpendicular distance of a point (x_0, y_0) to the line $N_1 x + N_2 y + C = 0$ where from the equation (10.52) the constant C is given by $C = -x_1 N_1 - y_1 N_2$.



Let $N_1 = A$ and $N_2 = B$ in equation (10.52) and write the equation of the line in two dimensions as $Ax + By + C = 0$, then as a special case of equation (10.53) in two dimensions one finds the perpendicular distance d from a point (x_0, y_0) to the given line is

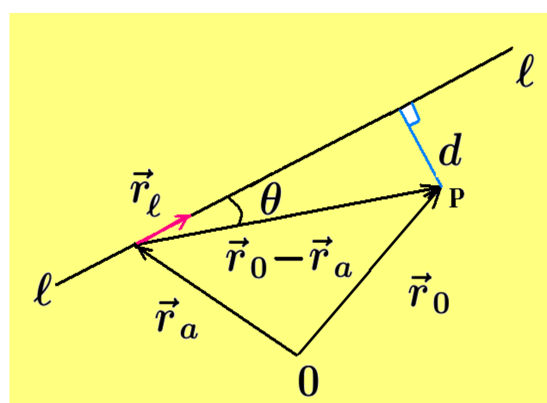
$$d = \left| \frac{Ax + By + C}{\sqrt{A^2 + B^2}} \right| \tag{10.54}$$

■

Example 10-19. Find the distance d from a point (x_0, y_0, z_0) to a given line ℓ represented by the vector $\vec{r} = \vec{r}_a + \lambda \vec{r}_\ell$.

Solution:

Let $\vec{r}_0 = x_0 \hat{\mathbf{e}}_1 + y_0 \hat{\mathbf{e}}_2 + z_0 \hat{\mathbf{e}}_3$ denote a vector from the origin to the given point P. The vector $\vec{r}_a$ is a vector to a point on the line ℓ when $\lambda = 0$ and $\vec{r}_\ell$ represents the direction vector of the line as illustrated in the accompanying figure. By vector addition $\vec{r}_0 = \vec{r}_a + (\vec{r}_0 - \vec{r}_a)$ as illustrated. The **projection of the vector** $\vec{r}_0 - \vec{r}_a$ onto the line perpendicular to ℓ and passing through the given point is



$$|\vec{r}_0 - \vec{r}_a| \cos(\frac{\pi}{2} - \theta) = |\vec{r}_0 - \vec{r}_a| \sin\theta = d \tag{10.55}$$

where θ is the angle between the vectors $(\vec{r}_0 - \vec{r}_a)$ and $\vec{r}_\ell$. One can use the definition of the cross product of two vectors

$$|(\vec{r}_0 - \vec{r}_a) \times \vec{r}_\ell| = |(\vec{r}_0 - \vec{r}_a)|\,|\vec{r}_\ell| \sin\theta$$

and write equation (10.55) in the form

$$d = \frac{|(\vec{r}_0 - \vec{r}_a) \times \vec{r}_\ell|}{|\vec{r}_\ell|} \tag{10.56}$$

which is how one calculates the distance from a point to a line in three dimensional space.

■

Example 10-20. Find the line which is defined by **the intersection of two given planes**

Solution:

Given the two planes

$$\begin{aligned} N_1 x + N_2 y + N_3 z + D_1 =&0 \\ M_1 x + M_2 y + M_3 z + D_2 =&0 \end{aligned} \quad \text{or} \quad \begin{aligned} \vec{N} \cdot (\vec{r} - \vec{r}_0) =&0 \\ \vec{M} \cdot (\vec{r} - \vec{r}_1) =&0 \end{aligned} \tag{10.57}$$

and assume these planes are not parallel[6] or $\vec{N} \neq k\vec{M}$, for k a nonzero constant.

In general, the vector equation of a line passing through a point P_0 and having a direction vector $\vec{r}_\ell$ is given by

$$\vec{r} = \vec{r}_{P_0} + \lambda\,\vec{r}_\ell$$

where $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ is a vector pointing to a variable point on the line determined by the parameter value of λ, $\vec{r}_{P_0} = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ is a vector pointing to a fixed point on the line and $\vec{r}_\ell = a\,\hat{\mathbf{e}}_1 + b\,\hat{\mathbf{e}}_2 + c\,\hat{\mathbf{e}}_3$ represents the vector direction associated with the line and λ is a parameter $-\infty < \lambda < \infty$. Our problem is to find a point P_0 on the line and to find a direction vector for the line which is along the intersection of two planes.

Since the equations for the planes are given, then one knows the normal vectors to the planes

$$\begin{aligned} \vec{N} =& N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3 \\ \vec{M} =& M_1\,\hat{\mathbf{e}}_1 + M_2\,\hat{\mathbf{e}}_2 + M_3\,\hat{\mathbf{e}}_3 \end{aligned}$$

[6] Two planes are parallel if and only if the normals $\vec{N}$ and $\vec{M}$ satisfy $\vec{N} \times \vec{M} = \vec{0}$ which implies $\vec{N} = k\vec{M}$ for some nonzero scalar k.

The cross product of these two vectors gives a direction vector for the line

$$\vec{r}_\ell = \vec{N} \times \vec{M} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ N_1 & N_2 & N_3 \\ M_1 & M_2 & M_3 \end{vmatrix} = \hat{\mathbf{e}}_1 \begin{vmatrix} N_2 & N_3 \\ M_2 & M_3 \end{vmatrix} - \hat{\mathbf{e}}_2 \begin{vmatrix} N_1 & N_3 \\ M_1 & M_3 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} N_1 & N_2 \\ M_1 & M_2 \end{vmatrix}$$

To find a point (x_0, y_0, z_0) on the line of intersection of the two planes given by the simultaneous equations (10.57), one can select a value for one of the variables x, y or z in the equations (10.57), then substitute this value into the simultaneous equations (10.57) and solve the resulting two equations in two unknowns. The algebra becomes easier if one of the quantities x, y or z is zero. This will make the line of intersection of the two planes cut across one of the planes $x = 0$, $y = 0$ or $z = 0$. For example, if one sets $z = 0$, the known point on the line becomes $(x, y, 0)$ which is on the line of intersection of the two given planes where x and y are the solutions associated with the system of equations

$$\begin{aligned} N_1 x + N_2 y + D_1 &= 0 \\ M_1 x + M_2 y + D_2 &= 0 \end{aligned} \tag{10.58}$$

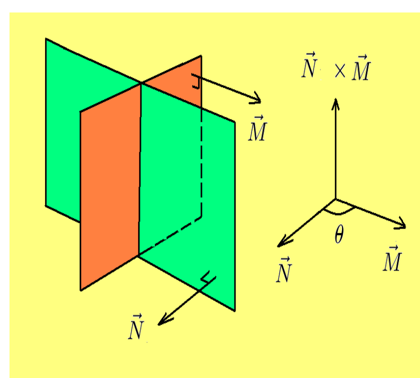
Solve this system of two equations and two unknowns and obtain a solution represented by $(x, y) = (x_0, y_0)$, then the point $P_0 = (x_0, y_0, 0)$ is a known point on the line and the equation of the line of intersection of the two planes is

$$\vec{r} = \vec{r}_{P_0} + \lambda \vec{r}_\ell, \qquad -\infty < \lambda < \infty$$

The angle formed by the intersection of two planes is called a **dihedral angle.** The line of intersection of the two planes is called the **edge of the dihedral angle.** The cosine of the dihedral angle can be calculated from the dot product formula

$$\vec{N} \cdot \vec{M} = \left|\vec{N}\right| \left|\vec{M}\right| \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\vec{N} \cdot \vec{M}}{\left|\vec{N}\right| \left|\vec{M}\right|} \tag{10.59}$$

when the origin of the normal vectors are made to coincide.



Caution, one can examine the given system of equations (10.58) to see if any plane is parallel to an axis. If this is the case, then one is restricted in the selection of the variable chosen to be zero. See for example the situations illustrated in the figures 10-15.

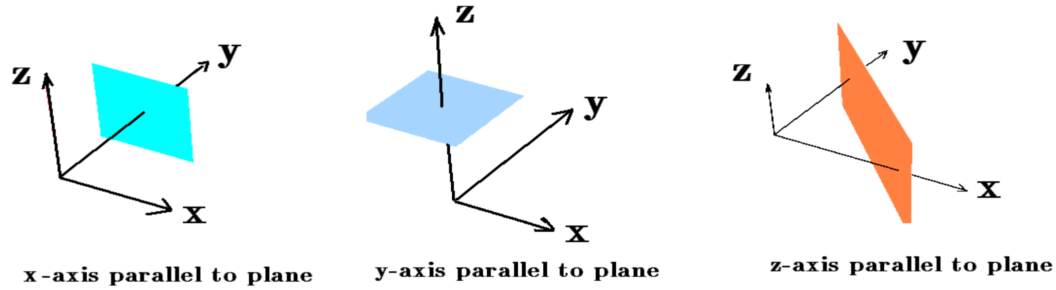


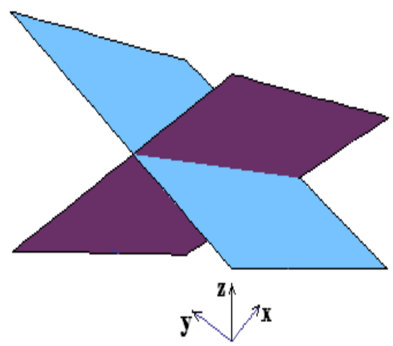
Figure 10-15. Situations where plane is parallel to an axis

One cannot select $x_0 = 0$ as a point on the intersection of two planes if the x-axis is parallel to one of the planes. Similar consideration must be made in the selection of $y_0 = 0$ or $z_0 = 0$ if these values are acceptable.

Example 10-21.

Given the planes

$$\begin{aligned} \text{plane 1:} \quad & 2x + 3y + z = 22 \\ \text{plane 2:} \quad & 2x - y + z = 6 \end{aligned} \tag{10.60}$$



Set $z = 0$ in the simultaneous equations (10.60) to obtain the system of two equations in two unknowns

$$\begin{aligned} 2x + 3y =& 22 \\ 2x - y =& 6 \end{aligned} \tag{10.61}$$

This system has the solution $x_0 = 5$ and $y_0 = 4$. Consequently, the point given by $(x_0, y_0, z_0) = (5, 4, 0)$ is on the line of intersection of the two given planes and the vector to this point is $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$. The normal vectors to the given planes are

$$\vec{N} = 2\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \qquad \text{and} \qquad \vec{M} = 2\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$

and from these normal vectors one can calculate the direction vector for the line by taking the cross product

$$\vec{N} \times \vec{M} = \vec{r}_\ell = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 2 & 3 & 1 \\ 2 & -1 & -1 \end{vmatrix} = \hat{\mathbf{e}}_1 \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - \hat{\mathbf{e}}_2 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix}$$

giving the direction numbers for the line of intersection (dihedral angle edge) as

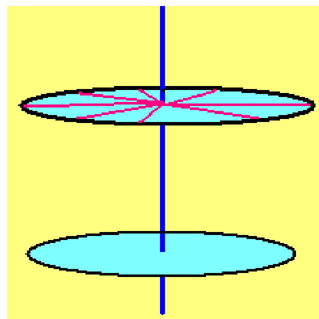
$$\vec{r}_\ell = \vec{N} \times \vec{M} = 4\,\hat{\mathbf{e}}_1 + 0\,\hat{\mathbf{e}}_2 - 8\,\hat{\mathbf{e}}_3$$

The vector equation for the line of intersection of the two planes is then represented

$$\vec{r} = \vec{r}_0 + \lambda\vec{r}_\ell = (5 + 4\lambda)\,\hat{\mathbf{e}}_1 + 4\lambda\,\hat{\mathbf{e}}_2 - 8\lambda\,\hat{\mathbf{e}}_3$$

■

Note a line in space has an infinite number of other lines perpendicular to any fixed point on the line. All these perpendicular lines lie in a plane which is perpendicular to the given line.



This fact can be used to show that two planes perpendicular to the same line at different points are parallel.

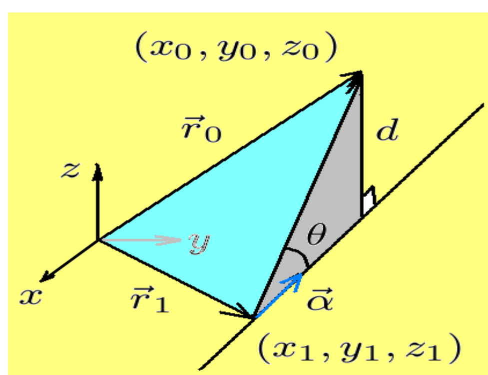
Example 10-22. Find the **perpendicular distance from a point** (x_0, y_0, z_0) **to a given line** ℓ defined by $\vec{r} = \vec{r}_1 + \vec{\alpha}t$. Assume the point $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ is not on the line and show the perpendicular distance is given by

$$d = \left| (\vec{r}_0 - \vec{r}_1) \times \frac{\vec{\alpha}}{|\vec{\alpha}|} \right| \quad \text{where} \quad \vec{\alpha} = \alpha_1\,\hat{\mathbf{e}}_1 + \alpha_2\,\hat{\mathbf{e}}_2 + \alpha_3\,\hat{\mathbf{e}}_3$$

and $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ are given vectors.

Solution

The **vector equation of the line** is $\vec{r} = \vec{r}_1 + \vec{\alpha}\,t$, where (x_1, y_1, z_1) is a point on the line described by the position vector $\vec{r}$ and $\vec{\alpha}$ is **the direction vector of the line** and t is a parameter. The vector $\vec{r}_0 - \vec{r}_1$ is a vector pointing from (x_1, y_1, z_1) to the



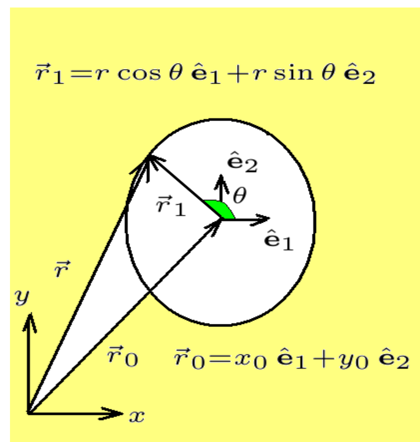
point (x_0, y_0, z_0). These vectors are illustrated in the accompanying figure.

Define the unit vector $\hat{\mathbf{e}}_\alpha = \frac{1}{|\vec{\alpha}|}\vec{\alpha}$ and construct the line segment from (x_0, y_0, z_0) which is perpendicular to the given line and label this distance d. Our problem is to find the distance d. From the geometry of the right triangle with hypotenuse $|\vec{r}_0 - \vec{r}_1|$ and side d one can write $\sin\theta = \dfrac{d}{|\vec{r}_0 - \vec{r}_1|}$. Use the fact that by definition of a cross product one can write

$$|(\vec{r}_0 - \vec{r}_1) \times \hat{\mathbf{e}}_\alpha| = |\vec{r}_0 - \vec{r}_1||\hat{\mathbf{e}}_\alpha| \sin\theta = d = \left|(\vec{r}_0 - \vec{r}_1) \times \frac{\vec{\alpha}}{|\vec{\alpha}|}\right|$$

■

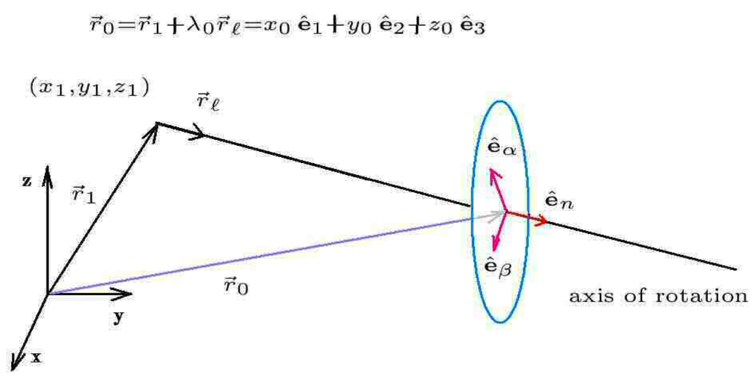
Example 10-23. (Circle in three dimensions)



In two dimensions the equation of the circle in vector form is $\vec{r} = \vec{r}_0 + \vec{r}_1$ where $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2$ is the position vector to the center of the circle and $\vec{r}_1 = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2$ is the vector equation of the circle. The parametric form for the equation of the circle of radius r centered at the point (x_0, y_0) is

$$x = x_0 + r\cos\theta, \qquad y = y_0 + r\sin\theta \tag{10.62}$$

In two dimensions a center and radius x_0, y_0, r are required to have a circle.



To construct a circle in three dimensions one needs in addition to a center (x_0, y_0, z_0) and radius r (i) an axis of rotation perpendicular to the circle center and (ii) two perpendicular unit vectors $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\beta$ in the plane of the circle. Let $\vec{r}_\ell$ denote a direction vector for the axis of rotation and let (x_1, y_1, z_1) denote an arbitrary point on this line. The vector equation of the line through the point (x_1, y_1, z_1) in the direction $\vec{r}_\ell$ is $\vec{r} = \vec{r}_1 + \lambda\vec{r}_\ell$. Let $\vec{r}_0 = \vec{r}_1 + \lambda_0\vec{r}_\ell = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ denote the center of the circle and let $\hat{\mathbf{e}}_n$ denote a unit vector in the direction $\vec{r}_\ell$. There are an infinite number of vectors perpendicular to $\hat{\mathbf{e}}_n$. One can construct a vector perpendicular to $\hat{\mathbf{e}}_n$ as follows. Let $\hat{\mathbf{e}}_n = n_1\,\hat{\mathbf{e}}_1 + n_2\,\hat{\mathbf{e}}_2 + n_3\,\hat{\mathbf{e}}_3$ and let $\vec{a} = a_1\,\hat{\mathbf{e}}_1 + a_2\,\hat{\mathbf{e}}_2 + a_3\,\hat{\mathbf{e}}_3$ denote the vector to be determined. We require

$$\hat{\mathbf{e}}_n \cdot \vec{a} = n_1 a_1 + n_2 a_2 + n_3 a_3 = 0 \tag{10.63}$$

Assume n_3 is different from zero and select a_1 and a_2 as positive numbers different from zero. One can then solve for a_3 to make equation (10.63) zero. One finds

$$a_3 = \frac{-n_1 a_1 - n_2 a_2}{n_3}$$

and so it is possible to construct a unit vector

$$\hat{\mathbf{e}}_\alpha = \frac{a_1\,\hat{\mathbf{e}}_1 + a_2\,\hat{\mathbf{e}}_2 + a_3\,\hat{\mathbf{e}}_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

which is perpendicular to $\hat{\mathbf{e}}_n$. The cross product $\hat{\mathbf{e}}_\beta = \hat{\mathbf{e}}_n \times \hat{\mathbf{e}}_\alpha$ produces another unit vector perpendicular to $\hat{\mathbf{e}}_n$ with the unit vector $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\beta$ in the plane of the circle. The vector equation for the circle in 3-space can now be expressed

$$\vec{r} = \vec{r}_0 + r\cos\theta\,\hat{\mathbf{e}}_\alpha + r\sin\theta\,\hat{\mathbf{e}}_\beta \tag{10.64}$$

If $\hat{\mathbf{e}}_\alpha = \alpha_1\,\hat{\mathbf{e}}_1 + \alpha_2\,\hat{\mathbf{e}}_2 + \alpha_3\,\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_\beta = \beta_1\,\hat{\mathbf{e}}_2 + \beta_2\,\hat{\mathbf{e}}_2 + \beta_3\,\hat{\mathbf{e}}_3$, then the parametric equation for a circle in 3-space is

$$\begin{aligned} x =& x_0 + \alpha_1 r\cos\theta + \beta_1 r\sin\theta \\ y =& y_0 + \alpha_2 r\cos\theta + \beta_2 r\sin\theta \\ z =& z_0 + \alpha_3 r\cos\theta + \beta_3 r\sin\theta \end{aligned} \tag{10.65}$$

■

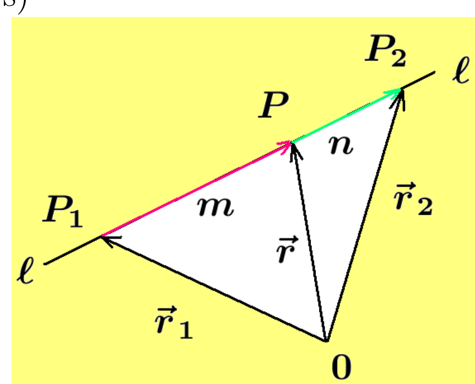
Example 10-24. (Section formula using vectors)

Given the vectors

$$\begin{aligned} \vec{r}_1 =& x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3 \\ \vec{r}_2 =& x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2 + z_3\,\hat{\mathbf{e}}_3 \end{aligned}$$

which defines the line ℓ with direction numbers

$$\vec{r}_\ell = \vec{r}_2 - \vec{r}_1$$



Find the point P with position vector $\vec{r}$ which divides the line segment $\overline{P_1P_2}$ into the ratio $m : n$.

Solution:

We want to obtain the result

$$\frac{|\overrightarrow{P_1P}|}{|\overrightarrow{PP_2}|} = \frac{m}{n} \qquad \Rightarrow \qquad n\overrightarrow{P_1P} = m\overrightarrow{PP_2}$$

Using vector addition one can verify that

$$\vec{r}_1 + \overrightarrow{P_1P} = \vec{r} \quad \text{or} \quad \overrightarrow{P_1P} = \vec{r} - \vec{r}_1$$
$$\text{and} \quad \vec{r} + \overrightarrow{PP_2} = \vec{r}_2 \quad \text{or} \quad \overrightarrow{PP_2} = \vec{r}_2 - \vec{r}$$

Therefore,

$$n\,(\vec{r} - \vec{r}_1) = m\,(\vec{r}_2 - \vec{r}) \qquad \Rightarrow \qquad (m+n)\,\vec{r} = m\,\vec{r}_2 + n\,\vec{r}_1$$

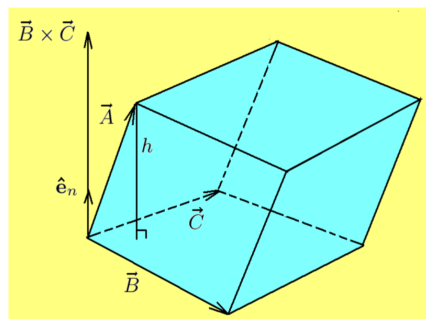
or

$$\vec{r} = \frac{m\,\vec{r}_2 + n\,\vec{r}_1}{m+n}$$

■

Example 10-25.

Find the volume of the parallelepiped illustrated which has the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ for its sides.



Solution

The volume of the parallelepiped is the area of the base times the altitude h. The area of the base is given by $|\vec{B} \times \vec{C}|$ and the altitude is given by $h = \vec{A} \cdot \hat{\mathbf{e}}_n$ which is the projection of the vector $\vec{A}$ onto the normal $\hat{\mathbf{e}}_n$. Note that $\hat{\mathbf{e}}_n$ is the unit vector perpendicular to the plane formed by the vectors $\vec{B}$ and $\vec{C}$. If the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are defined

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$$
$$\vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$$
$$\vec{C} = C_1\,\hat{\mathbf{e}}_1 + C_2\,\hat{\mathbf{e}}_2 + C_3\,\hat{\mathbf{e}}_3$$

then

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \hat{\mathbf{e}}_1(B_2C_3 - B_3C_2) - \hat{\mathbf{e}}_2(B_1C_3 - C_1B_3) + \hat{\mathbf{e}}_3(B_1C_2 - C_1B_2) \tag{10.66}$$

The unit normal vector to the plane formed by the vectors $\vec{B}$ and $\vec{C}$ is found to be

$$\hat{\mathbf{e}}_n = \frac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|} \tag{10.67}$$

and the altitude is found to be the projection of $\vec{A}$ onto the unit vector $\hat{\mathbf{e}}_n$ or

$$h = \vec{A}\cdot\hat{\mathbf{e}}_n = \vec{A}\cdot\frac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|} \tag{10.68}$$

The volume of the parallelepiped is the area of the base times the altitude or

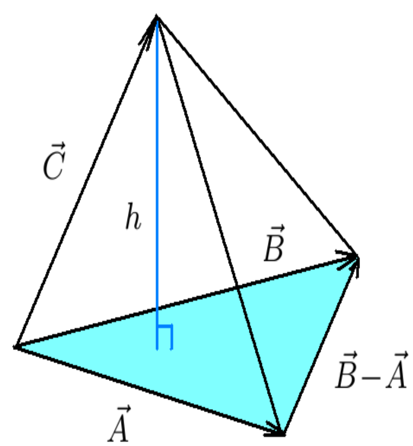
$$\begin{aligned}
\text{Volume} =& h\,|\vec{B}\times\vec{C}| = \vec{A}\cdot(\vec{B}\times\vec{C})\\
\text{Volume} =& A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - C_1B_3) + A_3(B_1C_2 - B_2C_1)\\
\text{Volume} =& \begin{vmatrix} A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}
\end{aligned}$$

Here the volume of the parallelepiped is found by calculating the triple scalar product $\vec{A}\cdot(\vec{B}\times\vec{C})$.

■

Example 10-26. Show that the tetrahedron with sides $\vec{A}$, $\vec{B}$, $\vec{C}$ has a volume $V = \frac{1}{6}|\vec{A}\cdot(\vec{B}\times\vec{C})|$ which equals one-sixth the volume of a parallelepiped.

Solution



We know that the triple scalar product

$$|\vec{A}\cdot(\vec{B}\times\vec{C})|$$

is the volume of a parallelepiped with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$. The tetrahedron is a polyhedron with 4 faces. The vectors $\vec{A}, \vec{B}$ and $\vec{B}-\vec{A}$ make up the base of the tetrahedron so one can express the area of the base in the form

$$\text{Area of base} = \frac{1}{2}|\vec{A}\times\vec{B}|$$

The vector $\vec{N} = \vec{A}\times\vec{B}$ is perpendicular to the vase and the vector $\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$ is a unit vector perpendicular to the base. The projection of vector $\vec{C}$ onto $\hat{n}$ is the height h of the tetrahedron or

$$h = \vec{C}\cdot\hat{n} = \frac{\vec{C}\cdot(\vec{A}\times\vec{B})}{|\vec{A}\times\vec{B}|}$$

The volume of the tetrahedron V_t is one-third the area of the base times the height giving

$$V_t = \frac{1}{3}\left(\frac{1}{2}|\vec{A}\times\vec{B}|\right)\left(\frac{\vec{C}\cdot(\vec{A}\times\vec{B})}{|\vec{A}\times\vec{B}|}\right) = \frac{1}{6}\vec{A}\cdot(\vec{B}\times\vec{C})$$

where we have used the cyclic shifting property of the triple scalar product.

■

Vector form for centroid

In general, if one has n-masses $m_1, m_2, \ldots, m_n$ having the position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$, then the position vector of the centroid is given by

$$\vec{r}_G = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i} \tag{10.69}$$

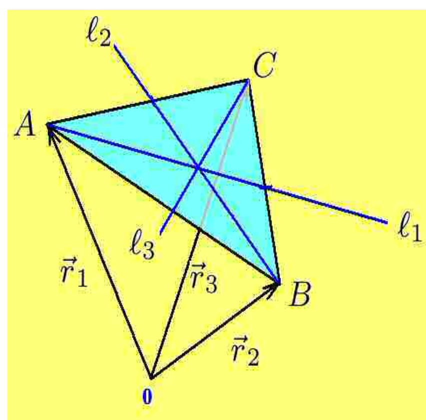
In the special case there exits 3 masses, where $m_1 = m_2 = m_3 = 1$, at position vectors

$$\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3, \quad \vec{r}_2 = x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2 + z_2\,\hat{\mathbf{e}}_3, \quad \vec{r}_3 = x_3\,\hat{\mathbf{e}}_1 + y_3\,\hat{\mathbf{e}}_2 + z_3\,\hat{\mathbf{e}}_3$$

which define the vertices of a three dimensional triangle, then the centroid of the triangle is given by

$$\vec{r}_G = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \tag{10.70}$$

This is in agreement with our original definition of a triangle centroid in two dimensions.



One can verify the position vectors to the midpoints of the triangle $\triangle ABC$ sides are given by

$$\begin{aligned}\vec{r}_{AB} =& \frac{1}{2}(\vec{r}_1 + \vec{r}_2)\\ \vec{r}_{BC} =& \frac{1}{2}(\vec{r}_2 + \vec{r}_2)\\ \vec{r}_{AC} =& \frac{1}{2}(\vec{r}_1 + \vec{r}_3)\end{aligned} \tag{10.71}$$

The vector equations for the median lines ℓ_1, ℓ_2, ℓ_3 have the form

$$\begin{aligned}\ell_1: \quad \vec{r} =& \vec{r}_1 + \lambda_1(\vec{r}_{BC} - \vec{r}_1)\\ \ell_2: \quad \vec{r} =& \vec{r}_2 + \lambda_2(\vec{r}_{AC} - \vec{r}_2)\\ \ell_3: \quad \vec{r} =& \vec{r}_3 + \lambda_3(\vec{r}_{AB} - \vec{r}_3)\end{aligned} \tag{10.72}$$

where $\lambda_1, \lambda_2, \lambda_3$ are parameters for the line.[7]

The median lines ℓ_1 and ℓ_2 intersect where

$$\vec{r}_1 + \lambda_1 (\vec{r}_{BC} - \vec{r}_1) = \vec{r}_2 + \lambda_2 (\vec{r}_{AC} - \vec{r}_2)$$

By equating like components in the above equation one obtains the following three scalar equations

$$x_1 + \lambda_1 \left(\frac{1}{2}(x_2 + x_3) - x_1 \right) = x_2 + \lambda_2 \left(\frac{1}{2}(x_1 + x_3) - x_2 \right) \tag{10.73}$$

$$y_1 + \lambda_2 \left(\frac{1}{2}(y_2 + y_3) - y_1 \right) = y_2 + \lambda_2 \left(\frac{1}{2}(y_1 + y_3) - y_2 \right) \tag{10.74}$$

$$z_1 + \lambda_1 \left(\frac{1}{2}(z_2 + z_3) - z_1 \right) = z_2 + \lambda_2 \left(\frac{1}{2}(z_1 + z_3) - z_1 \right) \tag{10.75}$$

The equations (10.73) and (10.74) represent two equations in the two unknowns λ_1, λ_2. One can verify the solution of these two equations produce the values $\lambda_1 = \frac{2}{3}$ and $\lambda_2 = \frac{2}{3}$ and these values are also solutions to the third equation (10.75). One can verify that the intersection of lines ℓ_1 and ℓ_3 also intersect at the same point given by the centroid

$$\begin{aligned} \vec{r}_G = &\vec{r}_1 + \frac{2}{3} (\vec{r}_{BC} - \vec{r}_1) \\ \vec{r}_G = &\vec{r}_2 + \frac{2}{3} (\vec{r}_{AC} - \vec{r}_1) \\ \vec{r}_G = &\vec{r}_3 + \frac{2}{3} (\vec{r}_{AB} - \vec{r}_3) \end{aligned}$$

all of which simplify to the centroid given by the point of concurrency given by the position vector

$$r_G = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)$$

Make note that this establishes the well known 2:1 ratio associated with the intersections of a triangles medians.

Vector form for incenter

The incenter occurs at the point of concurrency of the three angle bisectors of the three dimensional triangle formed by the position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ as illustrated in the figure 10-16.

[7] Note the parameters for the lines do not have to be the same.

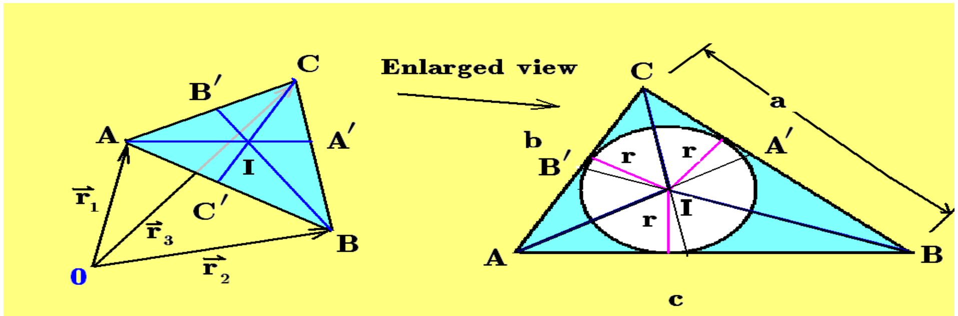


Figure 10-16. Intersection of angle bisectors at point I for 3D triangle

Let $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ denote the three angle bisectors which intersect at the point I representing the incenter and having the position vector $\vec{r}_I$. It will be demonstrated that this position vector can be represented in terms of the position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ to the triangle vertices and the triangle sides a, b, c opposite these vertices. The position vector $\vec{r}_I$ for the incenter will be found to have the form

$$\vec{r}_I = \frac{a\,\vec{r}_1 + b\,\vec{r}_2 + c\,\vec{r}_3}{a+b+c} \tag{10.76}$$

This is interpreted as a weighted sum[8] of the position vectors to the vertices of the three dimensional triangle. The weighting factors a, b, c being the length of the sides opposite each vertex.

To establish the above result begin by using the angle bisector theorem. The cevian $\overline{AA'}$ in triangle $\triangle ABC$ produces the ratios

$$\frac{|\vec{r}_3 - \vec{r}_{A'}|}{b} = \frac{|\vec{r}_2 - \vec{r}_{A'}|}{c} \tag{10.77}$$

so that by the section formula one obtains the position vector

$$\vec{r}_{A'} = \frac{b\,\vec{r}_2 + c\,\vec{r}_3}{b+c} \tag{10.78}$$

Similarly, the cevian $\overline{BB'}$ produces the triangle $\triangle BAA'$ and the ratio

$$\frac{|\vec{r}_I - \vec{r}_1|}{c} = \frac{|\vec{r}_{A'} - \vec{r}_I|}{|\vec{r}_2 - \vec{r}_{A'}|} \tag{10.79}$$

[8] A weighted sum of the quantities $a_1, a_2, \ldots, a_n$ using weighting factors $w_1, w_2, \ldots, w_n$ is the quantity $\bar{a} = \dfrac{w_1 a_1 + w_2 a_2 + \cdots w_n a_n}{w_1 + w_2 + \cdots + w_n}$. Note that if all the weights are 1, then the weighted sum becomes an average of the quantities $a_1, a_2, \ldots, a_n$.

resulting from the angle bisector theorem. Using the equation (10.77) one can write

$$\frac{|\vec{r}_3 - \vec{r}_{A'}|}{|\vec{r}_2 - \vec{r}_{A'}|} = \frac{b}{c} \quad \text{and} \quad \frac{|\vec{r}_3 - \vec{r}_{A'}|}{|\vec{r}_2 - \vec{r}_{A'}|} + 1 = \frac{b}{c} + 1$$

which simplifies to

$$\frac{|\vec{r}_3 - \vec{r}_{A'}| + |\vec{r}_2 - \vec{r}_{A'}|}{|\vec{r}_2 - \vec{r}_{A'}|} = \frac{b+c}{c} = \frac{|\vec{r}_3 - \vec{r}_2|}{|\vec{r}_2 - \vec{r}_{A'}|} = \frac{a}{|\vec{r}_2 - \vec{r}_{A'}|} \quad \Rightarrow \quad |\vec{r}_2 - \vec{r}_{A'}| = \frac{a\,c}{b+c}$$

Consequently

$$\frac{|\vec{r}_I - \vec{r}_1|}{|\vec{r}_{A'} - \vec{r}_I|} = \frac{b+c}{a}$$

and by the section formula

$$\vec{r}_I = \frac{a\vec{r}_1 + (b+c)\vec{r}_{A'}}{a+b+c} \tag{10.80}$$

Substituting the result from equation (10.78) one finds

$$\vec{r}_I = \frac{a\,\vec{r}_1 + (b+c)\left[\frac{b\,\vec{r}_2 + c\,\vec{r}_3}{b+c}\right]}{a+b+c}$$

which simplifies to the equation (10.76).

Vector relation between circumcenter and orthocenter

Let $\vec{r}_o$ denote the position vector of the triangle circumcenter with

$$\begin{aligned} \overrightarrow{OC} &= \vec{r}_3 - \vec{r}_o \\ \overrightarrow{OB} &= \vec{r}_2 - \vec{r}_o \\ \overrightarrow{OA} &= \vec{r}_1 - \vec{r}_o \end{aligned}$$

with

$$|\overrightarrow{OC}| = |\overrightarrow{OB}| = |\overrightarrow{OA}| \tag{10.81}$$

since $\vec{r}_o$ represents the circumcenter which is the intersection of the perpendicular bisectors of the triangle sides. Define the point H with vector position

$$\vec{r}_H = \vec{r}_o + \overrightarrow{OC} + \overrightarrow{OB} + \overrightarrow{OA} \tag{10.82}$$

as illustrated in the accompanying figure.

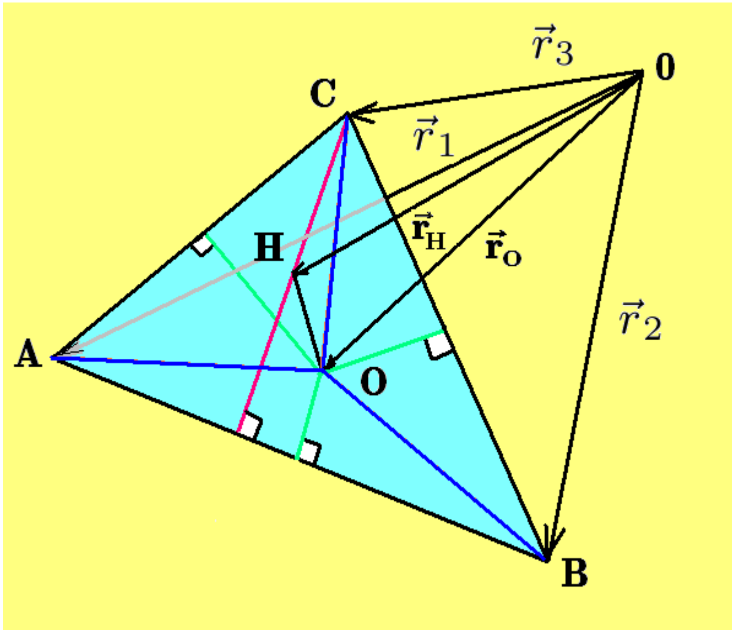


Figure 10-17. Circumcenter and orthocenter for general triangle.

Expand the equation (10.82) to form

$$\begin{aligned}\vec{r}_H =& \vec{r}_o + (\vec{r}_1 - \vec{r}_0) + (\vec{r}_2 - \vec{r}_o) + (\vec{r}_3 - \vec{r}_o)\\ \vec{r}_H =& \vec{r}_1 + \vec{r}_2 + \vec{r}_3 - 2\vec{r}_0\\ \vec{r}_H =& 3\vec{r}_G - 2\vec{r}_o\end{aligned} \tag{10.83}$$

where $\vec{r}_G = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)$ is the position vector to the triangle centroid. One can express the above vector addition as

$$\vec{r}_H = \vec{r}_o + \overrightarrow{OH} \quad \text{where} \quad \overrightarrow{OH} = 3\vec{r}_G - 3\vec{r}_o \tag{10.84}$$

It can now be demonstrated that the vector $\overrightarrow{CH}$ defined by the vector addition

$$\overrightarrow{OH} = \overrightarrow{OC} + \overrightarrow{CH} \quad \text{or} \quad \overrightarrow{CH} = \overrightarrow{OH} - \overrightarrow{OC} \tag{10.85}$$

is perpendicular to the triangle side $\overline{\text{AB}}$. To show this consider the dot product

$$\begin{aligned}\overrightarrow{CH} \cdot \overrightarrow{AB} =& \left(\overrightarrow{OH} - \overrightarrow{OC}\right) \cdot (\vec{r}_2 - \vec{r}_1) = [(\vec{r}_1 - \vec{r}_o) + (\vec{r}_2 - \vec{r}_o) + (\vec{r}_3 - \vec{r}_0) - (\vec{r}_3 - \vec{r}_0)] \cdot (\vec{r}_2 - \vec{r}_1)\\ =& [(\vec{r}_2 - \vec{r}_0) + (\vec{r}_1 - \vec{r}_o)] \cdot [(\vec{r}_2 - \vec{r}_o) - (\vec{r}_1 - \vec{r}_0)]\\ =& |\vec{r}_2 - \vec{r}_o|^2 - |\vec{r}_1 - \vec{r}_o|^2 = 0\end{aligned}$$

This result is zero because of equation (10.81). The above result shows the vector $\overrightarrow{CH}$ is perpendicular to $\overrightarrow{AB} = \vec{r}_2 - \vec{r}_1$ and demonstrates the point H lies on the triangle altitude from the vertex C to side $\overline{\text{AB}}$.

Using the same type of reasoning applied to all of the altitudes of triangle $\triangle ABC$, one can demonstrate that H is a point of concurrency which we call the orthocenter. The equation (10.83) shows that the position vector $\vec{r}_H$ of the orthocenter is determined once the position vector $\vec{r}_o$ of the circumcenter is known.

Vector form for circumcenter

Given the three dimensional triangle $\triangle ABC$ defined by the position vectors

$$\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3, \quad \vec{r}_2 = x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2 + z_2\,\hat{\mathbf{e}}_3, \quad \vec{r}_3 = x_3\,\hat{\mathbf{e}}_1 + y_3\,\hat{\mathbf{e}}_2 + z_3\,\hat{\mathbf{e}}_3$$

as illustrated in the figure 10-18.

Construct a vector $\vec{N}$ normal to the plane of the triangle by taking the cross product

$$\vec{N} = (\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)$$

and then make this vector a unit vector $\hat{n}$ given by

$$\hat{n} = \frac{(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)|}$$

Construct the vectors $\vec{V}_1$ and $\vec{V}_2$ defined by

$$\vec{V}_1 = \hat{n} \times (\vec{r}_2 - \vec{r}_1) \qquad \text{and} \qquad \vec{V}_2 = (\vec{r}_3 - \vec{r}_1) \times \hat{n}$$

and note that these vectors are perpendicular to the sides of the triangle. Finally, construct the unit vectors $\vec{e}_{AC}$ and $\vec{e}_{AB}$ defined by

$$\vec{e}_{AC} = \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|} \qquad \text{and} \quad \vec{e}_{AB} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

which are unit vectors on the sides $\overline{\text{AC}}$ and $\overline{\text{AB}}$. These new vectors will be used to find the position vector of the circumcenter $\vec{r}_o$ of the triangle.

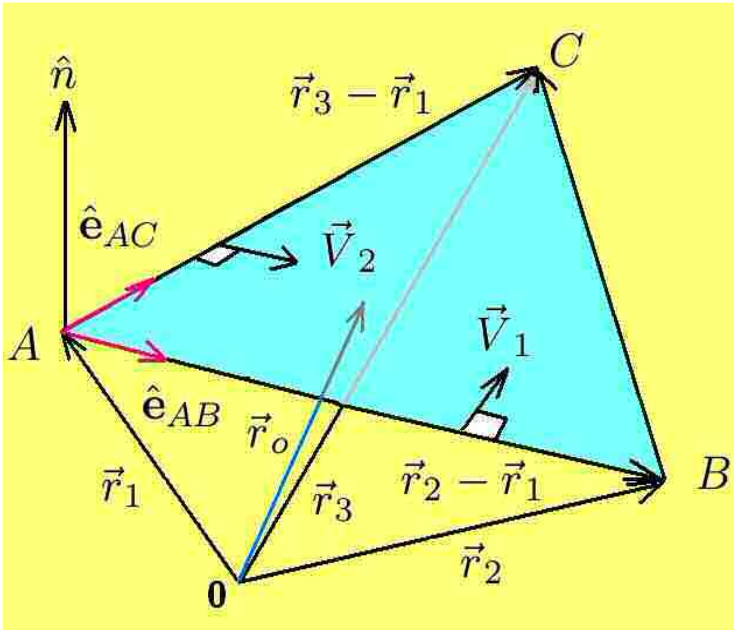


Figure 10-18. Find position vector $\vec{r}_o$ of the circumcenter

Assume the position vector $\vec{r}_o$ of the circumcenter is some linear combination of the vectors $\vec{V}_1$ and $\vec{V}_2$ and write

$$\vec{r}_o = \vec{r}_1 + \alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 \tag{10.86}$$

where α_1 and α_2 are coefficients to be determined.

Make note of the dot products

$$\begin{aligned} \vec{V}_1 \cdot \vec{e}_{AC} =& [\hat{n} \times (\vec{r}_2 - \vec{r}_1)] \cdot \vec{e}_{AC} & \qquad \vec{V}_1 \cdot \vec{e}_{AB} =& [\hat{n} \times (\vec{r}_2 - \vec{r}_1)] \cdot \vec{e}_{AB} = 0 \\ \vec{V}_2 \cdot \vec{e}_{AC} =& [(\vec{r}_3 - \vec{r}_1) \times \hat{n}] \cdot \vec{e}_{AC} = 0 & \qquad \vec{V}_2 \cdot \vec{e}_{AC} =& [(\vec{r}_3 - \vec{r}_1) \times \hat{n}] \cdot \vec{e}_{AB} \end{aligned}$$

which hold due to the cyclic properties of the triple scalar product of three vectors.

Taking the dot product of the vector $\vec{r}_o - \vec{r}_1$ with the unit vector $\vec{e}_{AC}$ one obtains

$$\begin{aligned} (\vec{r}_o - \vec{r}_1) \cdot \vec{e}_{AC} =& \frac{1}{2} |\vec{r}_3 - \vec{r}_1| = \alpha_1 \vec{V}_1 \cdot \vec{e}_{AC} + \alpha_2 \vec{V}_2 \cdot \vec{e}_{AC} \\ \frac{1}{2} |\vec{r}_3 - \vec{r}_1| =& \alpha_1 [\hat{n} \times (\vec{r}_2 - \vec{r}_1)] \cdot \vec{e}_{AC} \\ \frac{1}{2} |\vec{r}_3 - \vec{r}_1| =& \alpha_1 [\hat{n} \times (\vec{r}_2 - \vec{r}_1)] \cdot \frac{(\vec{r}_3 - \vec{r}_1)}{|\vec{r}_3 - \vec{r}_1|} \\ \frac{1}{2} |\vec{r}_3 - \vec{r}_1|^2 =& \alpha_1 \hat{n} \cdot [(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] \end{aligned}$$

or

$$\alpha_1 = \frac{1}{2} \frac{|\vec{r}_3 - \vec{r}_1|^2}{|(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)|} \tag{10.87}$$

Observe that the dot product $\vec{V}_2 \cdot \vec{e}_{AC} = 0$ and the dot product $(\vec{r}_o - \vec{r}_1) \cdot \vec{e}_{AC}$ represents the projection of the vector $\vec{r}_o - \vec{r}_1$ onto the side $\overline{\text{AC}}$ and has the length $\frac{1}{2}|\overline{\text{AC}}|$ or $\frac{1}{2}|\vec{r}_3 - \vec{r}_1|$ because of the definition of a circumcenter. Also note we have used the triple scalar product property to modify terms together with the fact that the dot product $\hat{n} \cdot [(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)]$ equals the magnitude of the first vector times the magnitude of the second vector times the cosine of the angle between them when their origins are made to coincide. Here this angle is zero and the cosine of zero is one.

In a similar fashion one finds that the dot product of the vector $\vec{r}_o - \vec{r}_1$ with the unit vector $\vec{e}_{AB}$ gives

$$\begin{aligned}
(\vec{r}_o - \vec{r}_1) \cdot \vec{e}_{AB} =& \frac{1}{2}|\vec{r}_2 - \vec{r}_1| = \alpha\,\vec{V}_1 \cdot \vec{e}_{AB} + \alpha_2\,\vec{V}_2 \cdot \vec{e}_{AB} \\
\frac{1}{2}|\vec{r}_2 - \vec{r}_1| =& \alpha_2\,[(\vec{r}_3 - \vec{r}_1) \times \hat{n}] \cdot \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \\
\frac{1}{2}|\vec{r}_2 - \vec{r}_1|^2 =& \alpha_2\,[(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] \cdot \hat{n} \\
\text{or} \quad \alpha_2 =& \frac{1}{2}\frac{|\vec{r}_2 - \vec{r}_1|^2}{|(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)|}
\end{aligned}$$

The above result show that the position vector of the circumcenter can be represented by the vector

$$\vec{r}_o = \vec{r}_1 + \frac{|\vec{r}_3 - \vec{r}_1|^2}{D}[(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] \times (\vec{r}_2 - \vec{r}_1) + \frac{|\vec{r}_2 - \vec{r}_1|^2}{D}(\vec{r}_3 - \vec{r}_1) \times [(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)]$$

where

$$D = 2|(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)|^2 \tag{10.88}$$

In the last two equations one can perform the cyclic rotation $1 \to 2 \to 3 \to 1$ to derive additional equations for calculation of the circumcenter.

Utilizing the vector identities (10.35) to (10.40) along with some vector algebra one can verify that the above equation for $\vec{r}_o$ can also be represented in the form

$$\vec{r}_o = \alpha\,\vec{r}_1 + \beta\,\vec{r}_2 + \gamma\,\vec{r}_3 \tag{10.89}$$

where

$$\begin{aligned}
\alpha =& \frac{|\vec{r}_2 - \vec{r}_3|^2\,(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_3)}{D} \\
\beta =& \frac{|\vec{r}_1 - \vec{r}_3|^2\,(\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_3)}{D} \\
\gamma =& \frac{|\vec{r}_1 - \vec{r}_2|^2\,(\vec{r}_3 - \vec{r}_1) \cdot (\vec{r}_3 - \vec{r}_2)}{D}
\end{aligned}$$

where D is given by equation (10.88).

Trigonometric form for circumcenter

The derivation for the trigonometric form for the vector representation of the circumcenter of a triangle $\triangle ABC$ uses the inscribed angle theorem, section formula for vectors, the law of sines and some trigonometry to show the position vector for the circumcenter is given by

$$\vec{r}_0 = \frac{\sin 2A\,\vec{r}_A + \sin 2B\,\vec{r}_B + \sin 2C\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C} \tag{10.90}$$

where $\vec{r}_A,\ \vec{r}_B,\ \vec{r}_C$ are the position vectors to the vertices A, B and C respectively and A, B, C are the vertex angles associated with given triangle.

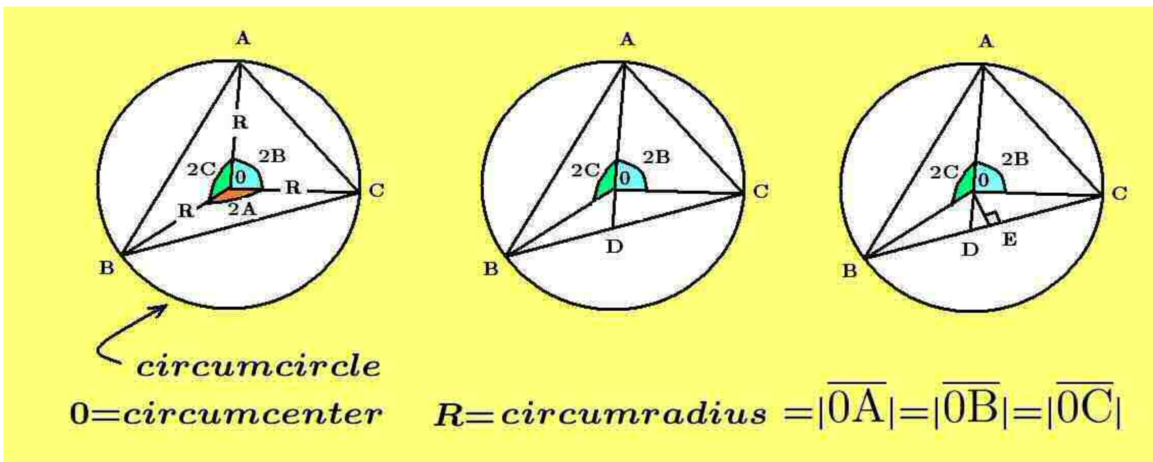


Figure 10-19. Triangle $\triangle ABC$ with circumcircle.

The figure 10-19 illustrates things to look for in this derivation.

(i) The distance from the circumcenter 0 to each vertex is R.

(ii) The inscribed angle theorem tells us the value of the central angles

$\angle A0B = 2C, \quad \angle A0C = 2B, \quad \angle C0B = 2A$

Extend the line $\overline{\text{A0}}$ to intersect side $\overline{\text{BC}}$ at point D and then construct the perpendicular bisector $\overline{\text{0E}}$ which bisects side $\overline{\text{BC}}$. If we can find the ratios $\frac{\overline{\text{BD}}}{\overline{\text{DC}}}$ and $\frac{\overline{\text{A0}}}{\overline{\text{0D}}}$, then we can employ the section formula for vectors to the line segments $\overline{\text{BD}}$ and $\overline{\text{AD}}$. These two results will then produce the equation (10.90).

In triangle $\triangle B0D$ the law of sines produces $\dfrac{\overline{\text{BD}}}{\sin\angle B0D} = \dfrac{\overline{\text{0D}}}{\sin\angle 0BD}$

In triangle $\triangle C0D$ the law of sines produces $\dfrac{\overline{\text{DC}}}{\sin\angle C0D} = \dfrac{\overline{\text{0D}}}{\sin\angle 0CD}$

The above results give the ratio

$$\frac{\overline{\mathrm{BD}}}{\overline{\mathrm{DC}}} = \frac{\sin\angle B0D}{\sin\angle C0D} = \frac{\sin(\pi-2C)}{\sin(\pi-2B)} = \frac{\sin 2C}{\sin 2B}$$

Hence the position vector $\vec{r}_D$ to the point D between the vectors $\vec{r}_B$ and $\vec{r}_C$ is given by

$$\vec{r}_D = \frac{\sin 2B\,\vec{r}_B + \sin 2C\,\vec{r}_C}{\sin 2B + \sin 2C} \tag{10.91}$$

In the last figure observe that triangles $\triangle B0E$ and $\triangle C0D$ are congruent because of Side-Side-Side and therefore $\angle B0E = \angle C0D = A$. One can show $\angle B0D = \pi - 2C$ and $\angle B0E = A$, so that $\angle D0E = A + 2C - \pi$. From the projections $\overline{\mathrm{0D}}\cos(\angle D0E) = \overline{\mathrm{0E}}$ and $\overline{\mathrm{0E}} = R\cos A$ one can write

$$\overline{\mathrm{0D}} = R\cos A\sec(A+2C-\pi)$$

Observe that one can write $A+2C-\pi$ as $A+2C-(A+B+C) = C-B$. This gives rise to the ratio

$$\frac{\overline{\mathrm{A0}}}{\overline{\mathrm{OD}}} = \frac{R}{R\cos A\sec(C-B)} = \frac{\cos(B-C)}{\cos A} = \frac{2\sin A\cos(B-C)}{2\sin A\cos A} = \frac{2\sin A\cos(C-B)}{\sin 2A}$$

Perform some trigonometry on the numerator term $2\sin A\cos(C-B)$ and show

$$\begin{aligned}
\sin A =& \sin(\pi-(B+C)) = \sin(B+C) \\
2\sin A\cos(B-C) =& 2\sin(B+C)\cos(B-C) \\
=& [\sin B\cos C + \cos B\sin C][\cos B\cos C + \sin B\sin C] \text{ expand and show} \\
=& 2\sin B\cos B + 2\sin C\cos C \\
=& \sin 2B + \sin 2C
\end{aligned}$$

This gives the ratio $\frac{\overline{\mathrm{A0}}}{\overline{\mathrm{0D}}} = \frac{\sin 2B + \sin 2C}{\sin 2A}$. The position vector $\vec{r}_0$ to the circumcenter between the vectors $\vec{r}_A$ and $\vec{r}_D$ is given by the vector section formula as

$$\vec{r}_0 = \frac{(\sin 2B + \sin 2C)\,\vec{r}_D + \sin 2A\,\vec{r}_A}{\sin 2A + \sin 2B + \sin 2C}$$

Now substitute for $\vec{r}_D$ from equation (10.91) and show

$$\vec{r}_0 = \frac{(\sin 2B + \sin 2C)\left[\frac{\sin 2C\,\vec{r}_C + \sin 2B\,\vec{r}_B}{\sin 2B + \sin 2C}\right] + \sin 2A\,\vec{r}_A}{\sin 2A + \sin 2B + \sin 2C}$$

which simplifies to the equation (10.90).

Trigonometric form for orthocenter

Recall the orthocenter is the point of concurrency of the three altitudes of a given triangle $\triangle ABC$. The derivation of the trigonometric form for the representation of the orthocenter uses the section formula for vectors, the law of sines and some trigonometry to obtain the position vector. One finds the position vector representing the orthocenter has the form

$$\vec{r}_H = \frac{\tan A\ \vec{r}_A + \tan B\ \vec{r}_B + \tan C\ \vec{r}_C}{\tan A + \tan B + \tan C} \tag{10.92}$$

where $\vec{r}_A,\ \vec{r}_B,\ \vec{r}_C$ are the position vectors to the vertices of the given triangle and A, B, C are the vertex angles.

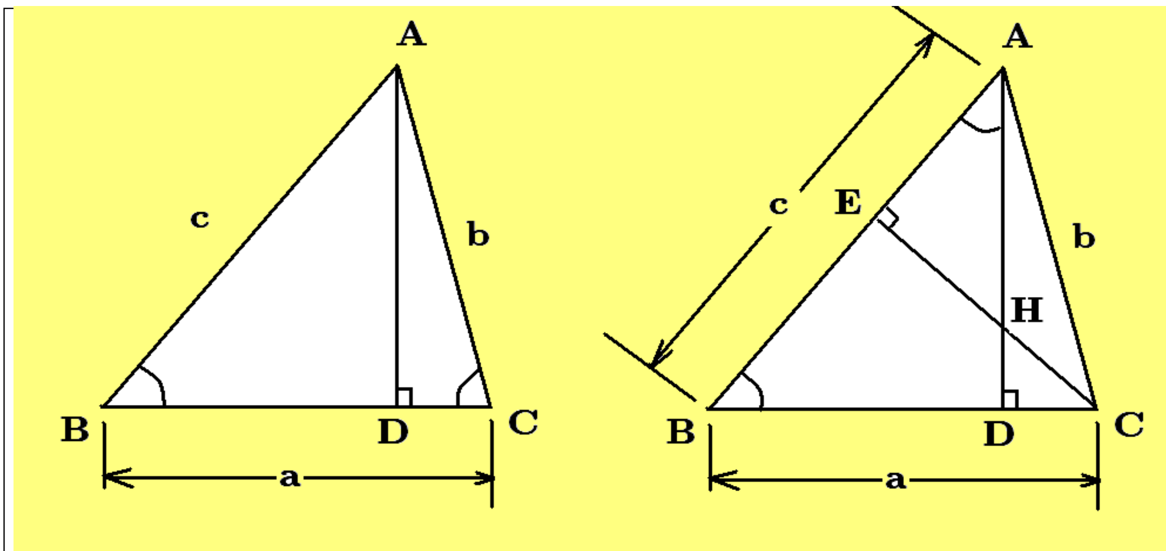


Figure 10-20. Altitudes and orthocenter for triangle $\triangle ABC$

Given triangle $\triangle ABC$, construct the altitude $\overline{\text{AD}}$ to side $\overline{\text{BC}}$. If we know the ratio $\frac{\overline{\text{BD}}}{\overline{\text{DC}}}$ and the position vectors $\vec{r}_B$ and $\vec{r}_C$, then one can use the section formula for vectors to calculate the position vector $\vec{r}_D$.

By constructing the altitude $\overline{\text{CE}}$ to side $\overline{\text{AB}}$, we can label the point of intersection of the two altitudes as the orthocenter H. Knowing the ratio $\frac{\overline{\text{AH}}}{\overline{\text{HD}}}$ and the position vector $\vec{r}_A$ and $\vec{r}_D$ one can use the section formula for vectors to calculate the position vector $\vec{r}_H$ of the orthocenter.

Working toward the accomplishment of these goals, examine the left triangle in figure 10-20 and calculate the projections of sides c and b onto the side $\overline{\text{BC}}$ and show

$$\overline{\text{BD}} = c\ \cos B \text{ and } \overline{\text{DC}} = b\ \cos C \ \Rightarrow\ \frac{\overline{\text{BD}}}{\overline{\text{DC}}} = \frac{c\ \cos B}{b\ \cos C}$$

Using the law of sines one can write

$$\frac{c}{\sin C} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{c}{b} = \frac{\sin C}{\sin B}$$

Consequently,

$$\frac{\overline{\text{BD}}}{\overline{\text{DC}}} = \frac{\sin C}{\sin B} \cdot \frac{\cos B}{\cos C} = \frac{\tan C}{\tan B}$$

Using the section formula for vectors one finds the position vector for point D is

$$\vec{r}_D = \frac{\tan B\ \vec{r}_B + \tan C\ \vec{r}_C}{\tan B + \tan C} \tag{10.93}$$

Next examine the second triangle in figure 10-20 where the altitude $\overline{\text{CE}}$ has been constructed. The projection of $\overline{\text{AH}}$ onto $\overline{\text{AB}}$ is $\overline{\text{AE}} = \overline{\text{AH}} \cos(\frac{\pi}{2} - B)$ and from triangle $\triangle HDC$ one finds $\tan(\frac{\pi}{2} - B) = \frac{\overline{\text{HD}}}{\overline{\text{DC}}}$. These results produce the ratio

$$\frac{\overline{\text{AH}}}{\overline{\text{HD}}} = \frac{\overline{\text{AE}}\ \sec(\frac{\pi}{2} - B)}{\overline{\text{DC}}\ \tan(\frac{\pi}{2} - B)} = \frac{b\ \cos A\ \sec(\frac{\pi}{2} - B)}{b\ \cos C\ \tan(\frac{\pi}{2} - B)} \tag{10.94}$$

where $\overline{\text{AE}}$ is the projection of side b onto the side $\overline{\text{AB}}$ and $\overline{\text{DC}}$ is the projection of side b onto the side $\overline{\text{BC}}$. Now one can use trigonometry to write equation (10.94) in a different form.

Using basic trigonometry one finds $\sec(\frac{\pi}{2} - B) = \csc B$, $\tan(\frac{\pi}{2} - B) = \cot B$. The sum of the angles of any triangle must sum to π radians and so one can write $A = \pi - (B + C)$ so that

$$\begin{aligned} \frac{\overline{\text{AH}}}{\overline{\text{HD}}} =& \frac{\cos(\pi - (B + C))}{\cos C} \cdot \frac{\csc B}{\cot B} = -\frac{\cos(B + C)}{\cos C} \cdot \frac{1}{\sin B} \cdot \frac{\sin B}{\cos B} \\ =& \frac{-(\cos B \cos C - \sin B \sin C)}{\cos B \cos C} = -(1 - \tan B \tan C) \end{aligned}$$

Recall the trigonometric identity $\tan(B + C) = \dfrac{\tan B + \tan C}{1 - \tan B \tan C}$ and write

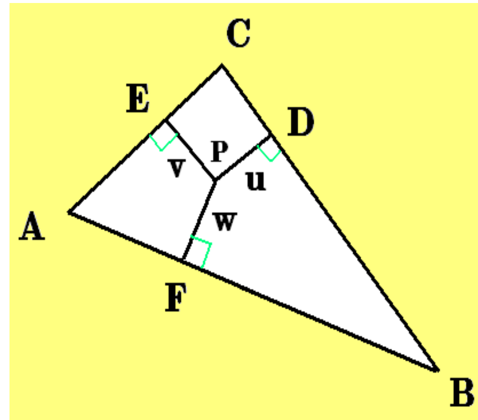
$$\frac{\overline{\text{AH}}}{\overline{\text{HD}}} = -\left(\frac{\tan B + \tan C}{\tan(B + C)}\right) = -\left(\frac{\tan B + \tan C}{\tan(\pi - A)}\right) = \frac{\tan B + \tan C}{\tan A}$$

Therefore, the position vector $\vec{r}_H$ of the orthocenter is obtained using the section formula for vectors, giving

$$\vec{r}_H = \frac{\tan A\ r_A + (\tan B + \tan C)\ \vec{r}_D}{\tan A + \tan B + \tan C} \tag{10.95}$$

By substituting for the position vector $\vec{r}_D$ from equation (10.93) one obtains the result given by equation (10.92).

Trilinear coordinates



For P a point inside a general triangle $\triangle ABC$ we have learned that one can construct lines through point P which are perpendicular to the sides of the triangle as illustrated in the accompanying figure. Let these perpendiculars intersect the triangle sides at the points D,E and F as illustrated. Define the positive distances

$$u = |\overline{\text{PD}}|, \quad v = |\overline{\text{PE}}|, \quad w = |\overline{\text{PF}}|$$

as lengths associated with the line segments $\overline{\text{PD}}, \overline{\text{PE}}, \overline{\text{PF}}$.

Define the exact trilinear[9] coordinates of the point P using the notation

$$P = [u : v : w] \tag{10.96}$$

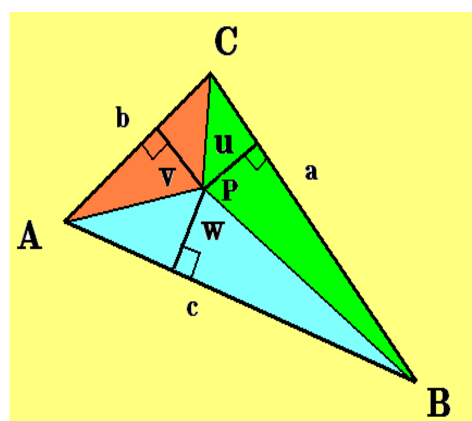
Any triple $[\lambda_1 : \lambda_2 : \lambda_3]$ is called a trilinear coordinate of point P if there exists a positive constant k different from zero such that

$$u = k\lambda_1, \quad v = k\lambda_2, \quad w = k\lambda_3 \tag{10.97}$$

then the trilinear coordinates $[u : v : w]$ and $[\lambda_1 : \lambda_2 : \lambda_3]$ are said to be equivalent because they have equivalent ratios

$$\frac{u}{v} = \frac{\lambda_1}{\lambda_2}, \quad \frac{u}{w} = \frac{\lambda_1}{\lambda_3}, \quad \frac{v}{w} = \frac{\lambda_2}{\lambda_3}$$

Any equivalent set of trilinear coordinates can be converted to exact trilinear coordinates by constructing a scale constant k as follows.



Construct lines from the point P to each of the triangle vertices to form three triangles as illustrated. The area of these triangles are

$$\begin{aligned}
[APB] &= \text{area } \triangle APB = \frac{1}{2}wc \\
[BPC] &= \text{area } \triangle PBC = \frac{1}{2}ua \\
[CPA] &= \text{area } \triangle CPA = \frac{1}{2}vb
\end{aligned}$$

[9] Trilinear coordinates were introduced by Julius Plücker (1801-1868), a German mathematician.

where a, b, c are the sides of the given triangle and the total area of the triangle $\triangle ABC$ is given by

$$[ABC] = \text{area } \triangle ABC = \frac{1}{2}(au + bv + cw) \tag{10.98}$$

Substitute for u, v, w from the equations (10.97) and show

$$2[ABC] = k\,(a\lambda_1 + b\lambda_2 + c\lambda_3)$$

so the scale factor k is found to be

$$k = \frac{2[ABC]}{a\lambda_1 + b\lambda_2 + c\lambda_3} \tag{10.99}$$

Here the proportionality constant k can be constructed by knowing (i) the total area of the triangle, (ii) the sides of the triangle and (iii) the trilinear coordinates of the point P.

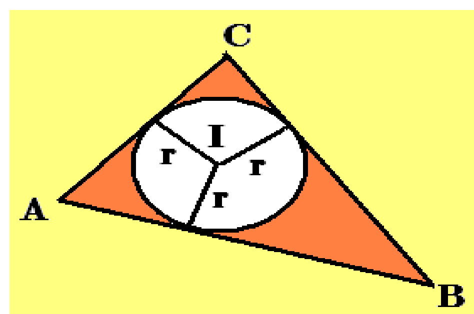
Example 10-27.

Find the trilinear coordinates of (i) the triangle vertices A,B,C and (ii) the triangle incenter I.

Solution:

It is easy to verify the vertices of the triangle $\triangle ABC$ have the following trilinear coordinates.

Vertex	Exact trilinear coordinates	Scaled trilinear coordinates
A	$[h_A : 0 : 0]$	$[1 : 0 : 0]$
B	$[0 : h_B : 0]$	$[0 : 1 : 0]$
C	$[0 : 0 : h_c]$	$[0 : 0 : 1]$



where h_A, h_B, h_C are the altitudes from the vertices A, B, C respectively.

The triangle incenter I is the point of concurrency where the angle bisectors meet. This point is equidistant from the triangle sides so that the exact trilinear coordinates for I are $[r : r : r]$. Removing the common factor r, one can say that the trilinear coordinates for the incenter are give by $[1 : 1 : 1]$.

∎

Example 10-28.

Find the trilinear coordinates for the orthocenter of a general triangle $\triangle ABC$.

Solution:

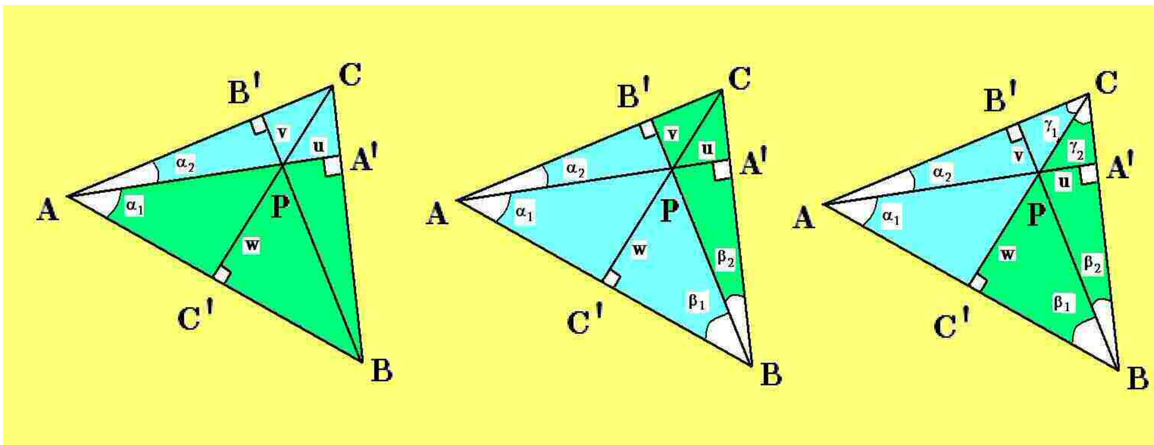


Figure 10-21. Orthocenter for triangle $\triangle ABC$

Examine the left side of figure 10-21 and show in the right triangle $\triangle AA'B$ that angle $\alpha_1 = \frac{\pi}{2} - B$ and in the right triangle $\triangle AA'C$ show angle $\alpha_2 = \frac{\pi}{2} - C$. It is then easy to show

$$\begin{aligned} \sin\alpha_1 &= \sin(\frac{\pi}{2} - B) = \cos B = \frac{w}{\overline{\text{AP}}} \\ \sin\alpha_2 &= \sin(\frac{\pi}{2} - C) = \cos C = \frac{v}{\overline{\text{AP}}} \end{aligned} \Rightarrow \quad \frac{w}{v} = \frac{\cos B}{\cos C} \tag{10.100}$$

Use the middle sketch from figure 10-21 to show in the right triangle $\triangle AB'B$ the angle $\beta_2 = \frac{\pi}{2} - A$ and in the triangle $\triangle BB'C$ the angle $\beta_1 = \frac{\pi}{2} - C$ so that

$$\begin{aligned} \sin\beta_1 &= \sin(\frac{\pi}{2} - C) = \cos C = \frac{u}{\overline{\text{PB}}} \\ \sin\beta_2 &= \sin(\frac{\pi}{2} - A) = \cos A = \frac{w}{\overline{\text{PB}}} \end{aligned} \Rightarrow \quad \frac{u}{w} = \frac{\cos C}{\cos A} \tag{10.101}$$

In a similar fashion one can use the last figure in figure 10-21 and demonstrate

$$\cos A = \frac{v}{\overline{\text{PC}}} \quad \text{and} \quad \cos B = \frac{u}{\overline{\text{PC}}} \Rightarrow \quad \frac{u}{v} = \frac{\cos B}{\cos A} \tag{10.102}$$

Therefore, the exact trilinear coordinates $[u : v : w]$ of the orthocenter is equivalent to $[\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}]$. This follows from the ratios established in equations (10.100), (10.101), and (10.102). That is

$$\frac{u}{v} = \frac{\frac{1}{\cos A}}{\frac{1}{\cos B}} = \frac{\cos B}{\cos A}, \quad \frac{u}{w} = \frac{\frac{1}{\cos A}}{\frac{1}{\cos C}} = \frac{\cos C}{\cos A}, \quad \frac{v}{w} = \frac{\frac{1}{\cos B}}{\frac{1}{\cos B}} \frac{1}{\cos C} = \frac{\cos C}{\cos B}$$

■

Example 10-29.

Find the trilinear coordinates of the circumcenter of a general triangle $\triangle ABC$.

Solution:

The circumcenter P is the point of concurrency of the perpendicular bisectors and center for the circumscribed circle. The situation is sketched in figure 10-22. Let $\beta = \angle APD$ so that

$$\cos\beta = \frac{w}{R} \tag{10.103}$$

where R is the radius of the circumscribed circle. Note that $\triangle APD \cong \triangle BPD$ by SSS, so that $\angle DPB = \beta$ also. Here 2β is a central angle with angle $\angle C = \frac{1}{2}\angle APB = \beta$. Consequently, equation (10.103) can be expressed as $R\cos C = w$. In a similar fashion, one can show

$$v = R\cos B \quad \text{and} \quad u = R\cos A \tag{10.104}$$

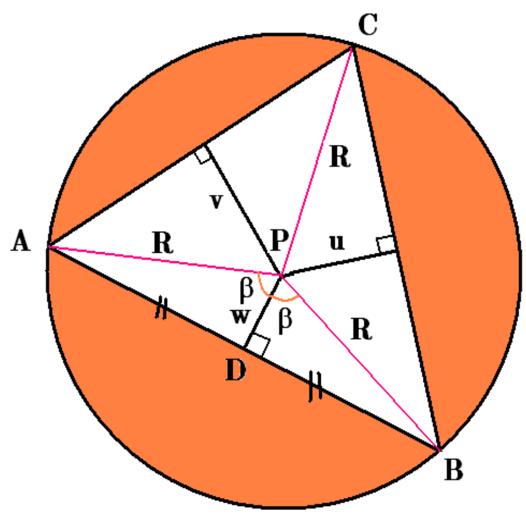


Figure 10-22. Circumcenter for triangle $\triangle ABC$

Therefore, the exact trilinear coordinates of the circumcenter is given by

$$P = [R\cos A : R\cos B : R\cos C] \tag{10.105}$$

Factoring out the common factor R one can say the trilinear coordinates of the circumcenter is given by

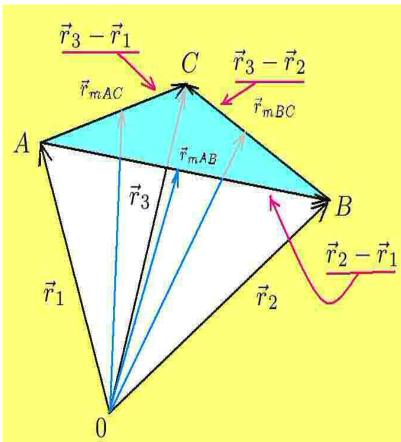
$$P = [\cos A : \cos B : \cos C] \tag{10.107}$$

■

Example 10-30.

Find the trilinear coordinates for the centroid of a general triangle $\triangle ABC$.

Solution:



Consider the 3-D triangle $\triangle ABC$ with vertices A,B,C described by position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ as illustrated in the accompanying sketch. The centroid of the triangle lies at the point of concurrency of the triangle medians. Recall that the position vectors to the midpoints of the triangle sides are

$$\vec{r}_{mAC} = \frac{1}{2}(\vec{r}_1 + \vec{r}_3)$$
$$\vec{r}_{mBC} = \frac{1}{2}(\vec{r}_2 + \vec{r}_3)$$
$$\vec{r}_{mAB} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

The median through the vertices to the midpoints of the opposite sides have the following equations

$$\text{median through vertex A} \quad \vec{r} = \vec{r}_1 + \mu_1(\frac{1}{2}(\vec{r}_2 + \vec{r}_3) - \vec{r}_1)$$
$$\text{median through vertex B} \quad \vec{r} = \vec{r}_2 + \mu_2(\frac{1}{2}(\vec{r}_1 + \vec{r}_3) - \vec{r}_2)$$
$$\text{median through vertex C} \quad \vec{r} = \vec{r}_3 + \mu_3(\frac{1}{2}(\vec{r}_1 + \vec{r}_2) - \vec{r}_3)$$

where μ_1, μ_2, μ_3 are parameters of the lines. At the point of intersection the lines meet. Setting the above vectors equal to one another one finds

$$\vec{r}_1 + \mu_1(\frac{1}{2}(\vec{r}_2 + \vec{r}_3) - \vec{r}_1) = \vec{r}_2 + \mu_2(\frac{1}{2}(\vec{r}_1 + \vec{r}_3) - \vec{r}_2) = \vec{r}_3 + \mu_3(\frac{1}{2}(\vec{r}_1 + \vec{r}_2) - \vec{r}_3)$$

The first equality requires that

$$1 - \mu_1 = \frac{1}{2}\mu_2, \quad \mu_1 = \mu_2, \quad \frac{\mu_1}{2} = 1 - \mu_2$$

with similar requirements for the second equality. These equalities require

$$\mu_1 = \mu_2 = \mu_2 = \frac{2}{3}$$

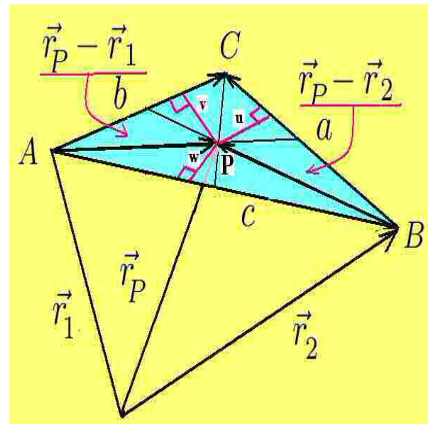
which produces the position vector

$$\vec{r}_P = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)$$

Define the vectors

$$\overrightarrow{BC} = \vec{r}_3 - \vec{r}_2, \quad \overrightarrow{AC} = \vec{r}_3 - \vec{r}_1, \quad \overrightarrow{AB} = \vec{r}_2 - \vec{r}_1$$

and use the results from example 10-19 to show the distances u, v, w from point P to the triangle sides are given by



$$u = \left| (\vec{r}_P - \vec{r}_2) \times \frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} \right|$$
$$v = \left| (\vec{r}_P - \vec{r}_1) \times \frac{\overrightarrow{AC}}{|\overrightarrow{AC}|} \right|$$
$$w = \left| (\vec{r}_P - \vec{r}_1) \times \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \right|$$

where $|\overrightarrow{BC}| = a, \ |\overrightarrow{AC}| = b, \ |\overrightarrow{AB}| = c$ are the triangle sides.

As an exercise expand the cross products in the numerators associated with the distances u, v, w and show they all have the same magnitude. This demonstrates the trilinear coordinates for the centroid are

$$[u : v : w] = [\frac{\alpha}{a} : \frac{\alpha}{b} : \frac{\alpha}{c}]$$

removing the constant factor α one finds the trilinear coordinates for the centroid are $[\frac{1}{a} : \frac{1}{b} : \frac{1}{c}]$. Remember that the trilinear coordinates can be scaled. If one multiplies by abc one finds $[bc : ac : ab]$ is another form for the trilinear coordinates. If one uses the law of sines, one can show $[\frac{1}{\sin A} : \frac{1}{\sin B} : \frac{1}{\sin C}]$ is still another form.

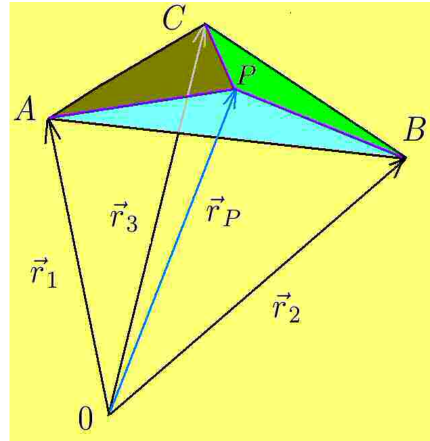
■

Barycentric coordinates for triangles

Barycentric coordinates[10] is the representation of a point P on the surface of a triangle in terms of areas determined by point P. For the general triangle illustrated let the vertices A, B, C be defined by position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ and let $[ABC]$ denote the total area of the triangle. Connect the point P to each vertex and define

$$[ABP] = \text{area } \triangle ABP$$
$$[BCP] = \text{area } \triangle BCP$$
$$[CAP] = \text{area } \triangle CAP$$

[10] Introduce by August Ferdinand Möbius (1790-1868) a German mathematician.



Define the barycentric coordinates of point P as $[\lambda_1 : \lambda_2 : \lambda_3]$ where

$$\lambda_1 = \frac{[BCP]}{[ABC]}, \quad \lambda_2 = \frac{[CAP]}{[ABC]}, \quad \lambda_3 = \frac{[ABP]}{[ABC]} \tag{10.107}$$

Observe that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and these ratios produce the vector

$$\vec{r}_P = \lambda_1\vec{r}_1 + \lambda_2\vec{r}_2 + \lambda_3\vec{r}_3 \tag{10.108}$$

which points to the point P in the plane of the triangle. Let $\overrightarrow{AB} = \vec{r}_2 - \vec{r}_1$ and $\overrightarrow{AC} = \vec{r}_3 - \vec{r}_1$ denote two independent vectors along the sides of triangle $\triangle ABC$ and express the vector $\vec{r}_P$ of equation (10.108) in the form

$$\begin{aligned}\vec{r}_P =& \lambda_1\vec{r}_1 + \lambda_2[(\vec{r}_2 - \vec{r}_1) + \vec{r}_1] + \lambda_3[(\vec{r}_3 - \vec{r}_1) + \vec{r}_1]\\ \vec{r}_P =& (\lambda_1 + \lambda_2 + \lambda_3)\vec{r}_1 + \lambda_2\overrightarrow{AB} + \lambda_3\overrightarrow{AC}\\ \vec{r}_P =& \vec{r}_1 + \lambda_2\overrightarrow{AB} + \lambda_3\overrightarrow{AC}\end{aligned}$$

since $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Relation between trilinear and barycentric coordinates

In triangle $\triangle ABC$ with sides a, b, c a point with trilinear coordinates $[t_1 : t_2 : t_3]$ has the barycentric coordinates $[a\,t_1 : b\,t_2 : c\,t_3]$. A point with barycentric coordinates $[\lambda_1 : \lambda_2 : \lambda_3]$ has the trilinear coordinates $[\frac{\lambda_1}{a} : \frac{\lambda_2}{b} : \frac{\lambda_3}{c}]$. Thus, one need only know the sides of triangle $\triangle ABC$ to convert from one set of coordinates to another.

Exercises

▶ **10-1.** If $\vec{A} \times (\vec{B} - \vec{C}) = \vec{0}$ what can be said about the vectors $\vec{A}$ and $\vec{B} - \vec{C}$?

▶ **10-2.** Given the vectors $\vec{A} = -\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2,\ \vec{B} = 4\,\hat{\mathbf{e}}_1,\ \vec{C} = -3\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2$

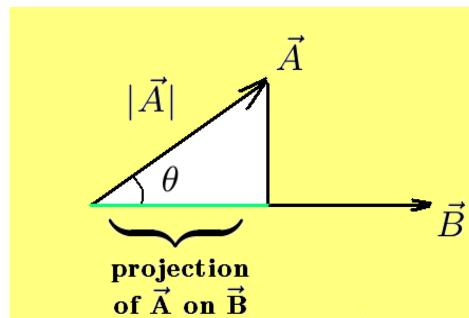
(a) Find the vector sum $\vec{D} = \vec{A} + \vec{B} + \vec{C}$

(b) Find the vector sum $\vec{E} = 2\vec{A} - \vec{B} + 3\vec{C}$

(c) Find unit vectors $\hat{\mathbf{e}}_A,\ \hat{\mathbf{e}}_B,\ \hat{\mathbf{e}}_C$ in the directions $\vec{A}, \vec{B}, \vec{C}$

(d) Show $\vec{C}$ is a linear combination of the vectors $\vec{B}$ and $\vec{C}$

▶ **10-3.**



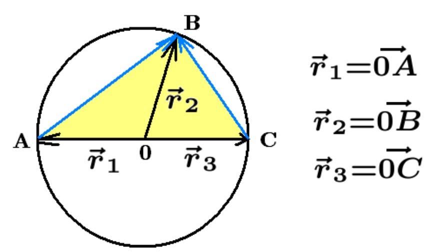
(a) Show the projection of $\vec{A}$ on $\vec{B}$ is $|\vec{A}|\cos\theta$

(b) Why is $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$?

(c) Show the projection of $\vec{A}$ on $\vec{B}$ is $\dfrac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$

(d) What is the projection of $\vec{B}$ on $\vec{A}$?

▶ **10-4.**



Let $\overline{\text{AC}}$ denote the diameter of a circle and B any point on the circular arc. Construct the vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ as illustrated.

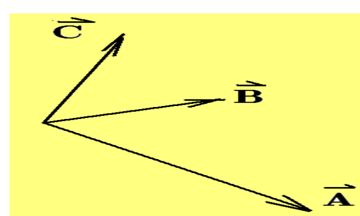
(a) Show $\overrightarrow{AB} = \vec{r}_2 - \vec{r}_1$

(b) Show $\overrightarrow{BC} = \vec{r}_2 - \vec{r}_3$

(c) Show $\vec{r}_1 + \vec{r}_3 = \vec{0}$

(d) Show $(\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_3) = 0 \ \Rightarrow\ \overline{\text{AB}} \perp \overline{\text{BC}} \ \Rightarrow\ \angle ABC = \dfrac{\pi}{2}$

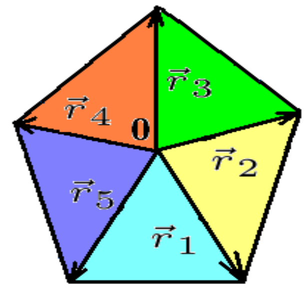
▶ **10-5.**



If vectors $\vec{A}, \vec{B}, \vec{C}$ have a common origin and represent the sides of a parallelogram, then show the diagonal of the parallelogram is given by $\vec{d} = \vec{A} + \vec{B} + \vec{C}$.

► **10-6.**

Given a regular pentagon with vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4, \vec{r}_5$ constructed from the center of the pentagon to each vertex as illustrated.

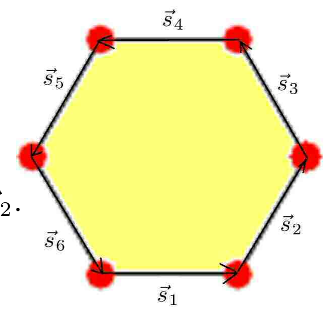


(a) Find representations for each vector.

(b) Show that $\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4 + \vec{r}_5 = \vec{0}$

► **10-7.**
The sides of a regular hexagon are defined by the six vectors

$$\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4, \vec{s}_5, \vec{s}_6$$



(a) Find representations for $\vec{s}_1$ and $\vec{s}_2$

(b) Find the vectors $\vec{s}_3, \vec{s}_4, \vec{s}_5, \vec{s}_6$ in terms of the vectors $\vec{s}_1$ and $\vec{s}_2$.

(c) Show $\vec{s}_1 + \vec{s}_2 + \vec{s}_3 + \vec{s}_4 + \vec{s}_5 + \vec{s}_6 = \vec{0}$

► **10-8.** Find the vector equation of the line through the given points.

(a) $(1, 2, 3)$ and $(7, 11, 30)$ (b) $(-2, 4, 6)$ and $(5, 1, 8)$ (c) $(0, 0, 0)$ and $(7, 11, 22)$

► **10-9.** Find the equation of the plane which passes through the given points.

(a) $(1, 2, 3),\ (7, 11, 4),\ (2, 3, 6)$

(b) $(4, 1, 2),\ (3, 6, 7),\ (9, 3, 2)$

(c) $(5, 3, 1),\ (3, 3, 8),\ (8, 1, -1)$

► **10-10.** Find the perpendicular distance from the given point to the given plane.

(a) $(3, 7, 10)$ to plane $(x-1)5 + (y-1)6 + (z-1)7 = 0$

(b) $(-1, -3, -5)$ to plane $(x-5) + (y-2)3 + (x-1)4 = 0$

(c) $(0, 0, 0)$ to plane $(x-2)2 + (x-3)5 + (z-4)(-1) = 0$

▶ **10-11.** Given the vectors $\vec{A} = 3\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + 5\,\hat{\mathbf{e}}_3$, $\vec{B} = 2\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3\,\hat{\mathbf{e}}_3$, $\vec{C} = 3\,\hat{\mathbf{e}}_1 - 3\,\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$. Calculate

$$\begin{array}{lll} (a)\ \vec{A}\times\vec{B} & (e)\ \vec{B}\times\vec{C} & (i)\ \vec{B}\cdot(\vec{C}\times\vec{A}) \\ (b)\ \vec{B}\times\vec{A} & (f)\ (\vec{A}\times\vec{B})\times\vec{C} & (j)\ \vec{C}\cdot(\vec{A}\times\vec{B}) \\ (c)\ (\vec{A}\times\vec{B})\cdot\vec{C} & (g)\ \vec{A}\times(\vec{B}\times\vec{C}) & (k)\ (\vec{A}\times\vec{B})\cdot\vec{C} \\ (d)\ (\vec{A}+\vec{B})\times(\vec{A}-\vec{C}) & (h)\ \vec{A}\cdot(\vec{B}\times\vec{C}) & (l)\ \vec{A}\cdot(\vec{A}\times\vec{B}) \end{array}$$

▶ **10-12.**

(a) Explain the importance of parentheses in the vector expression $\vec{A}\times\vec{B}\times\vec{C}$. That is, does $(\vec{A}\times\vec{B})\times\vec{C} = \vec{A}\times(\vec{B}\times\vec{C})$?

(b) Does $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$?

(c) Does $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$?

▶ **10-13.** Find the vector equation of the plane which passes through the given point P and is perpendicular to the given normal vector $\vec{N}$.

$$\begin{array}{lll} (a)\ P: & (1,2,3) & \vec{N} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \\ (b)\ P: & (-1,3,5) & \vec{N} = 2\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3\,\hat{\mathbf{e}}_3 \\ (c)\ P: & (0,4,5) & \vec{N} = \hat{\mathbf{e}}_3 \end{array}$$

▶ **10-14.** Find the equation of the plane which passes through the given points.

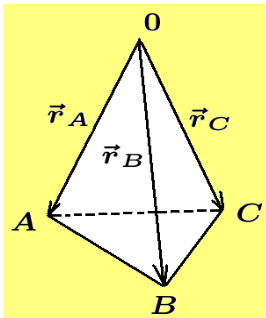
$$\begin{array}{l} (a)\ (1,2,4),\ (2,4,8),\ (-1,-2,1) \\ (b)\ (0,0,1),\ (1,0,0),\ (0,1,0) \\ (c)\ (0,0,1),\ (1,0,0),\ (0,-1,0) \end{array}$$

▶ **10-15.** Find the perpendicular distance from the given point P to the given plane.

$$\begin{array}{lll} (a)\ P: & (4,5,6) & 2(x-1)-(y-2)-3(z-3)=0 \\ (b)\ P: & (-2,3,-5) & (x-3)-(y-5)+2(z-3)=0 \\ (c)\ P: & (-1,-1,-1) & 3(x-2)+2(y-2)-(z-3)=0 \end{array}$$

▶ **10-16.**

Let vectors $\vec{r}_A,\ \vec{r}_B,\ \vec{r}_C$ define a tetrahedron $0ABC$.



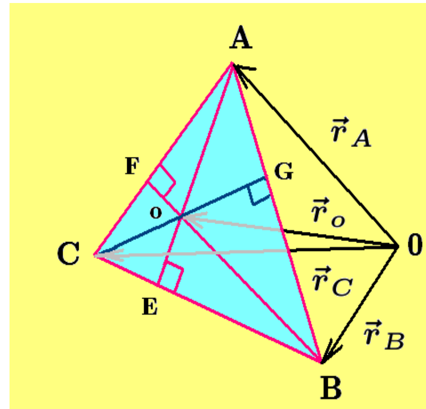
(a) Give a physical interpretation to the vector magnitudes

$$|\vec{v}_1| = |\frac{1}{2}\vec{r}_A\times\vec{r}_B|, \quad |\vec{v}_2 = |\frac{1}{2}\vec{r}_B\times\vec{r}_C|$$
$$|\vec{v}_3| = |\frac{1}{2}\vec{r}_C\times\vec{r}_A|, \quad |\vec{v}_4| = |\frac{1}{2}(\vec{r}_C-\vec{r}_A)\times(\vec{r}_B-\vec{r}_A)$$

(b) Show that $\vec{v}_1+\vec{v}_2+\vec{v}_3+\vec{v}_4 = \vec{0}$

▶ **10-17.**

Given the position vectors $\vec{r}_A, \vec{r}_B, \vec{r}_C$ which defines triangle $\triangle ABC$ in space. Construct the altitudes $\overline{\text{AE}}$ to side $\overline{\text{BC}}$ and $\overline{\text{BF}}$ to side $\overline{\text{CA}}$ with the altitudes intersecting at point o. Construct the vector $\vec{r}_o$ and then construct line $\overline{\text{Co}}$ and extend line to intersect side $\overline{\text{AB}}$ at point G. Show $\overline{\text{CG}} \perp \overline{\text{AB}}$ and altitudes meet at a point of concurrency. Give reasons for each of the following statements.



(a) $(\vec{r}_C - \vec{r}_B) \cdot (\vec{r}_0 - \vec{r}_A) = 0$

(b) $(\vec{r}_C - \vec{r}_A) \cdot (\vec{r}_0 - \vec{r}_B) = 0$

(c) Verify by subtraction $(\vec{r}_C - \vec{r}_0) \cdot (\vec{r}_A - \vec{r}_B) = 0$

(d) $\overline{\text{CG}} \perp \overline{\text{AB}}$

(e) Altitudes meet at point of concurrency.

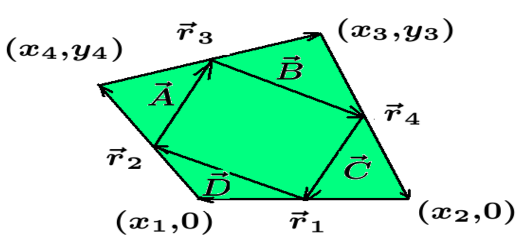
▶ **10-18.** Use vectors to prove the diagonals of a rhombus are perpendicular.

▶ **10-19.** Use vectors to prove the diagonals of parallelogram bisect one another.

▶ **10-20.** Given a triangle $\triangle ABC$ with the x, y-axes passing through the circumcenter of the triangle.

Show that $\vec{r}_0 = 0,\ \vec{r}_G = \frac{1}{3}(\vec{r}_A + \vec{r}_B + \vec{r}_C),\ \vec{r}_H = (\vec{r}_A + \vec{r}_B + \vec{r}_C),\ \vec{r}_I = \dfrac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}$ where $\vec{r}_A,\ \vec{r}_B,\ \vec{r}_C$ are the position vectors to the vertices A, B, C and a, b, c are the sides of the triangle $\triangle ABC$.

▶ **10-21.**



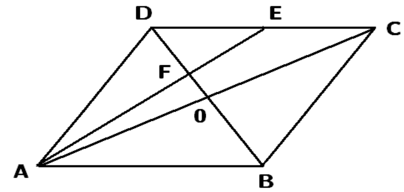
Given a general quadrilateral with sides $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$. (a) Find the vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ joining the midpoints of the quadrilateral sides as illustrated. (b) Show that the vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ define a parallelogram.

▶ **10-22.** Find the incenter and centroid of the triangles with vertices

(a) $(1, 1, 1),\ (5, 6, 10),\ (8, 8, 0)$

(b) $(0, 0, 0),\ (5, 5, 5),\ (2, -2, 0)$

▶ **10-23.**
Given the parallelogram ABCD with diagonals $\overline{\text{AC}}$ and $\overline{\text{DB}}$ intersecting at point 0. Let E denote the mid-point of line $\overline{\text{DC}}$ and then construct the line $\overline{\text{AE}}$ which intersect diagonal $\overline{\text{DB}}$ at point F. Define the vectors



$$\overrightarrow{AB} = \vec{r}_1 \qquad \overrightarrow{B0} = \vec{r}_3 \qquad \overrightarrow{AE} = \vec{r}_5$$
$$\overrightarrow{A0} = \vec{r}_2 \qquad \overrightarrow{AD} = \vec{r}_4$$

and use these vectors to show that $\overline{\text{AF}} = 2\,\overline{\text{FE}}$ and $\overline{\text{DF}} = 2\,\overline{\text{F0}}$
Hint: There exists scalars α and β such that $\alpha\vec{r}_5 + \beta\vec{r}_3 = \vec{r}_4$. Note vectors can be moved in the plane as long as they maintain their length and direction. For example, $\vec{r}_1 + \vec{r}_4 - \frac{1}{2}\vec{r}_1 = \vec{r}_5$ and $\vec{r}_1 + 2\vec{r}_3 = \vec{r}_4$.

▶ **10-24.** Given triangle $\triangle ABC$ with position vectors $\vec{r}_A, \vec{r}_B, \vec{r}_C$ to the vertices A, B, C. Show that

$$[ABC] = \frac{1}{2} \mid (\vec{r}_B - \vec{r}_A) \times (\vec{r}_C - \vec{r}_A) \mid$$
$$[ABC] = \frac{1}{2} \mid \vec{(}r_C - \vec{r}_B) \times (\vec{r}_A - \vec{r}_B) \mid$$
$$[ABC] = \frac{1}{2} \mid (\vec{r}_A - \vec{r}_C) \times (\vec{r}_B - \vec{r}_C) \mid$$

where $[ABC]$ denotes the area of triangle $\triangle ABC$.

▶ **10-25.** (Messy algebra) Show in the special two dimensional case $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2$, $\vec{r}_2 = x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2$, $\vec{r}_3 = x_3\,\hat{\mathbf{e}}_1 + y_3\,\hat{\mathbf{e}}_3$ the vector equation for the position of the circum-center

$$\vec{r}_o = \vec{r}_1 + \frac{|\vec{r}_3 - \vec{r}_1|^2}{D}[(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] \times (\vec{r}_2 - \vec{r}_1) + \frac{|\vec{r}_2 - \vec{r}_1|^2}{D}(\vec{r}_3 - \vec{r}_1) \times [(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)]$$

where

$$D = 2|(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)|^2$$

reduces to the equations (3.17) and (9.91).

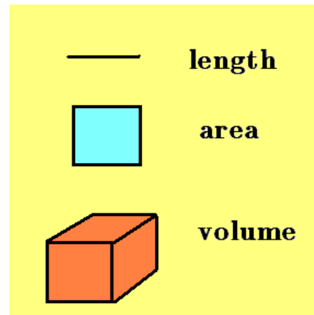
Geometry

Chapter 11

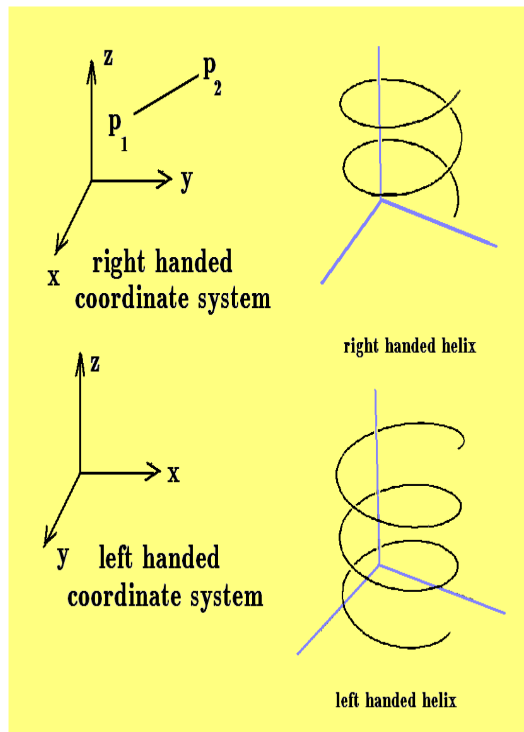
Solid Geometry I

Three dimensional solids

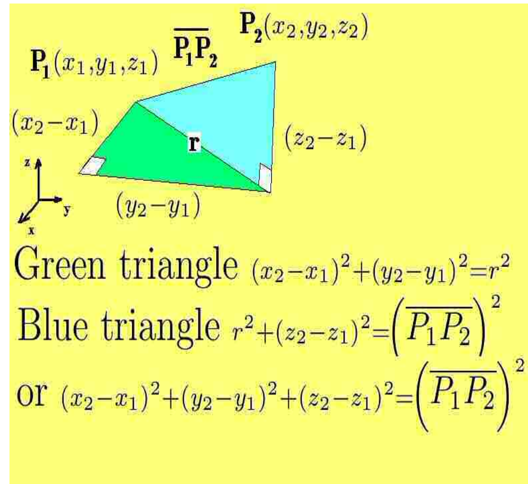
In previous chapters we have used line segments to measure one-dimensional length and we have used squares to measure two-dimensional area with units of length squared. Now we will use cubes to measure volume with units of length cubed. In this chapter we will learn how to define solids and then find their surface area and volume as well as other properties associated with various solid structures.



One can construct an (x, y, z) Cartesian coordinate system in three dimensional space. Consider three axes which meet at a point, called the origin, where all three axes are perpendicular to one another. The coordinate system can be right-handed or left-handed. Extend the fingers of the right hand to point along the positive x-axis and rotate the fingers toward the y-axis. The thumb then points in the z-direction. This then is a right-handed coordinate system. If you extend the fingers of the left hand in the direction of the x-axis and rotate the fingers toward the y-axis and the thumb points in the z-direction, then the coordinate system is left-handed. Many quantities in nature are either left-handed or right-handed. For example, in biology there are left and right-handed DNA helices.



In the Americas most books use a right-handed coordinate system, while in European countries many technical books use left-handed coordinate systems. Note that many equations in science and engineering from European countries will differ in sign from those in the Americas because of the coordinate system used.



The right-handed coordinate system is defined by three perpendicular planes, called the coordinate planes. The intersection of the coordinate planes produce the lines defining the coordinate axes. A point in space $p_1 = p_1(x_1, y_1, z_1)$ is defined by three coordinates (x_1, y_1, z_1) where

$$x_1 = \text{distance from the } yz\text{-plane}$$
$$y_1 = \text{distance from the } xz\text{-plane}$$
$$z_1 = \text{distance from the } xy\text{-plane}$$

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is found by using the Pythagorean theorem to produce the distant formula in three dimensions as

$$\overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{11.1}$$

for the line segment $\overline{P_1P_2}$. This result can be derived by examining the accompanying figure.

Polyhedra

Whenever a solid figure is constructed using polygons, the resulting solid is called a **polyhedron**[1] (plural polyhedra). Anytime one deals with two congruent polygons, called bases, that lie in parallel planes directly over one another and their vertices are properly aligned and are connected by straight lines, then the resulting polyhedron is called **a prism.** A parallelepiped is an example of a prism having bases which are parallelograms and faces which are parallelograms.

The figure 11-1 is a sketch of two parallel planes a perpendicular distance h separating them along with a two dimensional polygon in each plane **where the polygons are congruent and directly in line with one another.** By connecting the vertices of the two figures with lines perpendicular to both planes, one can construct the prisms illustrated. The names associated with the prisms are determined by the shape of the polygon base. For example, a rectangular prism would have a

[1] In Greek the word "poly" means many and the suffix "hedron" is the Greek word for surface or face of a figure. Thus, a polyhedron describes a figure with many faces.

rectangular base, a triangular prism would have a triangle for its base, etc. Note that the rectangles on the sides of the prisms are called faces.

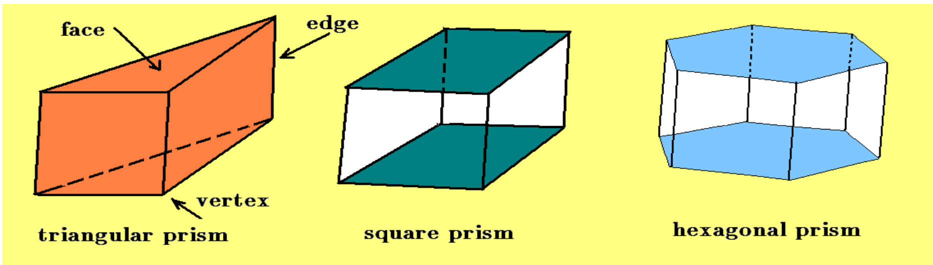


Figure 11-1. Polyhedra

In geometry the word "side" is sometimes used to describe a part of a three dimensional solid. The preferred terminology to be used in describing polyhedra is to refer to the number of "faces (F)", "edges (E)" and "vertices (V)". The faces (F), edges (E) and vertices (V) of polyhedra are related by the Euler[2] formula

$$\underbrace{F}_{Faces} + \underbrace{V}_{Vertices} = \underbrace{E}_{Edges} + 2 \tag{11.2}$$

For example, the triangular prism in figure 11-1 has 5 faces, 9 edges and 6 vertices, giving 5+6=9+2 for the Euler formula.

Prisms are solids which can be unfolded.

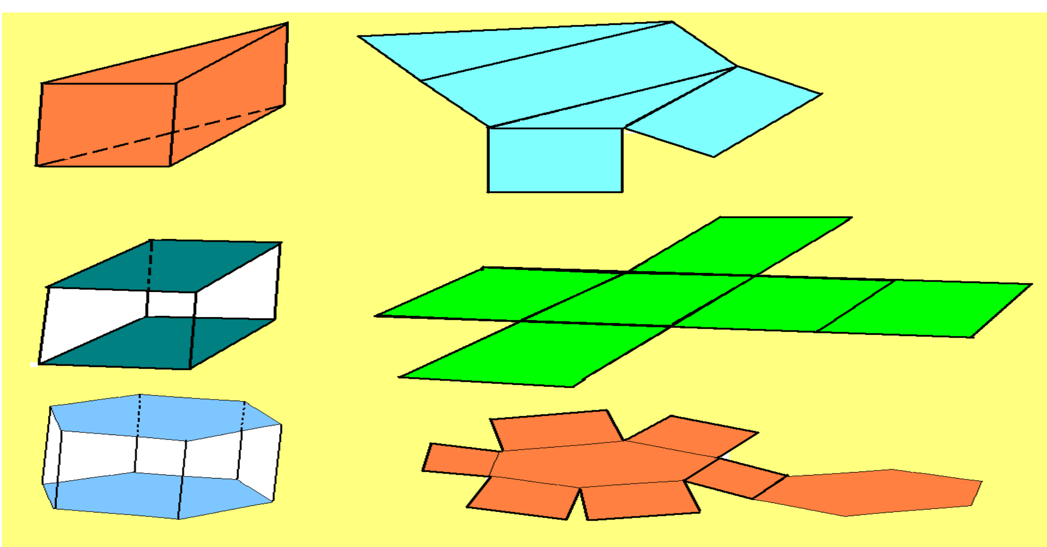


Figure 11-2. Unfolding polyhedra

[2] Leonhard Euler (1707-1783) A famous Swiss mathematician.

For example, by unfolding the figures of figure 11-1 one obtains the sketch in figure 11-2. By unfolding the solid one can obtain the surface area associated with each part of the polyhedra. The sum of these surface areas then give the total surface area associated with the prism. This unfolding technique can be applied to many other solids having straight edges.

Volume of a solid

A rectangular prism is a box with width w, length ℓ and height h. One can use cubes to define volume by defining a unit cube C such that mC, m an integer, will fit exactly into the box. For example, how many one inch cubes will fit into a box 6 inches long, 5 inches wide and 3 inches high? The answer will determine the volume held by the box.

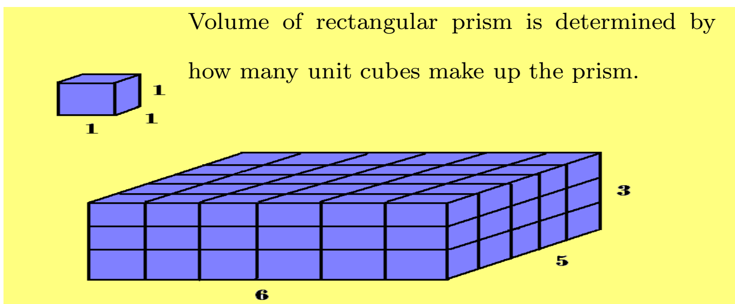


Figure 11-3. Volume of a box

If the box has length ℓ, width w and height h, the volume would be

$$V = (w\ell)h = (base)(height)$$

This shows the volume is given by the area of the base times the height. If the box is cut in half by a plane through the diagonal on the surface, then the area of the triangular prism is $V = (\frac{1}{2}w\ell)h$ which is again the area of the triangular base times the height.

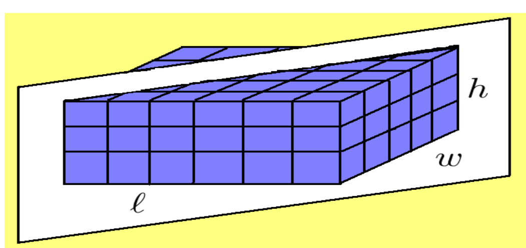
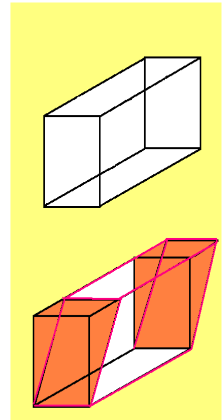


Figure 11-4. Volume of a triangular prism



The volume of a rectangular prism is $V = w\ell h$ as demonstrated above. If one adds and subtracts a triangular prism to each end of the rectangular prism one can create a parallelepiped. The volume of the parallelepiped will also be $V = w\ell h$ which is the area of the base times the height.

In general, the volume of a prism will always equal the area of the base times the height. One can use this fact along with the knowledge that as the number of sides of the regular polygonal base of a prism increases it gets closer and closer to a circle. In the limit as the number of sides of a regular polygonal base increases one obtains the volume of a cylinder where the top and bottom regular polygons become circles. The volume of a cylinder is found to be equal to the **area of the base times the height** or $V_{cylinder} = \pi r^2\, h$.

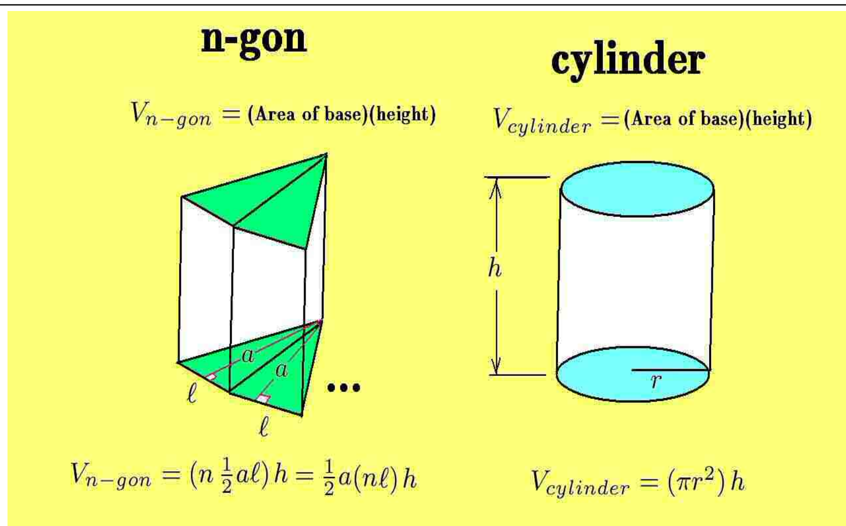


Figure 11-5. Volume of a n-gon becomes volume of cylinder

That is, if a is the apothem of the regular n-gon and ℓ is the length of a side of the n-gon, then the area of one triangle is $\frac{1}{2}a\ell$ and the area of the base is n times the area of one triangle. In the limit as n increases

$$\lim_{n\to\infty} n\ell = 2\pi r \qquad \text{and} \qquad \lim_{n\to\infty} a = r \quad \text{so that} \quad \lim_{n\to\infty} \frac{1}{2}a(n\ell)h = \frac{1}{2}r(2\pi r)h = \pi r^2 h$$

Platonic solids

The Platonic solids are named after the Greek mathematician and philosopher Plato (427-347)BCE who founded an Academy devoted to the study of philosophy and science. The Platonic solids all have congruent **regular polygons for their faces**. There are only five such solids to be investigated. By unfolding these solids one can find methods for their construction. In the following representation of the Platonic solids there is a left and right sketch. Make note that by folding the right-hand figure along the lines given, one can construct the left-hand figure.

Tetrahedron

The tetrahedron is bounded by four congruent equilateral triangles.

Figure 11-6. Tetrahedron

The tetrahedron has 4 faces, 4 vertices and 6 edges. It can be demonstrated that if ℓ is the length of each edge of the equilateral triangles, then the tetrahedron has the following volume and surface area.

$$\text{Volume} = \frac{\sqrt{2}}{12}\,\ell^3 \qquad\qquad \text{Surface area} = \sqrt{3}\,\ell^2$$

Hexahedron

The hexahedron or cube is bounded by six congruent squares. It has six faces, 8 vertices and 12 edges.

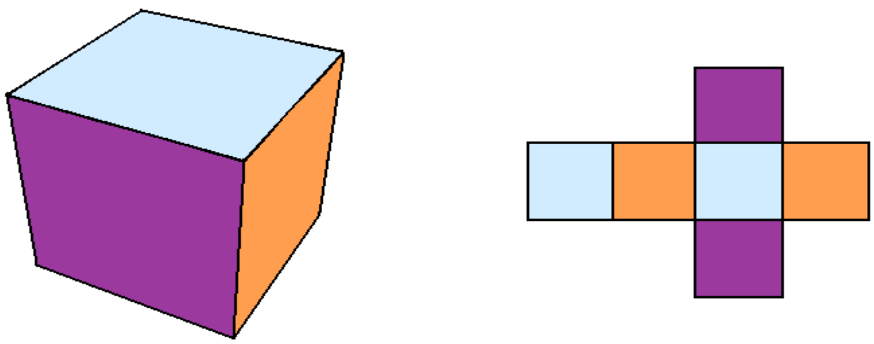


Figure 11-7. Hexahedron

If ℓ is the length of the squares edges, then it can be demonstrated that the hexahedron has the following volume and surface area.

$$\text{Volume} = \ell^3 \qquad\qquad \text{Surface area} = 6\,\ell^2$$

Octahedron

The octahedron is bounded by eight congruent equilateral triangles. It has eight faces, 12 edges and 6 vertices.

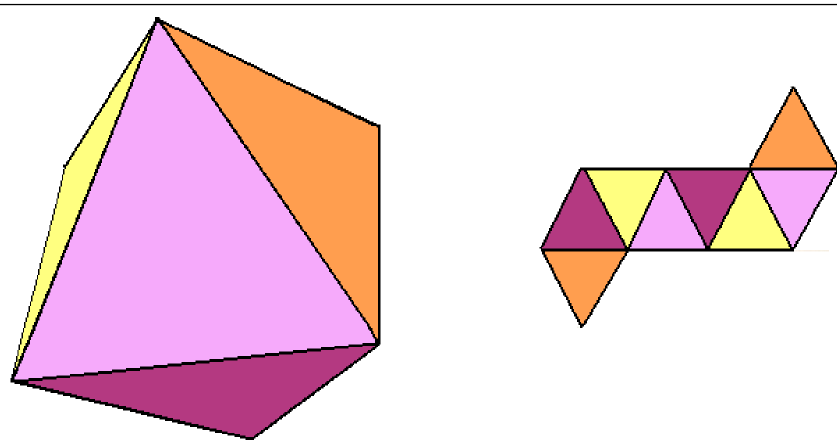


Figure 11-8. Octahedron

If ℓ is the length of each edge of the equilateral triangles, then it can be demonstrated that the volume and surface area are given by

$$\text{Volume} = \frac{\sqrt{2}}{3}\,\ell^3 \qquad\qquad \text{Surface area} = 2\sqrt{3}\,\ell^2$$

Dodecahedron

The dodecahedron is bounded by twelve congruent regular pentagons. If has 12 faces, 30 edges and 20 vertices.

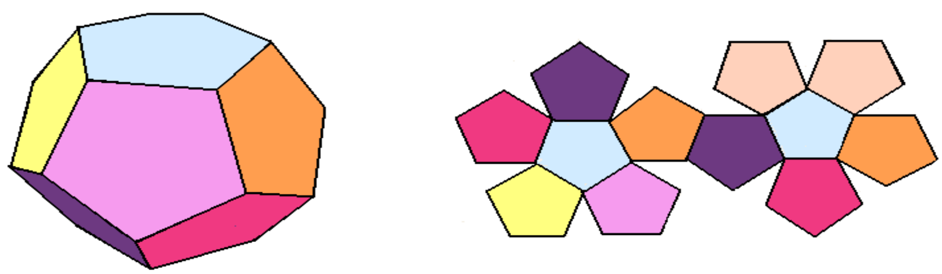


Figure 11-9. Dodecahedron

If ℓ is the length of an edge on the regular pentagons, then it can be demonstrated that the dodecahedron has the following volume and surface area

$$\text{Volume} = \frac{15+7\sqrt{5}}{4}\,\ell^3 \qquad\qquad \text{Surface area} = 3\sqrt{5(5+2\sqrt{5})}\,\ell^2$$

Example 11-1.

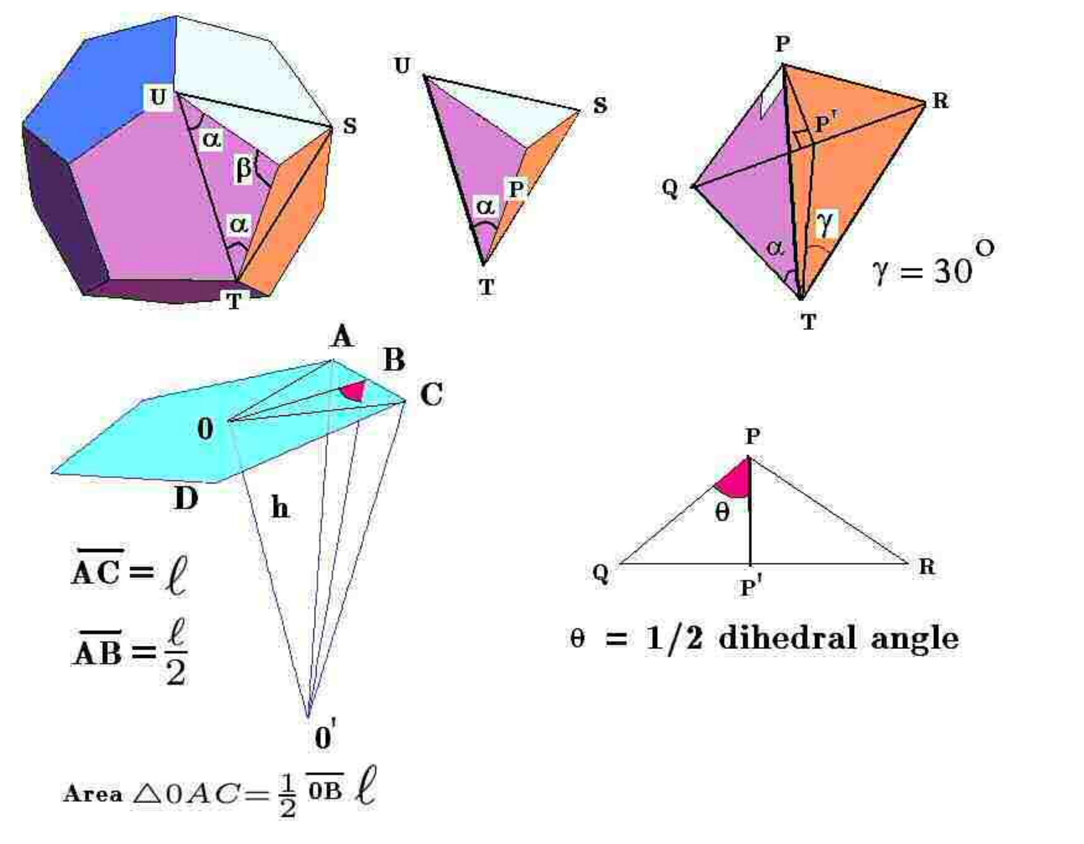


Figure 11-10. The dodecahedron

The volume of a dodecahedron can be considered as a combining of 12 pyramids with regular pentagons for their bases. One can calculate the volume of the pyramids and thus determine the volume of the dodecahedron. Consider the section TSU from the dodecahedron. Select a point P on the edge and construct lines $\overline{\text{PQ}}$ and $\overline{\text{PR}}$ which are both perpendicular to line $\overline{\text{PT}}$ so that triangles $\triangle QPT$ and $\triangle RPT$ are right triangles. Follow this by constructing the perpendicular line $\overline{PP'}$.

Examine the above figure and observe that $\beta = 108°$, $\alpha = 36°$, TSU is an equilateral triangle, $\gamma = 30°$ which implies $\overline{P'R}$ is one-half $\overline{\text{RT}}$ implying triangle $\triangle QRT$ is also equilateral with $\overline{\text{QT}} = \overline{\text{QR}} = \overline{\text{RT}}$.

Using trigonometry one can verify that

$$\overline{\text{0B}} = \frac{\ell}{2}\tan 54° \quad \text{and} \quad \sin 36° = \frac{\overline{\text{QP}}}{\overline{\text{QT}}}$$

Also note the sine of one-half the dihedral angle is

$$\sin\theta = \frac{\overline{0P'}}{\overline{\mathrm{QP}}} = \frac{\frac{1}{2}\overline{\mathrm{QR}}}{\overline{\mathrm{QP}}} = \frac{\overline{\mathrm{QT}}}{2\overline{\mathrm{QP}}} = \frac{1}{2\frac{\overline{\mathrm{QP}}}{\overline{\mathrm{QT}}}} = \frac{1}{2\sin 36^\circ} = \frac{1}{\sqrt{\frac{(5+\sqrt{5})}{10}}}$$

and consequently one can find h from the equation

$$\tan\theta = \frac{h}{\overline{0\mathrm{B}}} = \frac{h}{\frac{\ell}{2}\tan 54^\circ}$$

The area of triangle $\triangle 0AC = \frac{1}{2}\ell\,\overline{0\mathrm{B}} = \frac{\ell^2}{4}\tan 54^\circ$ so the area of the regular pentagon is $\frac{5}{4}\ell^2\tan 54^\circ$. Note that from exercises 7-29 and 7-30 one can find $\tan 54^\circ$ and $\sin 36^\circ$.

The volume of one pyramid is $\frac{1}{3}$(area of base)(height) or

$$V_1 = \frac{1}{3}(\frac{5}{4}\ell^2\tan 54^\circ)(\frac{\ell}{2}\tan 54^\circ\tan\theta) = \frac{5}{24}\ell^3(\tan 54^\circ)^2\tan\left(Arcsin(\frac{1}{2\sin 36^\circ})\right) = \frac{5+2\sqrt{5}}{12\sqrt{6-2\sqrt{5}}}\,\ell^3$$

Hence the volume of the dodecahedron is

$$V_{12} = 12V_1 = \frac{5+2\sqrt{5}}{\sqrt{6-2\sqrt{5}}}\,\ell^3 = \frac{\sqrt{(5+2\sqrt{5})^2}}{\sqrt{6-2\sqrt{5}}}\cdot\frac{\sqrt{6+2\sqrt{5}}}{\sqrt{6+2\sqrt{5}}}\,\ell^3 = \frac{15+7\sqrt{5}}{4}\,\ell^3$$

■

Icosahedron

The icosahedron is bounded by twenty congruent equilateral triangles. It has twenty faces, 12 vertices and 30 edges.

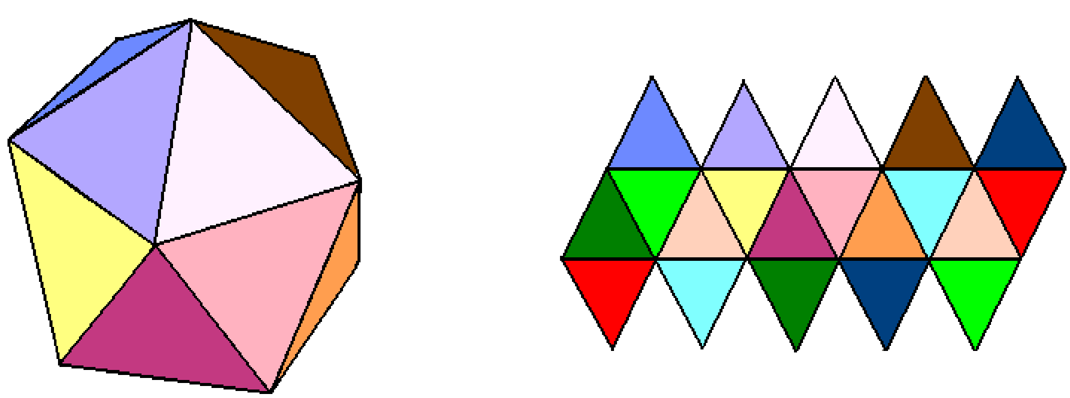


Figure 11-11. Icosahedron

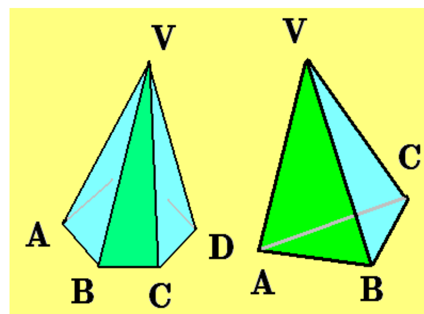
If ℓ is the length of an edge on the equilateral triangles, then the volume and surface area are given by

$$\text{Volume} = \frac{5}{12}(3+\sqrt{5})\,\ell^3 \qquad \text{Surface area} = 5\sqrt{3}\,\ell^2$$

Ruled Surfaces

A ruled surface is generated by moving a line or line segment to sweep out the surface. The moving line segment can be fixed at one point and move along a base curve called a directrix. The moving line segment can be made to move between two nonintersecting curves in three dimensional space called a directrix and director. A ruled surface is said to exist when every point of a surface can be made to connect to at least one other point on the surface to make a straight line segment. The lines which sweep out and generate the surface are called rulings. The figures 11-12 and 11-13 are illustrations of some ruled surfaces. One can find examples of ruled surfaces in many architectural structures.

A polyhedral angle is formed by a line segment moving through a fixed point called the vertex and the boundary of a polygon. The angle is sometimes referred to as a solid angle. The moving line is called the generatrix and the fixed polygon is called the directrix. A **trihedral angle** is a polyhedral angle whose directrix is a triangle.



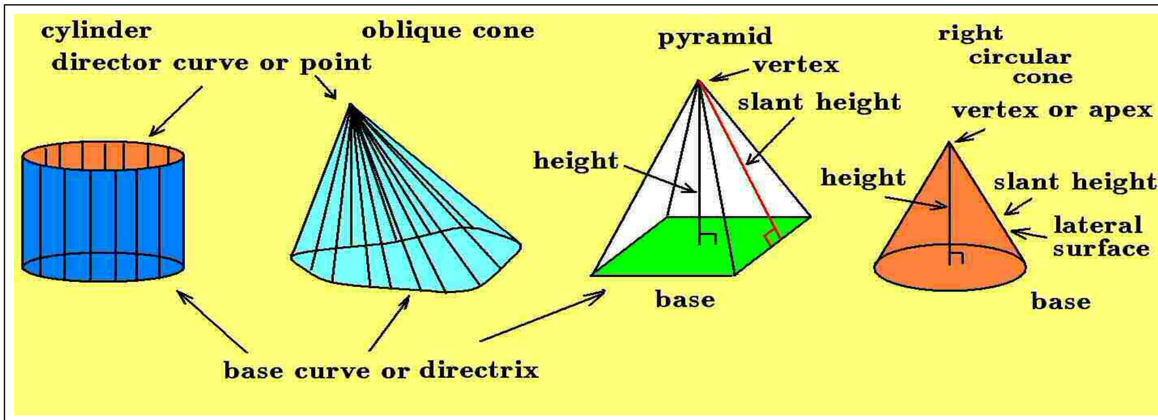
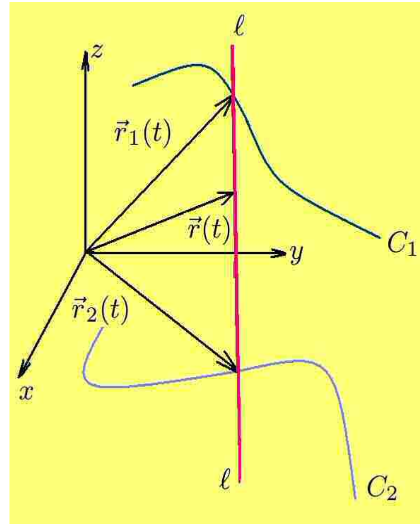


Figure 11 -12. Four examples of ruled surfaces

One method for constructing a ruled surface is to have two curves in space, say C_1 and C_2 defined by position vectors $\vec{r} = \vec{r}_1(t)$ and $\vec{r} = \vec{r}_2(t)$ having a common parameter t. Then the vector

$$\vec{r}(t, \lambda) = \vec{r}_2(t) + \lambda\,(\vec{r}_1(t) - \vec{r}_2(t))$$

is a two parameter ruled surface. Here for each value of the parameter t there results a line ℓ as t is held fixed and λ is allowed to vary between two fixed values, say $\lambda_1 \leq \lambda \leq \lambda_2$.



Definitions The **vertex** of a right circular cone of height h is a point directly over the center of the circular base.

The **height of a right circular cone** is the perpendicular distance between the vertex point and the base of the cone. The **slant height** of a right circular cone is the straight line distance from the vertex of the cone to a point on the circumference of the circular base.

A **regular pyramid** has a regular polygon for its base and has triangles for its lateral faces. All the face triangles have a common vertex point.

The **height of a regular pyramid** is the perpendicular distance between the vertex and the base.

The **slant height of a regular pyramid or cone** is the perpendicular distance from the vertex to an edge of the base.

The **apex of a pyramid or cone** is the tip, vertex or highest point of the figure which is opposite the base. The **frustum** (plural frustums) of a pyramid or cone is that portion of the figure lying between the base of the figure and a plane parallel to the base and cutting the figure.

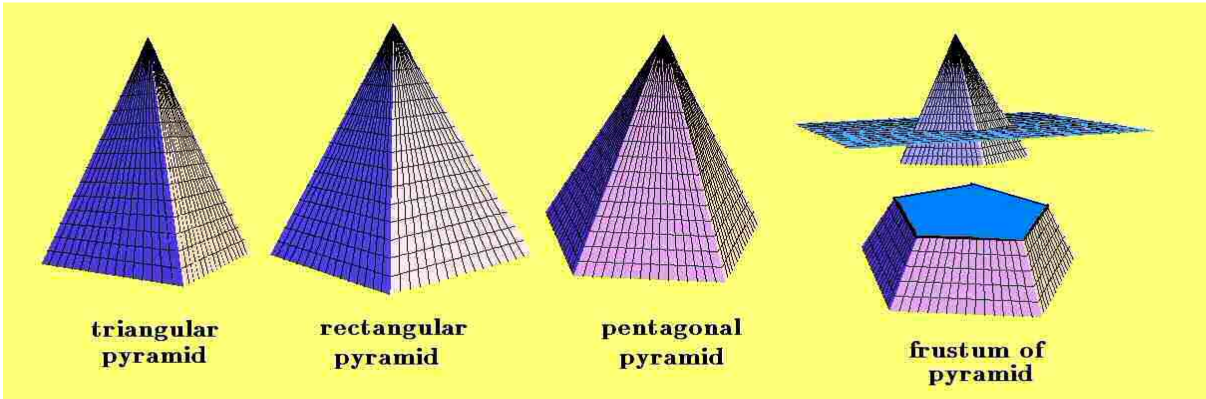


Figure 11-13. Pyramids as ruled surfaces

Curved surfaces

A curved surface can be created by **rotating a curve about an axis**.

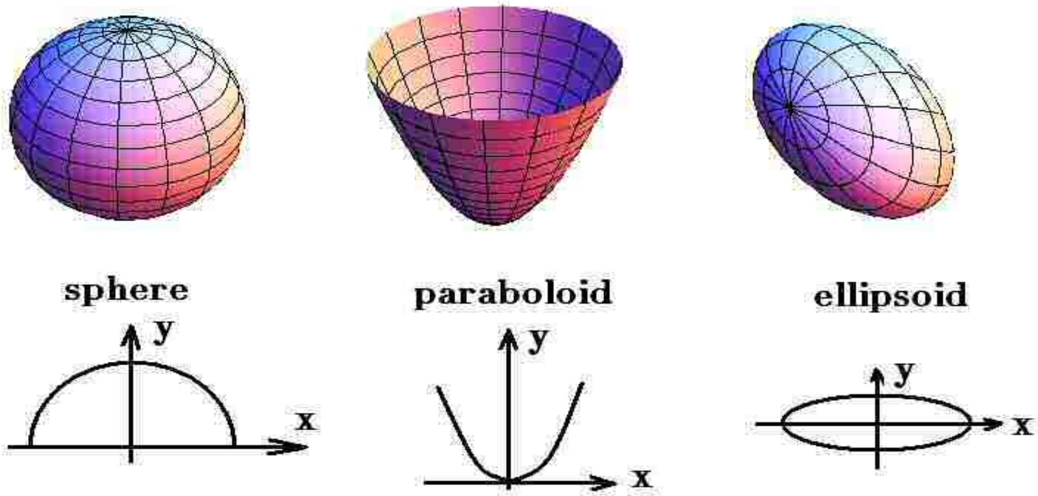


Figure 11-14. Rotation of curves to produce a solid

Consider the semicircle $y = \sqrt{r^2 - x^2}$ in figure 11-14. If this semicircle is rotated about the x-axis, then a sphere results. If the parabola in the figure 11-14 is rotated about the y-axis, then a paraboloid results. If the ellipse in the above figure is rotated about the x-axis an ellipsoid results. Many other solid shapes can be produced by the rotation of a curve or line about an axis.

Volume of right circular cone

To find the volume associated with a right circular cone such as the one illustrated in the figure 11-15 one can first approximate the volume and then take a limiting process to find the true volume of the cone. One can construct rectangles and rotate these rectangles about the z-axis to form disks, then a summation of the volumes associated with each disk will give an approximation for the volume of the cone.

As the number of disks are increased, the approximation will approach the true volume. Observe that in the y-z plane the equation of the line passing through the points $(0,h)$ and $(r,0)$ is given by $z-h=-(\frac{h}{r})y$ and solving for y one obtains $y=r-(\frac{r}{h})z$. If the height h of the cone is divided into n-parts each of length h/n, then the radius of the disk at any height z will be given by $y=r-(\frac{r}{h})z$ and the volume of the disk is the area of the base times the height (h/n). Consider the ith disk at height $z_i=ih/n$ with radius $y_i=r-(\frac{r}{h})ih/n$.

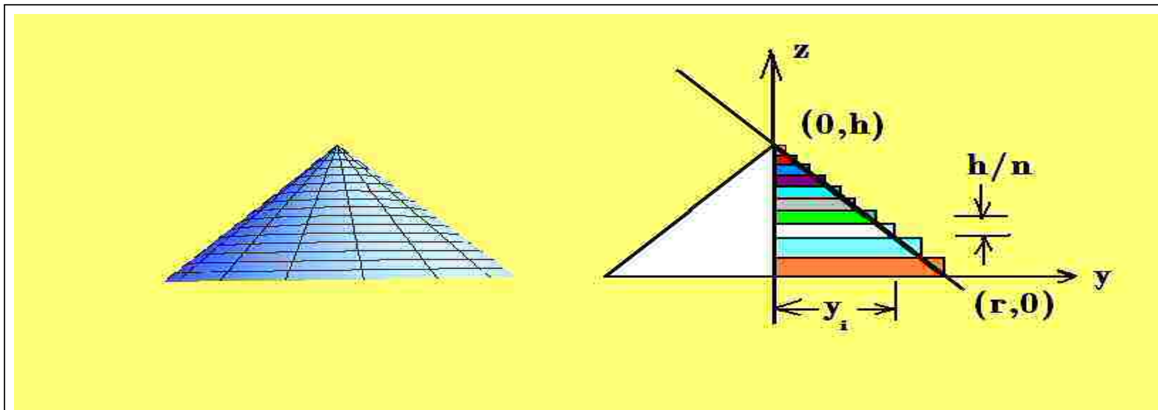


Figure 11-15. Summation of disks to approximate volume

The volume of this disk is

$$\text{Volume of } i\text{th disk} = V_i = \pi y_i^2 (h/n) = \pi \left[r - (\frac{r}{h}) ih/n \right]^2 (h/n)$$

and a summation of these disks gives the approximate volume of the cone as

$$V_{approx} \approx \sum_{i=1}^{n} \frac{h}{n} \pi r^2 \left[1 - \frac{i}{n} \right]^2 = \sum_{i=1}^{n} \frac{h}{n} \pi r^2 \left[1 - \frac{2i}{n} + \frac{i^2}{n^2} \right] \tag{11.3}$$

Recall from chapter 6 we developed the summation formulas

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1) \tag{11.4}$$

where $\dfrac{n(n+1)}{2} = \dfrac{n^2+n}{2}$ and $\dfrac{n}{6}(n+1)(2n+1) = \dfrac{2n^3+3n^2+n}{6}$. Expressing the equation (11.3) in the form

$$V_{approx} \approx h\pi r^2 \left[\frac{1}{n}\sum_{i=1}^{n} 1 - \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2\right] \tag{11.5}$$

and using the results from equations (11.4) one can show

$$\lim_{n\to\infty} V_{approx} = \lim_{n\to\infty} \pi r^2 h \left[\frac{1}{n}n - \frac{2}{n^2}\left(\frac{n^2+n}{2}\right) + \frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)\right]$$

giving the volume of the cone as

$$V_{cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}(\text{area of base})(\text{height}) \tag{11.6}$$

Volume of frustum associated with right circular cone

If a right circular cone is cut by a plane parallel to its base at a distance h above the base, then a frustum of the right circular cone is created. The situation is illustrated in the figure 11-16 where the top portion of the right circular cone is removed leaving the frustum. Let

$$\begin{aligned} V_f =& \text{ volume of frustum of right circular cone} \\ V_t =& \text{ volume of top cone which is removed} \\ V_c =& \text{ volume of original cone with height } (h_1+h) \end{aligned}$$

then one can write

$$V_f = V_c - V_t$$

which states the volume of the frustum is volume of original cone minus the top portion which is removed. Using equation (11.6) for the volume of a cone, we find

$$\begin{aligned} V_f = V_c - V_t =& \frac{\pi}{3} r_2^2 (h_1+h) - \frac{\pi}{3} r_1^2 h_1 \\ =& \frac{\pi}{3} r_2^2 h + \frac{\pi}{3}(r_2^2 - r_1^2) h_1 \end{aligned} \tag{11.7}$$

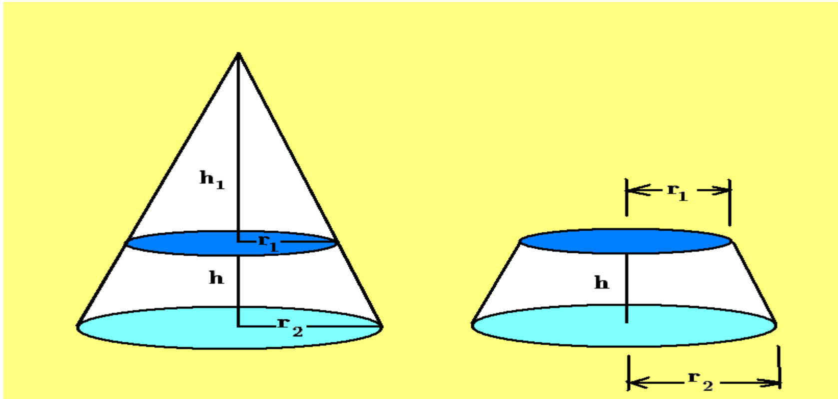


Figure 11-16. Frustum of right circular cone

Observe that by similar triangles one finds the proportions

$$\frac{h_1}{r_1} = \frac{h_1 + h}{r_2} \quad \text{or} \quad h_1 = \frac{r_1}{r_2 - r_1} h \tag{11.8}$$

Substituting for h_1 from equation (11.8) into equation (11.7) one obtains

$$\begin{aligned} V_f =& \frac{\pi}{3} r_2^2 h + \frac{\pi}{3}(r_2^2 - r_1^2) \frac{r_1}{r_2 - r_1} h \\ V_f =& \frac{\pi}{3} r_2^2 h + \frac{\pi}{3}(r_2 + r_1) r_1 h \\ V_f =& \frac{\pi}{3} h \left(r_2^2 + r_1 r_2 + r_1^2 \right) \end{aligned} \tag{11.9}$$

which represents the equation for obtaining the volume for the frustum of a right circular cone.

Surface area of right circular cylinder

The surface area of a right circular cylinder can be obtained by cutting it vertically on a side and unfolding it. The unfolded section is a rectangle with length $2\pi r$ and width h. This gives the lateral surface area of a cylinder as $S = 2\pi r h$

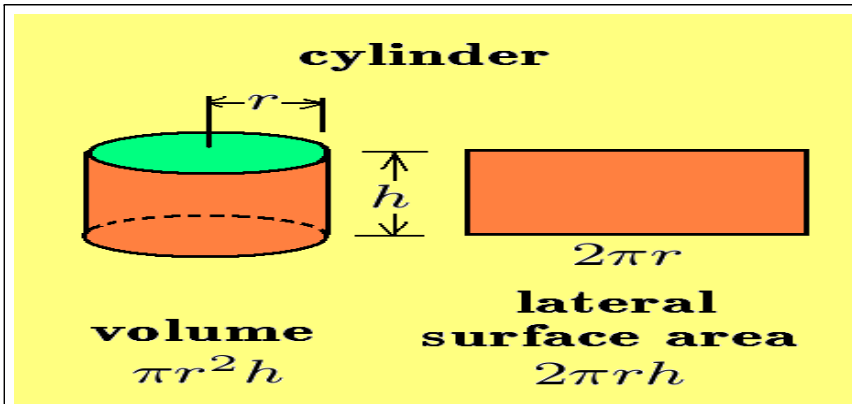


Figure 11-17. Unfolding cylinder to find surface area

The total surface area associated with a cylinder is obtained by adding the surface areas of the top and bottom circles to the lateral surface area. This gives the total surface area of the right circular cylinder as

$$S_t = 2\pi r^2 + 2\pi r h$$

Surface area of right circular cone The lateral surface area of a right circular cone with base radius r and slant height ℓ can be obtained by cutting along the slant height and unfolding the cone. This produces the figure 11-18. An examination of this figure produces the following proportions

$$\begin{aligned} &\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{angle of sector}}{2\pi} \\ \text{and}\quad &\frac{\text{circumference of cone base}}{\text{circumference of circle with radius } \ell} = \frac{\text{angle of sector}}{2\pi} \end{aligned} \tag{11.10}$$

Let A_s denote the area of the sector which is also the lateral surface area of the cone, then the proportions given in equation (11.10) can be written as

$$\frac{A_s}{\pi \ell^2} = \frac{\theta}{2\pi} \qquad \text{and} \qquad \frac{2\pi r}{2\pi \ell} = \frac{\theta}{2\pi} \tag{11.11}$$

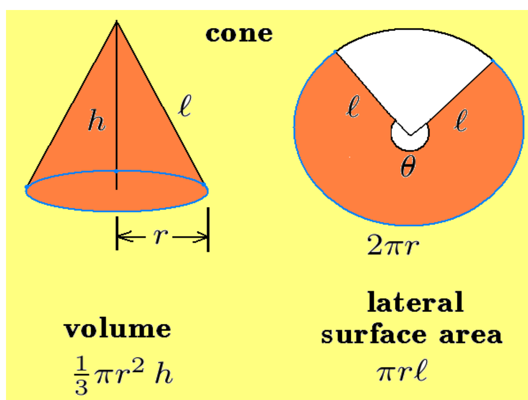


Figure 11-18. Unfolding cone to find surface area

The equations (11.11) simplify to

$$A_s = \frac{1}{2}\theta \ell^2 = \frac{1}{2}(\theta \ell)\,\ell \quad \text{and} \quad 2\pi r = \theta \ell$$

which gives

$$A_s = \pi r \ell \tag{11.12}$$

That is, the lateral surface area of the right circular cone is π times the radius times the slant height. To find the total surface area one must add to the lateral surface area the area of the base which is πr^2.

Surface area of frustum of a right circular cone

The lateral surface area associated with the frustum of a right circular cone illustrated in the figure 11-19 can be obtained by subtracting the the lateral surface area of two cones. The larger cone with base radius r_2 has the lateral surface area $\pi r_2(\ell_1 + \ell_2)$ obtained using the lateral surface area relation from equation (11.12). The smaller cone with base radius r_1 has the lateral surface area $\pi r_1 \ell_1$. The figure 11-19 also shows two similar triangles giving the proportions

$$\frac{r_1}{r_2} = \frac{\ell_1}{\ell_1 + \ell_2} \qquad \Rightarrow \qquad r_2\ell_1 = r_1(\ell_1 + \ell_2) \tag{11.13}$$

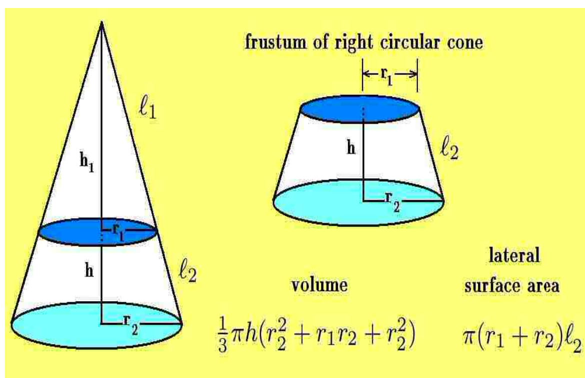


Figure 11-19. Subtracting the lateral surface area of two cones

Subtracting the lateral surface area of the smaller cone from that of the larger cone gives the lateral surface area S_L for the frustum associated with the larger cone. One finds

$$S_L = \pi r_2(\ell_1 + \ell_2) - \pi r_1 \ell_1$$

One can rearrange terms and use equation (11.13) to express this result in the following form

$$\begin{aligned} S_L =& \pi(r_1 + r_2)\ell_2 + \pi\left[r_1\ell_1 - r_1(\ell_1 + \ell_2)\right] \\ S_L =& \pi(r_1 + r_2)\ell_2 \end{aligned} \tag{11.14}$$

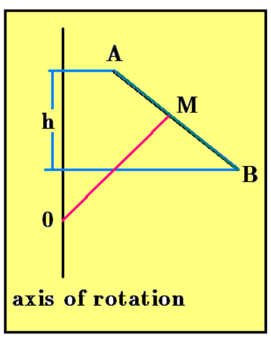
where the term inside the square brackets is zero because of equation (11.13). The lateral surface area of the frustum is π times the sum of the top and bottom radius all multiplied by the slant height. To find the total surface area of the frustum one must add to the lateral surface area πr_1^2 plus πr_2^2 which represents the surface area of the top and bottom of the frustum.

Rotation of line segment to calculate the surface area

There is another way that one can calculate the lateral surface area associated with **a right cylinder, right cone and frustum of a right cone.** The method requires that one analyze **the rotation of a line segment $\overline{\text{AB}}$ about an axis of rotation** as follows. First find the midpoint M of the line segment $\overline{\text{AB}}$, then construct the perpendicular bisector of the line segment $\overline{\text{AB}}$ labeling the point where the perpendicular bisector intersects the axis of rotation as point 0. One can then calculate the distance $\overline{\text{M0}}$ from point 0 to M. **For each solid it will be demonstrated that the**

$$\text{Lateral surface area} = 2\pi\,\overline{\text{M0}}\,h \tag{11.15}$$

where h **is a length obtained by the projection of the line segment $\overline{\text{AB}}$ onto the axis of rotation.** In using this result we will refer to it as **the surface area by rotation theorem.**



Rotation of line segment $\overline{\text{AB}}$ about an axis
$\overline{\text{M0}}$ is perpendicular bisector of line segment $\overline{\text{AB}}$.
h is the projection of $\overline{\text{AB}}$ onto axis of rotation.
0 is point of intersection of $\overline{\text{M0}}$ with axis of rotation.

Figure 11-23. Line segment $\overline{\text{AB}}$ rotated about an axis

The cylinder

In figure 11-20 the line segment $\overline{\text{AB}}$ is parallel to the axis of rotation. The line segment $\overline{\text{M0}}$ is the perpendicular bisector of line segment $\overline{\text{AB}}$ which intersects the axis of rotation at point 0. The distance $\overline{\text{CD}} = h$ represents the projection of line segment $\overline{\text{AB}}$ onto the axis of rotation. The cylinder formed when $\overline{\text{AB}}$ is rotated about the axis of rotation has the lateral surface area $S = 2\pi r h$ where r is congruent to $\overline{\text{M0}}$ and h is congruent to $\overline{\text{CD}}$. Therefore, $S = 2\pi\overline{\text{M0}}\,h$

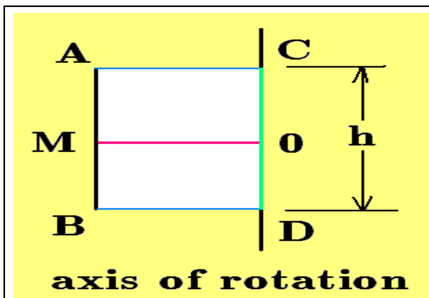


Figure 11-20. Cylinder

The cone

In figure 11-21 the line segment $\overline{\text{AB}}$ has one end attached to the axis of rotation. The line segment $\overline{\text{M0}}$ is the perpendicular bisector of the line segment $\overline{\text{AB}}$. The cone that results when the line segment $\overline{\text{AB}}$ rotates about the axis of rotation has the lateral surface area $S = \pi r \ell$ where $r = \overline{\text{AC}}$ and $\ell = \overline{\text{AB}}$. In the triangle $\triangle ABC$ of figure 11-21 construct the line segment $\overline{\text{ME}} \parallel \overline{\text{AC}}$. Note point M is a midpoint so that $\overline{\text{AC}} = 2\overline{\text{ME}}$. Also triangles $\triangle 0ME$ and $\triangle ACD$ ($\overline{\text{BC}} = \overline{\text{CD}}$) are similar so that one can write the proportional statement

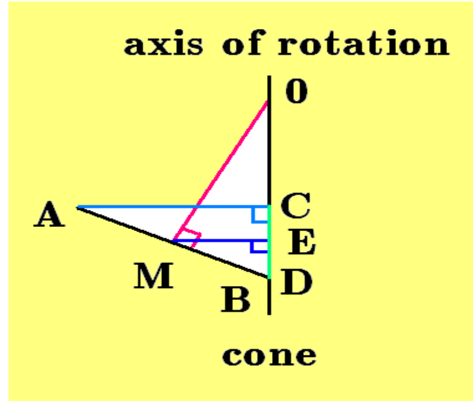


Figure 11-21. Cone

$$\frac{\overline{\text{AB}}}{\overline{\text{M0}}} = \frac{\overline{\text{CD}}}{\overline{\text{ME}}} \quad \Rightarrow \quad \overline{\text{AB}} \cdot \overline{\text{ME}} = \overline{\text{M0}} \cdot \overline{\text{CD}} = \overline{\text{M0}}\, h$$

Therefore, the surface area can be represented

$$S = \pi r \ell = \pi\, \overline{\text{AC}} \cdot \overline{\text{AB}} = \pi(2\overline{\text{ME}}) \cdot \overline{\text{AB}} = 2\pi \overline{\text{AB}} \cdot \overline{\text{ME}} = 2\pi\, \overline{\text{M0}}\, h$$

The frustum

Here the line segment $\overline{\text{AB}}$ is offset at an angle away from the axis of rotation. The rotation of the line segment about the axis produces the frustum that we wish to examine. The frustum produced has the surface area

$$S_f = \pi(r_1 + r_2)\ell = S_f = \pi(\overline{\text{AC}} + \overline{\text{BD}}) \cdot \overline{\text{AB}} \tag{11.16}$$

Construct the line $\overline{\text{ME}}$ parallel to both the line segments $\overline{\text{AC}}$ and $\overline{\text{BD}}$ which represents the median of the trapezoid ABDC. The median line is the average of the top base and bottom base of the trapezoid so one can write $\overline{\text{ME}} = \frac{1}{2}(\overline{\text{AC}} + \overline{\text{BD}})$ and so equation (11.16) can be written in the form

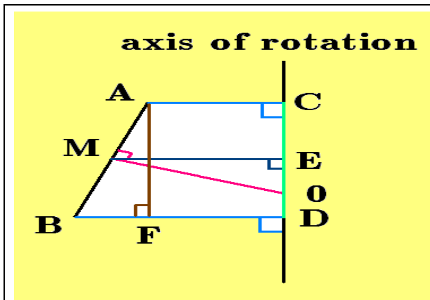


Figure 11-22. Frustum

$$S_f = \pi(2\overline{\text{ME}}) \cdot \overline{\text{AB}} \tag{11.17}$$

By dropping a perpendicular line from point A which intersect line segment $\overline{\text{BD}}$

at point F, one can make note that the triangles $\triangle AFB$ and $\triangle M0E$ are similar ($\triangle AFB \sim \triangle M0E$) so that one can form the ratios

$$\frac{\overline{\text{AB}}}{\overline{\text{M0}}} = \frac{\overline{\text{AF}}}{\overline{\text{ME}}} \quad \Rightarrow \quad \overline{\text{ME}} \cdot \overline{\text{AB}} = \overline{\text{M0}} \cdot \overline{\text{AF}} \tag{11.18}$$

But $\overline{\text{AF}}$ is congruent to $\overline{\text{CD}} = h$ which is the projection of line segment $\overline{\text{AB}}$ onto the axis of rotation. Therefore, from equations (11.17) and (11.18) one finds

$$S_f = \pi(2\overline{\text{ME}}) \cdot \overline{\text{AB}} = 2\pi\, \overline{\text{M0}} \cdot \overline{\text{AF}} = 2\pi\, \overline{\text{M0}} \cdot \overline{\text{CD}} = 2\pi\, \overline{\text{M0}}\, h \tag{11.19}$$

Volume of the sphere

Consider a sphere of radius r as illustrated in the figure 11-24. If the sphere is divided in half, then two hemispheres result. Consider the problem of finding the volume of a hemisphere. Recall that the area of a circle was approximated by circumscribing and inscribing n-gons inside a circle and then adding up the areas of triangles associated with the n-gons and then letting the integer n get very large. One can proceed in a similar fashion to obtain the volume of a hemisphere.

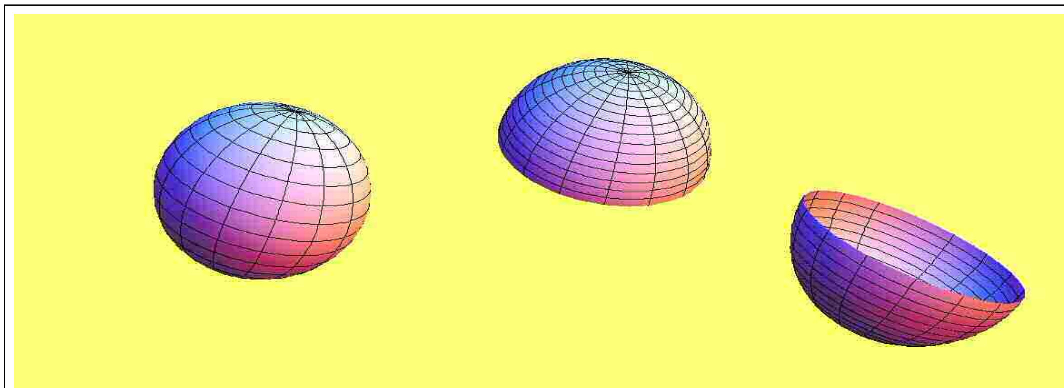


Figure 11-24. Sphere divided in half.

Imagine the approximation for the volume of a hemisphere is obtained by stacking different sized **cylinders** to approximate the hemisphere. These cylinders can be generated by examining the figure 11-25 where the circle $y = \sqrt{r^2 - x^2}$ is sketched in the first quadrant. If this curve is **rotated about the y-axis**, then a hemisphere results. If the radial distance from 0 to r is divided into say 6 parts, each of length $h = \frac{r}{6}$, then a series of 6 rectangles can be constructed each of height h. As each of these rectangles is **rotated about the y-axis**, then 6 cylinders are formed. The volume of each cylinder is obtained by multiplying **the area of the base times the height h.**

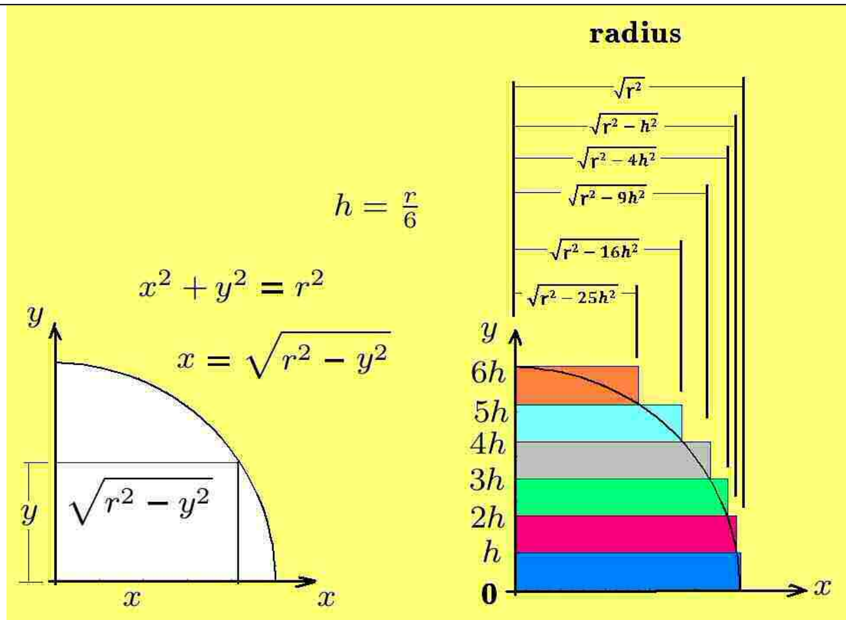


Figure 11-25. **Use of 6 cylinders to approximate volume of hemisphere**

One finds

$$\begin{aligned}\text{Volume of blue cylinder } &= \pi r^2\, h\\ \text{Volume of red cylinder } &= \pi(r^2-h^2)\, h\\ \text{Volume of green cylinder } &= \pi(r^2-4h^2)\, h\\ \text{Volume of gray cylinder } &= \pi(r^2-9h^2)\, h\\ \text{Volume of light blue cyclinder } &= \pi(r^2-16h^2)\, h\\ \text{Volume of orange cylinder } &= \pi(r^2-25h^2)\, h\end{aligned}$$

Summation of these volumes gives the following approximation for the volume of a hemisphere

$$V_{hemisphere} \approx \pi r^2\, h+\pi(r^2-h^2)\, h+\pi(r^2-4h^2)\, h+\pi(r^2-9h^2)\, h+\pi(r^2-16h^2)\, h+\pi(r^2-25h^2)\, h$$

Using the summation convention one can express the above result as

$$V_{hemisphere} \approx \pi\sum_{i=1}^{6}\left[r^2-(i-1)^2h^2\right] h \tag{11.20}$$

or substituting $h=\frac{r}{6}$ into equation (11.20) one obtains

$$V_{hemisphere} \approx \pi\sum_{i=1}^{6}\left[r^2-(i-1)^2\left(\frac{r}{6}\right)^2\right]\frac{r}{6} = \pi r^3\sum_{i=1}^{6}\left[1-\left(\frac{i-1}{6}\right)^2\right]\frac{1}{6} \tag{11.21}$$

Instead of 6 cylinders produced when the curve $y = \sqrt{r^2 - x^2}$ is rotated about the y-axis, suppose we create N cylinders. The formula approximating the volume of the hemisphere will look exactly like equation (11.21) except now the radial distance is divided by N instead of 6. In equation (11.21) replace 6 by N **everywhere** to obtain the approximation

$$
\begin{aligned}
V_{hemisphere} &\approx \pi r^3 \sum_{i=1}^{N} \left[1 - \left(\frac{i-1}{N} \right)^2 \right] \frac{1}{N} \\
V_{hemisphere} &\approx \pi r^3 \left[\frac{1}{N} \sum_{i=1}^{N} 1 - \frac{1}{N^3} \sum_{i=1}^{N} (i-1)^2 \right]
\end{aligned}
\tag{11.22}
$$

Recall the summation formulas from chapter 6

$$\sum_{j=1}^{N} 1 = N \tag{11.23}$$

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2} = \frac{N^2}{2} + \frac{N}{2} \tag{11.24}$$

$$\sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \tag{11.25}$$

Examine the equation (11.22) and note that we desire to find the summations

$$\sum_{i=1}^{N} 1 \qquad \text{and} \qquad \sum_{i=1}^{N} (i-1)^2$$

The first sum is obtained from equation (11.23). In order to find the second summation make a change of variable. Let $j = i - 1$ and note that when $i = 1$, then $j = 0$ and when $i = N$, then $j = N - 1$. This change of the summation index produces

$$\sum_{i=1}^{N} (i-1)^2 = \sum_{j=0}^{N-1} j^2 = \sum_{j=1}^{N-1} j^2 \tag{11.26}$$

This last summation is the same as equation (11.25) with N replaced by $N - 1$. Substituting $N - 1$ into equation (11.25) replacing N everywhere produces

$$\sum_{i=1}^{N} (i-1)^2 = \sum_{j=1}^{N-1} j^2 = \frac{(N-1)(N)(2N-1)}{6} = \frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6} \tag{11.27}$$

Substitute the summation formulas

$$\sum_{i=1}^{N} 1 = N \qquad \text{and} \qquad \sum_{j=1}^{N} (i-1)^2 = \frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6}$$

into the equation (11.22)and show

$$\begin{aligned} V_{hemisphere} &\approx \pi r^3 \left[\frac{1}{N} N - \frac{1}{N^3} \left(\frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6} \right) \right] \\ V_{hemisphere} &\approx \pi r^3 \left(1 - \frac{1}{3} + \left[\frac{1}{2N} - \frac{1}{6N^2} \right] \right) \end{aligned} \tag{11.28}$$

Observe that as N gets very large all the terms inside the square brackets

$$\left[\frac{1}{2N} - \frac{1}{6N^2} \right] \tag{11.29}$$

get very small and eventually approach zero for very large values of N. Therefore, the volume for the approximation of the hemisphere becomes

$$V_{hemisphere} = \pi r^3 \left(1 - \frac{1}{3} \right) = \frac{2}{3} \pi r^3 \tag{11.30}$$

and so the volume of a sphere is given by twice the volume of a hemisphere so that

$$V_{sphere} = \frac{4}{3} \pi r^3 \tag{11.31}$$

Surface area of a sphere

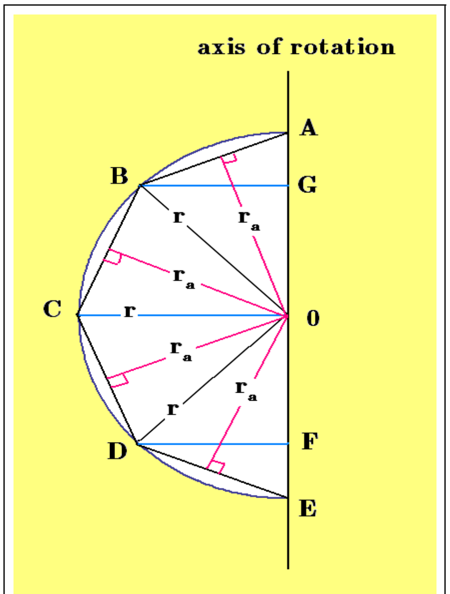


Figure 11-26.
Rotation of octagon

We will use the surface area by rotation theorem to find the surface area of a sphere. The surface area can be approximated by constructing an octagon inside a semicircle and then rotating the figure about the diameter of the circle as illustrated in the figure 11-26. Recall from equation (11.15) that when the line segment $\overline{AB}$ is rotated about the axis of rotation the surface area of the cone generated is

$$S_{AB} = 2\pi r_a \overline{AG} \tag{11.32}$$

where r_a is the apothem of the polygon and $\overline{AG}$ is the projection of $\overline{AB}$ onto the axis of rotation.

Using the results from equation (11.15) one can rotate the other line segments $\overline{\text{BC}}, \overline{\text{CD}}, \overline{\text{DE}}$ about the axis of rotation to obtain the surface areas

$$\begin{aligned} S_{BC} =& 2\pi r_a \overline{\text{G0}} \\ S_{CD} =& 2\pi r_a \overline{\text{0F}} \\ S_{DE} =& 2\pi r_a \overline{\text{FE}} \end{aligned} \tag{11.33}$$

The total surface area associated with the rotation of the polygon is

$$S = 2\pi r_a \left(\overline{\text{AG}} + \overline{\text{G0}} + \overline{\text{0F}} + \overline{\text{FE}}\right) \tag{11.34}$$

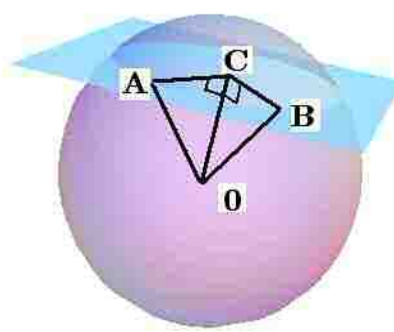
Observe that $(\overline{\text{AG}} + \overline{\text{G0}} + \overline{\text{0F}} + \overline{\text{FE}}) = 2r$ where r is the circumradius of the polygon and resulting radius of the sphere when the figure 11-26 is rotated. The rotation of the figure 11-26 gives the following approximation for the surface area of the sphere

$$S_{sphere} \approx 2\pi r_a (2r) \tag{11.35}$$

In the limit as the number of sides of the polygon increases the apothem distance r_a approaches the circumradius distance r reducing the equation (11.35) to the formula for the surface area of a sphere given by

$$S_{sphere} = \lim_{n\to\infty} 2\pi r_a \,(2r) = 2\pi r \,(2r) = 4\pi r^2 \tag{11.36}$$

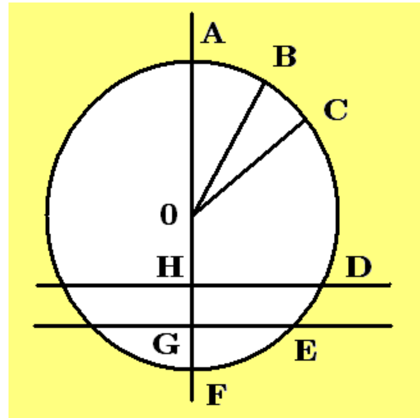
Plane and sphere



A plane intersecting a sphere results in a circular cross-section. The proof is as follows. One can construct the line $\overline{\text{0C}}$ which is perpendicular to the plane. Select any two points A and B on the curve of intersection associated with the plane and sphere and then construct the line segments $\overline{\text{CA}}$ and $\overline{\text{CB}}$ which lie in the plane. The line $\overline{\text{0C}}$ is common to both right triangles and so triangles $\triangle 0CA$ and $\triangle 0CB$ are two right triangles having an equal leg and hypotenuse. Consequently, these triangles are congruent and indicate that $\overline{\text{CA}} = \overline{\text{CB}}$. The points A and B are arbitrary having an equal distance from C. Hence, the curve of intersection is a circle.

A corollary to the above is that two circles on a sphere are equal if their planes are equidistant from the center.

Parts of a sphere



When the circle illustrated is rotated about the diameter $\overline{\text{AF}}$ then a sphere results and (i) the rotation of an arc generates a surface area called a zone and (ii) the rotation of an area generates a volume associated with the zone. Consider the following:
(a) The area GEDH generates a zone called a spherical segment having an upper and lower circular base with $\overline{\text{GH}}$ called an altitude and represents the distance between the parallel planes defining the upper and lower base. The resulting spherical segment has both a volume and suface area.
(b) The area FGE generates a zone called a spherical cap. Both the spherical segment and spherical cap can be thought of as having two parallel planes intersecting the sphere at different points where in one case one of the planes is tangent to the sphere.
(c) The areas $0BA$ and $0CA$ generate spherical cones.
(d) The area $0CB$ generates a spherical sector when rotated.
(e) When the arc $\overset{\frown}{\text{BC}}$ is rotated a zone results which is the base of the spherical sector.

Spherical Cap

When a plane intersects a sphere it creates a spherical cap. The volume and surface area associated with a spherical cap can be determined from the limit of an approximation. Consider a cross section of a sphere of radius r and let the line $y = H$ denote the plane cutting the sphere.

To approximate the volume of the spherical cap consider the rotation of n-disks about the y-axis. Divide the distance $r - H$ into n-parts giving each of the disks a thickness of $d = (r - H)/n$. The volume element given by the ith disk is

$$V_i = \pi x_i^2\, d = \pi(r^2 - y_i^2)\, d = \pi\left(r^2 - [r - id]^2\right) d = \pi\left(r^2 - \left[r - i\frac{(r-H)}{n}\right]^2\right)\frac{(r-H)}{n}$$

Expand the above equation and show

$$V_i = 2\pi r\left(\frac{r-H}{n}\right)^2 i - \pi\left(\frac{r-H}{n}\right)^3 i^2 \tag{11.37}$$

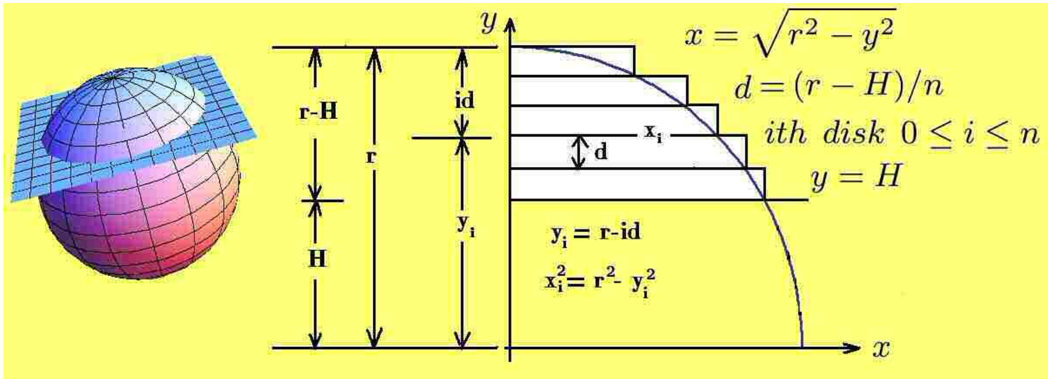


Figure 11-27.
Approximation for volume of spherical cap

A summation of the volume elements from 1 to n gives the volume approximation

$$v_a = \sum_{i=1}^{n} V_i = 2\pi r \left(\frac{r-H}{n}\right)^2 \sum_{i=1}^{n} i - \pi \left(\frac{r-H}{n}\right)^3 \sum_{i=1}^{n} i^2 \tag{11.38}$$

Using the summation formulas

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

previously derived in chapter 6, one finds

$$V_a = \sum_{i=1}^{n} V_i = 2\pi r \frac{(r-H)^2}{n^2}\left[\frac{n^2}{2} + \frac{n}{2}\right] - \pi \frac{(r-H)^3}{n^3}\left[\frac{n^3}{3} + \frac{n^2}{3} + \frac{n}{6}\right]$$

Taking the limit as n gets very large, the volume approximation produces the true result for the volume of the spherical cap

$$V_{cap} = \pi r (r-H)^2 - \frac{\pi}{3}(r-H)^3 = \frac{\pi}{3}(r-H)^2\,(2r+H) \tag{11.39}$$

Letting $h = r - H$ denote the height of the spherical cap, the equation (11.39) can be represented in the form

$$V_{cap} = \frac{\pi}{3} h^2 (3r - h) \tag{11.40}$$

One can replace the rectangles in figure 11-27 by trapezoids and then find the surface area of a spherical cap by using the rotation of line segments about the y-axis (rotation of line segments theorem) to approximate the surface area. As the number of line segments increase one can demonstrate the lateral surface area associated with a spherical cap is given by

$$S_{cap} = 2\pi r h \tag{11.41}$$

Great circles and spherical angles

The **great circle of a sphere** is the intersection of the sphere surface with a plane which passes through the center of the sphere. A **small circle on a sphere** is created by the intersection of the spherical surface with a plane not passing through the center of the sphere. A spherical angle is formed when planes associated with two great circles pass through the same point P on the surface and form a dihedral angle which cuts out a spherical angle on the surface of the sphere. The spherical angle equals the arc $\widehat{AB}$ formed by the planes creating the angle intersecting the equator of the sphere. A **zone on the sphere** is the surface area on the sphere between two parallel planes cutting the surface of the sphere. A **spherical cone** or **sector of a sphere** is constructed by adding a cone with vertex at the sphere center and the cone based attached to the spherical cap base. This gives the volume of the spherical cone as

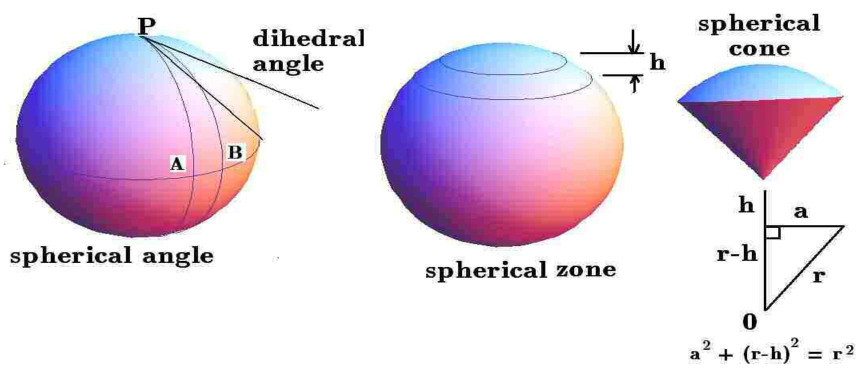


Figure 11-28.
Spherical angle, spherical zone, spherical cone or sector of sphere

$$\text{Volume of spherical cone} = \text{Volume of spherical cap} + \text{Volume of cone}$$
$$V = \frac{\pi}{6}h(3a^2 + h^2) + \frac{\pi}{3}a^2(r-h)$$
$$V = \frac{\pi}{6}h(3(2rh - h^2) + h^2) + \frac{\pi}{3}\left[(2rh - h^2)(r-h)\right]$$

which simplifies to

$$V = \frac{2}{3}\pi r^2 h$$

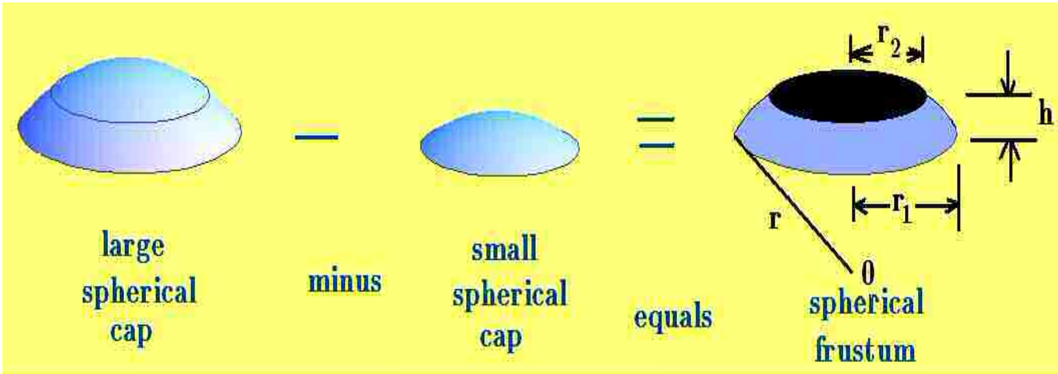


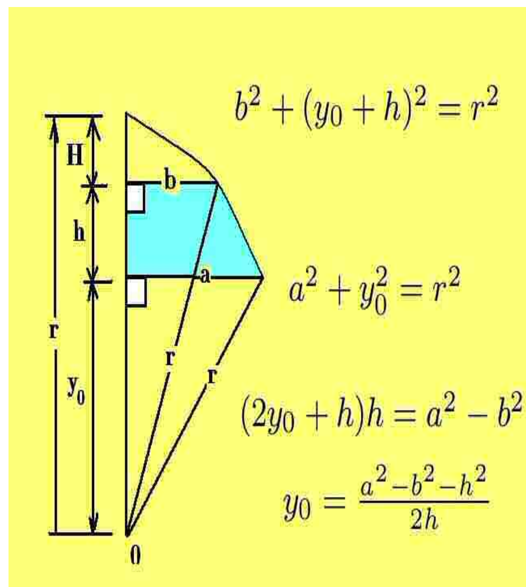
Figure 11-29.
Spherical zone, spherical segment or spherical frustum

A spherical segment, spherical zone or spherical frustum is obtained by removing a small spherical cap from a large spherical cap as illustrated in the figure 11-29. The volume of the spherical frustum can be represented

$$V_f = \frac{\pi}{6}(H+h)(3a^2+(H+h)^2) - \frac{\pi}{6}H(3b^2+H^2)$$

Make the substitution $R = H+h$ and show the volume of a spherical frustum can be expressed

$$V_f = \frac{\pi}{6}\left(R(3a^2+R^2)\right) - \frac{\pi}{6}(R-h)(3b^2+(R-h)^2)$$



A cross section of the spherical frustum shows the top radius b and bottom radius a satisfy the equations

$$a^2+y_0^2=r^2, \qquad b^2+(y_0+h)^2=r^2, \;\Rightarrow\; y_0=\frac{a^2-b^2-h^2}{2h}$$

Substituting $R = H+h = r-y_0$ into the equation for the spherical frustum volume one obtains after simplification

$$V_f = \frac{\pi}{6}\left[3b^2h+h^3+(3a^2-3b^2-3h^2)r+3hr^2-(3a^2+3b^2+3h^2-6hr)y_0+3hy_0^2\right] \qquad (11.42)$$

Now substitute into the above equation

$$y_0 = \frac{a^2 - b^2 - h^2}{2h} \quad \text{and} \quad r = \sqrt{a^2 + y_0}$$

and simplify to obtain the result

$$V_f = \frac{\pi}{6} h \left[3a^2 + 3b^2 + h^2\right] \tag{11.43}$$

In a similar fashion one can use the rotation of line segments to calculate the surface area associated with a spherical frustum. One finds the surface area of a spherical frustum is given by

$$S_f = 2\pi r\, h \tag{11.44}$$

The total surface area of the spherical frustum is obtained by adding the surface area of the two bases giving

$$S_{fT} = 2\pi r h + \pi a^2 + \pi b^2$$

Right pyramid with square base

Consider the right pyramid with square base illustrated in the figure 11-30, where the height of the pyramid is h and the square base has a bottom length of $2b$ on any side.

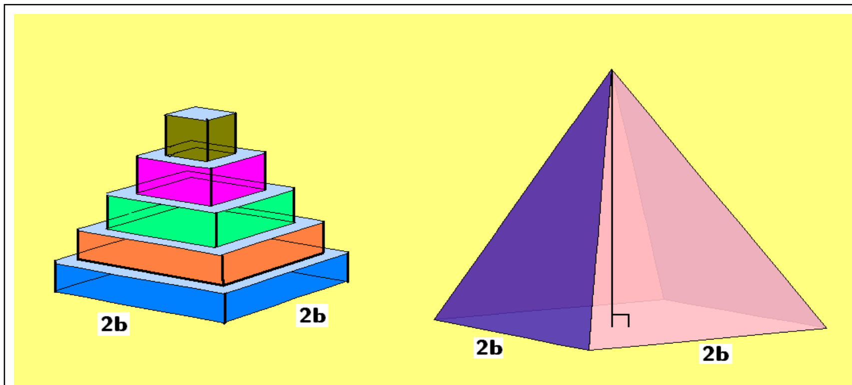


Figure 11-30.

Square pyramid approximation of volume

To find the volume of the pyramid one can start off by first getting an approximate value for the volume and then improve on the approximate value obtained. For example, suppose the volume of the pyramid is approximated by 5 prisms stacked one upon the other as illustrated in the figure 11-30. The dimensions of the prisms used in the approximation can be obtained by taking a cross section of the pyramid and showing a section in the $y-z$ plane produces the line $z-h=-\left(\frac{h}{b}\right)y$ or $y=b\left(1-\frac{z}{h}\right)$.

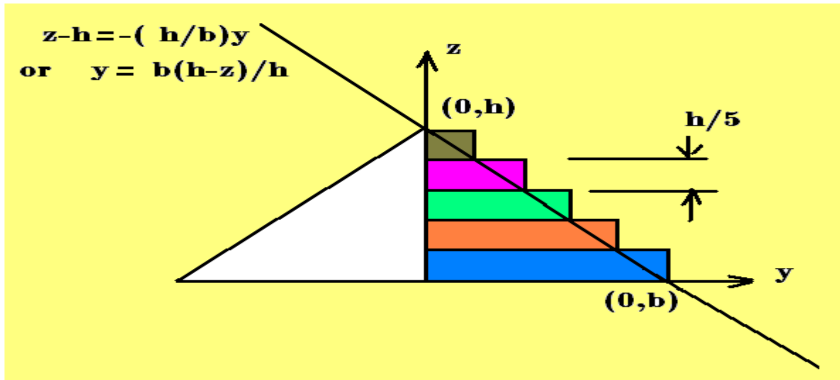


Figure 11-31.

Cross section of pyramid.

If the height h of the pyramid is divided into 5 parts, then the volume of each disk is the area of each base times the height of the disk or $(h/5)$. Using the equation of the line describing the slant line of the prism one can examine the figure 11-31 to determine the length of the sides associate with each base of the stacked prisms.

Use $y=b\left(1-\frac{z}{h}\right)$ to determine length of $\frac{1}{2}$side		
z	y	length of side
0	b	2b
h/5	b(1-1/5)	2b(1-1/5)
2h/5	b(1-2/5)	2b(1-2/5)
3h/5	b(1-3/5)	2b(1-3/5)
4h/5	b(1-4/5)	2b(1-4/5)

By squaring the length of the prism side and multiplying by the height $h/5$ one can obtain the volume of each prism. Addition of these volumes gives an approximation V_a for the volume of the pyramid as

$$V_a \approx (2b)^2\frac{h}{5} + [2b\,(1-1/5)]^2\,\frac{h}{5} + [2b\,(1-2/5)]^2\,\frac{h}{5} + [2b\,(1-3/5)]^2\,\frac{h}{5} + [2b\,(1-4/5)]^2\,\frac{h}{5}$$

Using the summation convention introduced in chapter 6, this approximation can be expressed

$$V_a \approx \sum_{i=1}^{5}\left[2b\left(1-\frac{(i-1)}{5}\right)\right]^2\frac{h}{5} \tag{11.45}$$

Instead of 5 prisms, suppose one used n prisms, where n is a large integer. The volume approximation would be the same as equation (11.45) except that all the 5's in equation (11.45) would be replaced by the integer n to obtain

$$V_a \approx \sum_{i=1}^{n} \left[2b\left(1 - \frac{(i-1)}{n}\right)\right]^2 \frac{h}{n} \tag{11.46}$$

Expand the equation (11.46) and represent it in the form

$$\begin{aligned} V_a \approx &= \sum_{i=1}^{n} 4b^2 \left[1 - 2\frac{(i-1)}{n} + \frac{(i-1)^2}{n^2}\right] \frac{h}{n} \\ V_a \approx &= 4b^2 \frac{h}{n} \sum_{i=1}^{n} 1 - 8b^2 \frac{h}{n^2} \sum_{i=1}^{n} (i-1) + 4b^2 \frac{h}{n^3} \sum_{i=1}^{n} (i-1)^2 \end{aligned} \tag{11.47}$$

We have previously shown the summation terms in equation (11.47) can be represented

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} (i-1) = \frac{n^2}{2} - \frac{n}{2}, \qquad \sum_{i=1}^{n} (i-1)^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

Using these results in the equation (11.47) one finds the volume approximation for the pyramid with n prisms can be represented

$$V_a \approx 4b^2h - 4b^2h\left(1 - \frac{1}{n}\right) + 4b^2h\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right)$$

Observe that as the integer n gets very large, then the terms that are divided by n all get very small, so that in the limit as $n \to \infty$ one obtains

$$V_{pyramid} = \lim_{n\to\infty} V_a = \frac{1}{3}(4b^2)h = \frac{1}{3}(base)(height) \tag{11.48}$$

The surface area of the pyramid is four times the area of one triangular face. A single triangular face as the area

$$A_s = \frac{1}{2}(2b)\ell$$

where ℓ is the slant height and $2b$ is the length of the triangle base. Therefore, the total lateral surface area of the pyramid is $S_{pyramid} = 4A_s = 2(2b)\ell$. One can use the Pythagorean theorem to calculate the slant height since $\ell^2 = b^2 + h^2$. Observe that the lateral surface area of the pyramid can be written in the alternative form

$$S_{pyramid} = \frac{1}{2}[2b\ell + 2b\ell + 2b\ell + 2b\ell] = \frac{1}{2}P\ell = \frac{1}{2}P\sqrt{b^2 + h^2} \tag{11.49}$$

where $P = 8b$ is the perimeter of the base and ℓ is the slant height of the pyramid. To obtain the total surface area associated with the pyramid one must add the area associated with the square base $(2b)^2$ to the lateral surface area.

Frustum of a square pyramid

The frustum of a square pyramid is obtained by a plane parallel to the base of the pyramid and cutting the top off. The situation is illustrated in the figure 11-32. One can calculate the volume of a large pyramid and then calculate the volume of a smaller pyramid, where the smaller pyramid represents the top of the pyramid that was removed to form the frustum. The volume of the frustum of a pyramid is then the volume of the larger pyramid minus the volume of the smaller pyramid. Let the height of the original square pyramid be $(h_2 + h)$ and its base denoted by b_1. Let the height of the pyramid cut off to form the frustum be h_2 and its base denoted by b_2, then one can express the volume of the pyramid frustum as

$$V_{frustum} = \frac{1}{3}(b_1)^2(h_2 + h) - \frac{1}{3}(b_2)^2(h_2) \tag{11.50}$$

which represents the volume of the original pyramid minus the portion removed.

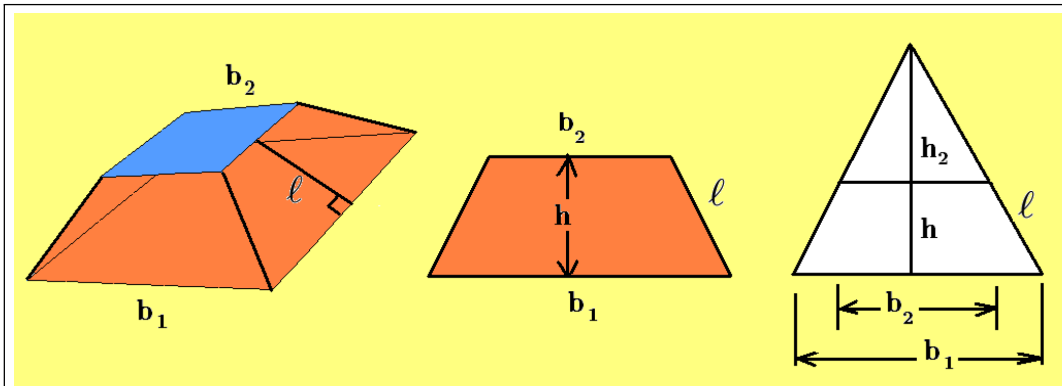


Figure 11-32.

Frustum of pyramid and its cross section.

This result can be expressed in a different form involving only h, the height of the frustum and also the quantities A_1, A_2 representing the areas of the top base and bottom base of the frustum. If one makes use of similar triangles one can write the proportion

$$\frac{h_2}{h_2 + h} = \frac{b_2}{b_1} \quad \Rightarrow \quad h_2 = \frac{b_2}{b_1 - b_2} h \tag{11.51}$$

Replace the h_2 in equation (11.50) using the results from equation (11.51) and show

$$\begin{aligned} V_{frustum} =&\frac{1}{3}b_1^2\left[\frac{b_2}{b_1-b_2}h+h\right]-\frac{1}{3}b_2^2\left[\frac{b_2}{b_1-b_2}h\right] \\ V_{frustum} =&\frac{1}{3}h\left[\frac{b_2(b_1^2-b_2^2)+(b_1-b_2)b_1^2}{b_1-b_2}\right] \\ V_{frustum} =&\frac{1}{3}h\left[A_1+A_2+\sqrt{A_1A_2}\right]=\frac{1}{3}h\left(b_1^2+b_1b_2+b_2^2\right) \end{aligned} \tag{11.52}$$

where $A_1 = b_1^2$ and $A_2 = b_2^2$ are the areas of the top and bottom of the frustum.

Let ℓ denote the slant height of each face of the frustum. Each face of the frustum is a trapezoid with lateral surface area

$$S_{frustum} = 4\left[\frac{1}{2}(b_1+b_2)\ell\right]$$

Let $4b_1 = p_1$ and $4b_2 = p_2$ denote the perimeter of the top and bottom faces, then the lateral surface area can be expressed

$$S_{frustum} = \frac{1}{2}(p_1+p_2)\ell \tag{11.53}$$

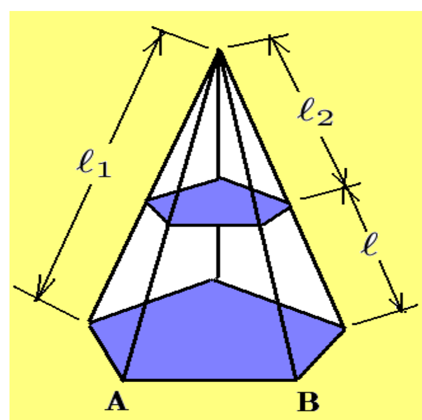
The total surface area of the frustum requires the addition of the top and bottom areas to the lateral surface area.

Lateral surface area of pyramid

The lateral surface area of a regular pyramid with n-sides is obtained by first finding the area of one triangular face

$$S_1 = \frac{1}{2}\overline{\text{AB}}\,\ell_1 = \frac{1}{2}(\text{base})(\text{height})$$

where $\overline{\text{AB}}$ is base of triangle and ℓ_1 is the slant height as illustrated in the accompanying figure.



The total lateral surface area of n-faces is

$$S_t = n\,S_1 = \frac{n}{2}\overline{\text{AB}}\,\ell_1 = \frac{1}{2}p_1\ell$$

where $p_1 = n\,\overline{\text{AB}}$ is the perimeter of the base.

If the pyramid is cut by a plane parallel to the base, then a frustum of a regular pyramid is created. The lateral surface area for this frustum is

$$S = \frac{1}{2}p_1\ell_1 - \frac{1}{2}p_2\ell_2 \tag{11.54}$$

where p_1 is the perimeter of the lower base and p_2 is the perimeter of the upper base. Note that the original pyramid and top pyramid cut by the plane are similar figures so that one can write

$$\frac{p_2}{p_1} = \frac{\ell_2}{\ell_1} \quad \text{or} \quad p_2\ell_1 = p_1\ell_2 \tag{11.55}$$

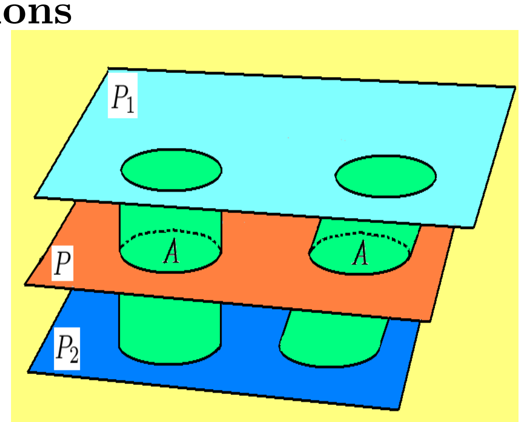
Therefore, the equation (11.54) can be expressed

$$S = \frac{1}{2}\left[p_1\ell_1 - p_1\ell_2 + p_2\ell_1 - p_2\ell_2\right] = \frac{1}{2}(p_1+p_2)(\ell_1-\ell_2) = \frac{1}{2}(p_1+p_2)\ell \tag{11.56}$$

where $\ell = \ell_1 - \ell_2$ is the slant height of the frustum. The equation (11.56) shows the lateral surface area associated with a frustum of a regular pyramid having n-faces is given by the average perimeter of the top and bottom bases multiplied by the slant height ℓ which is the same as our previous result for a square pyramid.

Cavalier's principle in three dimensions

Consider two parallel planes P_1 and P_2 with two solids between them. Cavalier's **first principle** states that if every plane P parallel to both planes P_1 and P_2 cuts the two solids **to produce equal area cross-sections**, then the volumes of the two solids will be equal. Cavalier's **second principle** states that if the cross-sections have areas in the ratio $\frac{A_1}{A_2}$, then the volumes of the solids will have the ratio $\frac{V_1}{V_2} = \frac{A_1}{A_2}$.

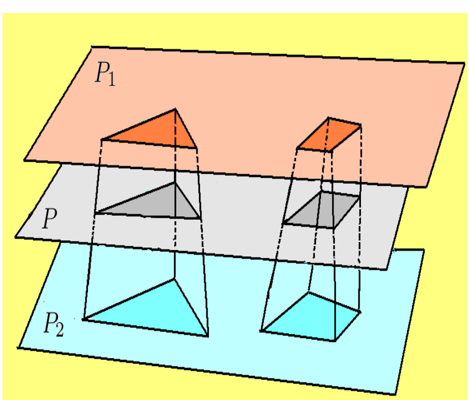


Cavalier's principle implies the following. If the bases at top and bottom are of equal area, then

A right cylinder and slanted right cylinder of the same height have equal volume.

A right prism and slanted right prism of the same height have equal volumes

A right pyramid and slanted right pyramid of the same height have equal volumes



The Cavalier's principle does not require that the solids be similar in shape. The only thing required is that as the plane P moves up and down and remains parallel to planes P_1 and P_2, then the cross-sectional areas of the two solids must be equal for all possible cross-sections. Whenever this occurs, then the solids have equal volume.

Note that Cavalier's principle says the volumes are the same. However, the lateral surface area changes as is illustrated in the following examples.

Example 11-2.

Find the volume and surface area for the cylinder and slanted cylinder.

Solution:

Let V denote volume, S the lateral surface area, h the height of the solid and a the radius of circle perpendicular to the sides of the oblique cylinder.

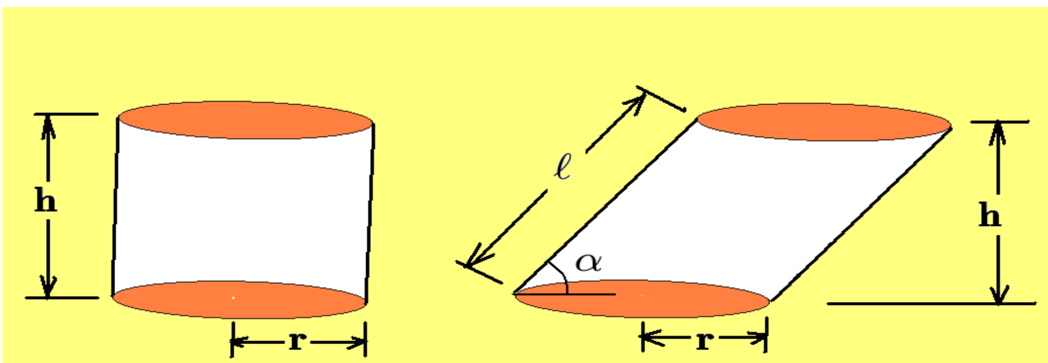


Figure 11-33. Cylinder and slanted cylinder

cylinder

Volume =(area of base)(height)

$$V = (\pi r^2)\, h$$

lateral surface area =(perimeter)(height)

$$S = 2\pi r\, h$$

total surface area $= S_t = 2\pi r\, h + 2(\pi r^2)$

oblique or slanted cylinder

Volume =(area of base)(height)

$$V = (\pi r^2)\ell \sin\alpha = \pi r^2\, h$$

lateral surface area

$$S = (2\pi r)\ell = \frac{2\pi r\, h}{\sin\alpha}$$

total surface area

$$S_t = 2\pi r\,\ell + 2(\pi r^2) = 2\pi r\,(\ell + r)$$

■

Example 11-3.

Find the volume and surface area for a prism and oblique prism.

Solution:

Let V denote volume, A the area of the base, S the lateral surface area, p the perimeter of the base, ℓ the slant height of the oblique prism, p_N the perimeter of the prism and A_N the area of a normal section to sides of oblique prism and h the height of the prisms.

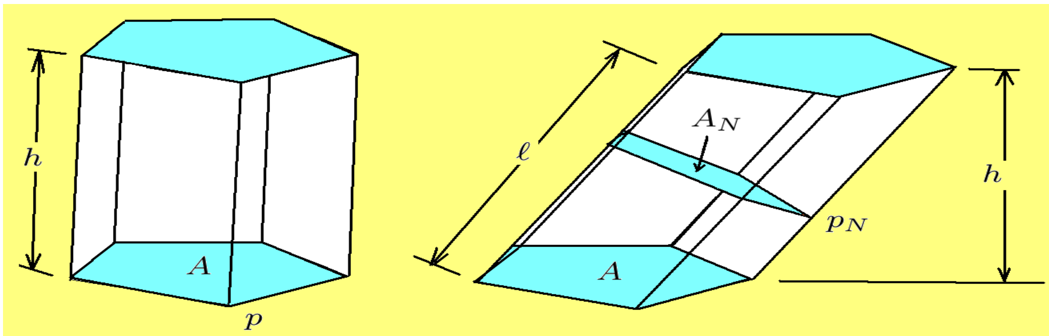


Figure 11-34. Prism and oblique prism

One can then verify the following.

prism	oblique prism
$V =$(area of base)(height)	$V =$(area of base)(height)
$V = A\,h$	$V = A\,h$ or or $V = A_N\,\ell$
$S =$(perimeter of base)(height)	
$S = p\,h$	$S = p_N\,\ell$
total surface area	total surface area
$S_t = p\,h + 2A$	$S_t = p_N\,\ell + 2A$

■

Cavalier's principle implies each of the solids in figure 11-35 have a volume given by

$$\text{Volume} = \frac{1}{3}(\text{area of base})(\text{height})$$

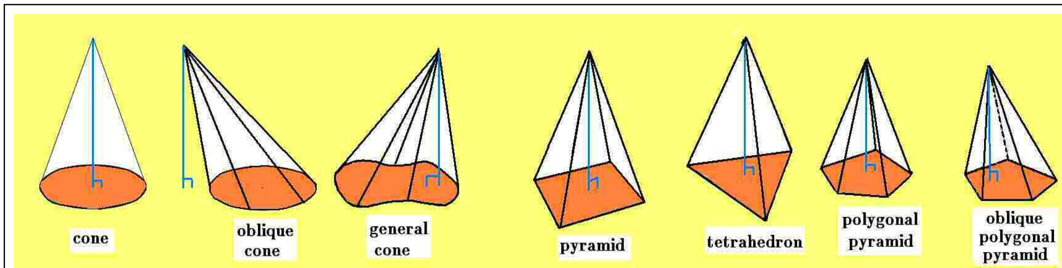
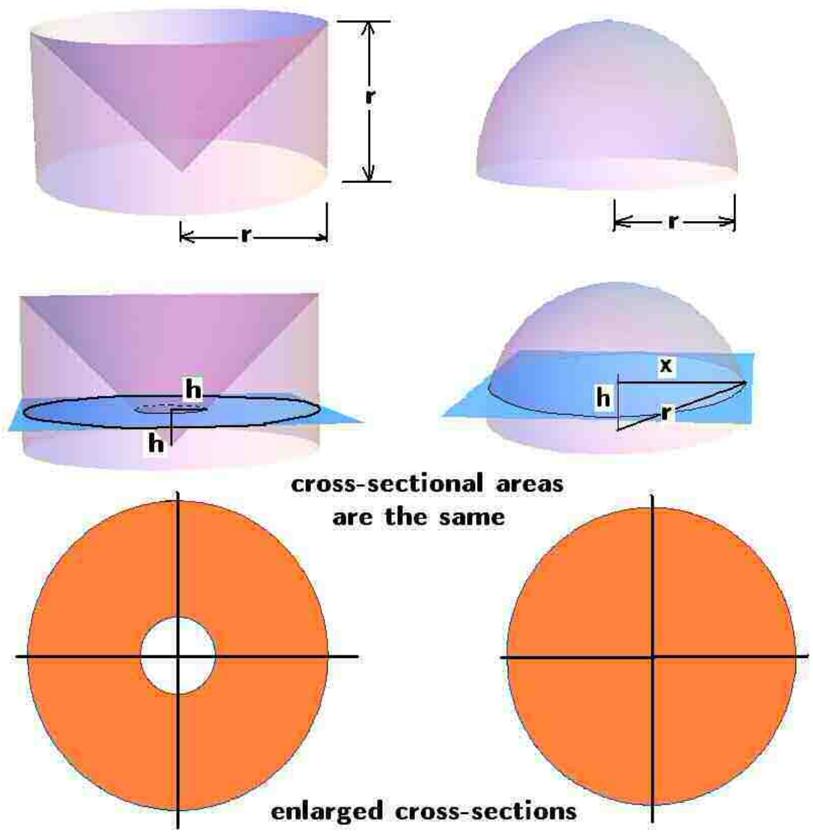


Figure 11-35. All solids have volume $\frac{1}{3}$(area of base)(height)

That is, we have demonstrated the cone has a volume given by $V = \frac{1}{3} b\, h$ so that if the area of all the bases in figure 11-35 are the same and all heights and cross sections are the same, then they all must have the same volume given by the relation

$$V = \frac{1}{3}(\text{area of base})(\text{height})$$

Example 11-4. Consider a cylinder with base πr^2 and height r. Place a cone within the cylinder as illustrated in the figure below.



Place next to this cylinder a hemisphere with radius r and then cut both figures by a plane at a height h. The cross-section produced through the cylinder is a washer shaped area. This area equals $A = \pi(r^2 - h^2)$ since the slope of the cone is unity. The cross-sectional area of the sphere is a circle with radius $x = \sqrt{r^2 - h^2}$. The cross-sectional area of the sphere is also $A = \pi(r^2 - h^2)$. By Cavalier's theorem the volumes of these two solids are equal. The volume of the cone is $V_{cone} = \frac{1}{3}(base)(height) = \frac{1}{3}(\pi r^2)(r)$

and the volume of the cylinder is $V_{cylinder} = (base)(height) = (\pi r^2)(r)$ Therefore the volume associated with the cross-sectional area is

$$V = V_{cylinder} - V_{cone} = \pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3 = V_{hemisphere}$$

Therefore, by Cavalier's principle and symmetry, the volume of the sphere is given by $V_{sphere} = \frac{4}{3}\pi r^3$.

■

Theorems of Pappus

Pappus[3] of Alexandria derived two propositions or theorems for determining the volume and surface area associated with solids created by rotation of an area about an axis.

The first proposition of Pappus states that the **surface area** associated with a solid of revolution equals the product

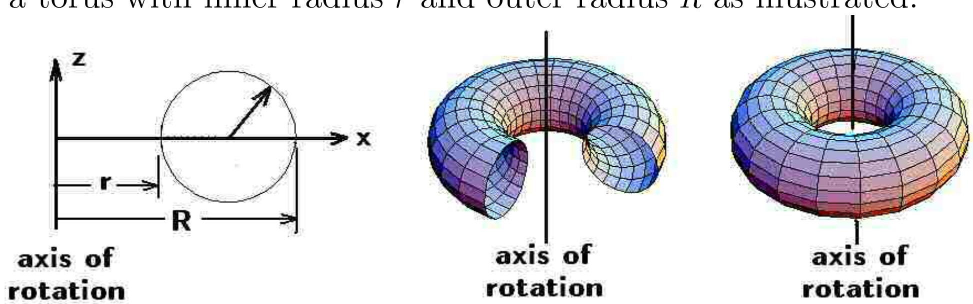
(Length of arc around area rotated)(distance traveled by the centroid of the arc)

The second proposition of Pappus states that the **volume associated** with a solid of revolution equals the product

(cross-sectional area rotated)(distance traveled by the centroid of the area)

Example 11-5.

Consider the circle $(x - \frac{r+R}{2})^2 + z^2 = \left(\frac{R-r}{2}\right)^2$ which is revolved about the $z-$axis to form a torus with inner radius r and outer radius R as illustrated.



The **surface area** is

$S =$(Length of arc around area rotated)(distance traveled by the centroid of the arc)

$$S = \left(2\pi(\frac{(R-r)}{2}\right)\left(2\pi\frac{R+r}{2}\right) = \pi^2(R^2 - r^2)$$

[3] Pappus of Alexandria (290-350)CE mathematician and philosopher.

since the origin of the circle is the centroid for the arc.

The **volume** is found to be

$$V = (\text{cross-sectional area rotated})(\text{distance traveled by the centroid of the area})$$

$$V = \left(\pi\left(\frac{R-r}{2}\right)^2\right)\left(2\pi\frac{R+r}{2}\right) = \frac{\pi^2}{4}(R-r)(R^2-r^2)$$

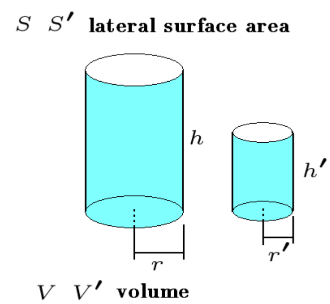
■

Similar solids

In the following discussions make note that if two solids are similar, then their ratio of areas is proportional to the square of the similarity ratio and the ratio of their volumes is proportional to the cube of the similarity ratios.

Cylinders

Two similar cylinders provides one with the similarity ratios $\frac{r}{r'} = \frac{h}{h'}$. The lateral surface area of the two similar cylinders are $S = 2\pi r h$ and $S' = 2\pi r' h'$ giving the ratio



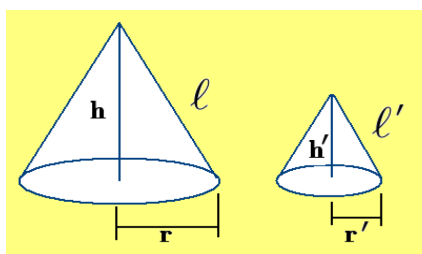
$$\frac{S}{S'} = \frac{2\pi r h}{2\pi r' h'} = \frac{\left(r'\frac{h}{h'}\right)h}{r'h'} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2}$$

The volumes of the two similar cylinders are $V = \pi r^2 h$ and $V' = \pi r'^2 h'$ giving the ratio

$$\frac{V}{V'} = \frac{\pi r^2 h}{\pi r'^2 h'} = \frac{\left(\frac{h}{h'}r'\right)^2 h}{r'^2 h'} = \frac{h^3}{h'^3} = \frac{r^3}{r'^3}$$

Right circular cones

Examine the two similar right circular cones and produce the similarity ratios $\frac{r}{r'} = \frac{h}{h'} = \frac{\ell}{\ell'}$. The lateral surface area of the two similar cones are $S = \pi r \ell$ and $S' = \pi r' \ell'$ giving the ratio

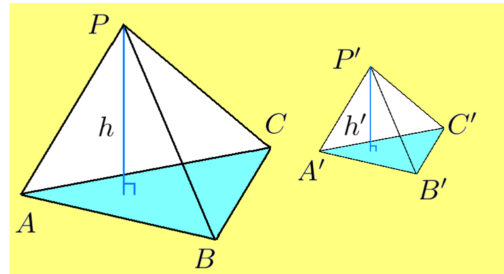


$$\frac{S}{S'} = \frac{\pi r \ell}{\pi r' \ell'} = \frac{\left(\frac{h}{h'}r'\right)\left(\frac{h}{h'}\ell'\right)}{r'\ell'} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2}$$

The volumes of the similar cones are $V = \frac{1}{3}\pi r^2 h$ and $V' = \frac{1}{3}\pi r'^2 h'$ giving the ratio

$$\frac{V}{V'} = \frac{\frac{1}{3}\pi r^2 h}{\frac{1}{3}\pi r'^2 h'} = \frac{\left(r'\frac{h}{h'}\right)^2 h}{r'^2 h'} = \frac{h^3}{h'^3} = \frac{r^3}{r'^3}$$

Tetrahedrons



Two similar tetrahedrons have the similarity ratios $\frac{\overline{\text{AB}}}{\overline{A'B'}} = \frac{\overline{\text{AC}}}{\overline{A'C'}} = \frac{\overline{\text{BC}}}{\overline{B'C'}} = \frac{h}{h'}$. The surface areas of the two similar tetrahedrons are given by

$$S = 4[ABC] \text{ and } S' = 4[A'B'C']$$

giving the ratio

$$\frac{S}{S'} = \frac{[ABC]}{[A'B'C']} = \text{similarity ratio squared} = \frac{h^2}{h'^2}$$

from previous investigation involving ratio of areas.

The volumes of the similar tetrahedrons are $V = \frac{1}{3}[ABC]\,h$ and $V' = \frac{1}{3}[A'B'C']\,h'$ giving the ratio

$$\frac{V}{V'} = \frac{\frac{1}{3}[ABC]\,h}{\frac{1}{3}[A'B'C']\,h'} = \frac{[ABC]}{[A'B'C']}\,\frac{h}{h'} = \frac{h^2}{h'^2}\,\frac{h}{h'} = \frac{h^3}{h'^3}$$

Two similar solids

In general if two solids are similar, then the ratio of the surface areas will be proportional to the similarity ratio squared and the ratio of the volumes will be proportional to the similarity ratio cubed. This can be proven as follows. One can select a certain size small tetrahedron and then approximate the similar solids by a summation of tetrahedrons. If T_i is the volume of the ith tetrahedron, then one can write the fact that the two solids are similar by writing

$$T_1 + T_2 + \cdots + T_n \sim\ T_1' + T_2' + \cdot + T_n'$$

and the ratio of the volumes is

$$\frac{V}{V'} = \frac{T_1 + T_2 + \cdots + T_n}{T_1' + T_2' + \cdots + T_n'} = \frac{T_1}{T_1'} = \frac{h^3}{h'^3}$$

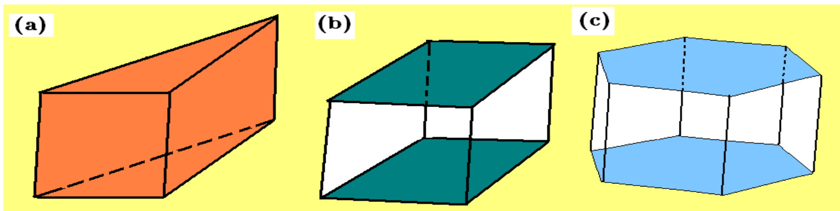
Similarly if S_i is the surface area of the ith tetrahedron, then the ratio of surface areas becomes

$$\frac{S}{S'} = \frac{S_1 + S_2 + \cdots + S_n}{S_1' + S_2' + \cdots + S_n'} = \frac{S_1}{S_1'} = \frac{h^2}{h'^2}$$

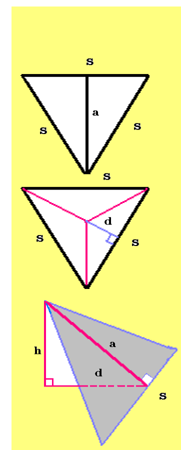
These ratios holding even in the limit as n gets very large.

Exercises

► **11-1.** Name the solid and test Euler's formula.



► **11-2.**



The volume and surface area of a tetrahedron with edge of length s are given by $V = \frac{\sqrt{2}}{12}s^3$ and $S = \sqrt{3}\,s^2$ To verify this proceed as follows.

(a) Find the altitude a associated with one face.

(b) Find the area of one face.

(c) Find the distance d from an edge to the center of one face.

(d) Find the height h of the pyramid.

(e) Find the volume equal to one-third the area of base times the height.

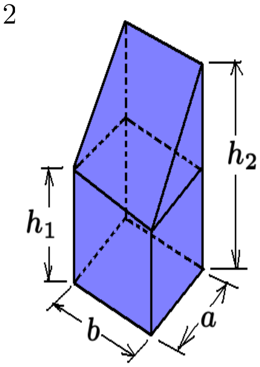
(f) Find the surface area of the tetrahedron.

► **11-3.**

(a) Find the volume and surface area associated with one-half of an octahedron having an edge length s.

(b) Show the total volume is $V = \frac{\sqrt{2}}{3}\,s^3$ and the surface area is $S = 2$

► **11-4.** Find the volume of the solid illustrated.

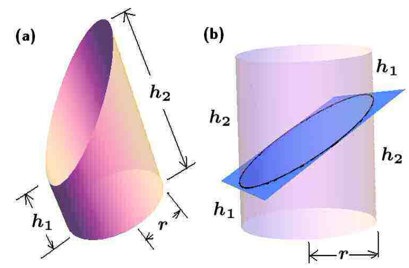


► **11-5.** Find the diagonal of the cube having each side of length 8.

► **11-6.** Prove the diagonals of a parallelepiped meet at a point of concurrency.

▶ **11-7.**

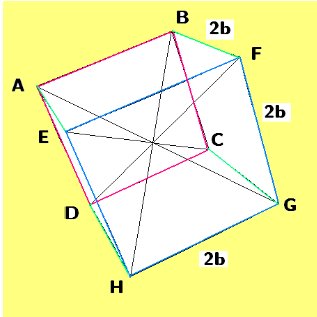
Find the volume of the cylinder which has been cut by a plane as illustrated in figure (a) where h_1 is the smallest height and h_2 is the largest height associated with the cross-section and r is the radius of the cylinder.
Hint: Make it into a problem with symmetry.



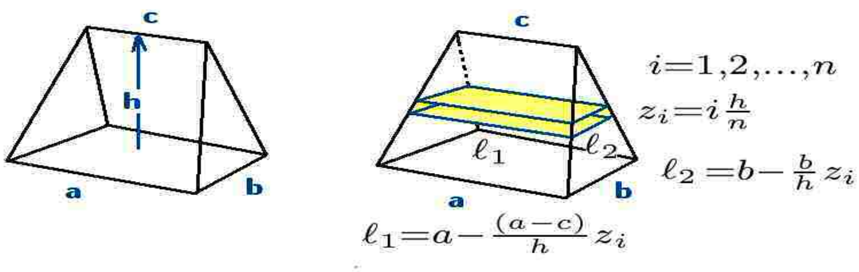
▶ **11-8.**

For the cube illustrated,
(a) Prove any three diagonals intersect at a point.
(b) Use symmetry and find the number of pyramids created by the diagonals.
(c) Verify the volume of these pyramids is given by

$$\frac{1}{3}(base)(height)$$



▶ **11-9.**



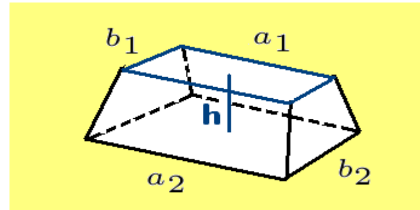
Consider the solid formed by a horizontal slicing of the dihedral angle formed by two identical isosceles triangles and a rectangular base as illustrated with top line c centered over the base. Divide the height h into $n-$parts and form a sandwich-shaped rectangle at height $z_i = i\frac{h}{n}$.
(a) Show the sides of the sandwich-shaped rectangle are given by

$$\ell_1 = a - \frac{(a-c)}{h}z_i, \qquad \ell_2 = b - \frac{b}{h}z_i, \qquad i = 1, 2, \ldots, n$$

(b) Show the volume associated with the sandwich-shaped element is $V_i = \ell_1\ell_2\frac{h}{n}$
(c) Use the summation formulas from chapter 6 to show the total volume of the dihedral angle solid is

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} V_i = \frac{bh}{6}(2a+c)$$

▶ **11-10.**

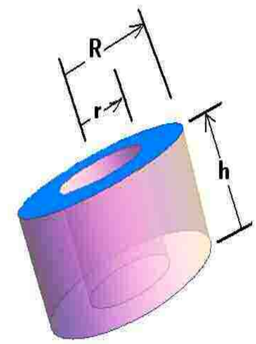


The figure at the right is called an obelisk and results when one examines a frustum of the dihedral angle solid illustrated in the previous problem. Use the results from the previous problem to show the volume of the obelisk is given by

$$V = \frac{h}{6}\left[(2a_2 + a_1)b_2 + (2a_1 + a_2)b_1\right]$$

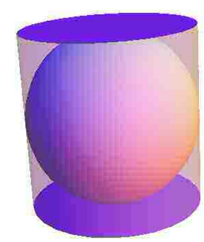
Hint: The whole equals the sum of its parts.

▶ **11-11.**



Find the volume and surface area of the hollow cylinder with height h, inner radius r and outer radius R.

▶ **11-12.**

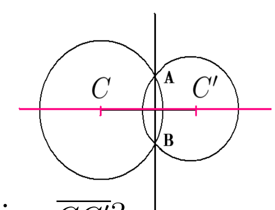


On Archimedes tomb is the picture of a sphere with radius r placed inside a cylinder of radius r and height $2r$.

(a) Find the relation between volume of sphere and volume of cylinder.

(b) Find the relation between total surface area of sphere and total surface area of cylinder.

▶ **11-13.**



The figure is a sketch of two circles intersecting with chord $\overline{\text{AB}}$ joining the points of intersection. If C and C' are the centers of the circles, then prove that two spherical surfaces intersecting will always produce a circle.
Hint: What happens if the attached figure is rotated about the line $\overline{CC'}$?

▶ **11-14.** (**The octahedron**)

Let s denote the length of the sides of the eight congruent equilateral triangles.

(a) Show by symmetry the octahedron can be made up of two pyramids.

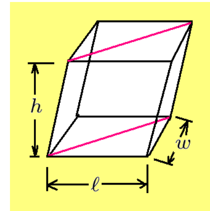
(b) Prove the surface area of the octahedron is $S = 2\sqrt{3}\, s^2$

(c) Prove the volume of the octahedron is $V = \dfrac{\sqrt{2}}{3}\, s^3$

▶ **11-15.**

(a) Find the diagonal and volume of the parallelepiped.

(b) Find the volume of a triangular slanted prism.



▶ **11-16.** The dodecahedron has a surface area which is made up of twelve congruent regular pentagons. Assume the pentagons have each side of length ℓ.

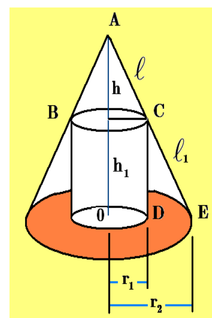
(a) Show the area of one face is $S_1 = \frac{1}{4}\sqrt{5(5+2\sqrt{5})}\,\ell^2$

(b) Show the total surface area of the dodecahedron is $S_{12} = 3\sqrt{5(5+2\sqrt{5}}\,\ell^2$

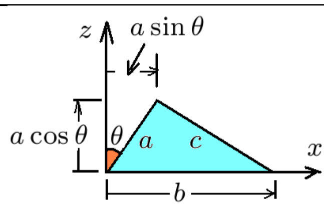
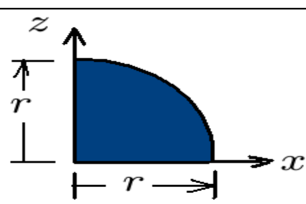
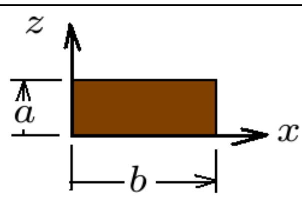
▶ **11-17.** A cylinder is constructed inside a right circular cone as illustrated.

(a) Find the similarity ratio between the upper right cone and larger right cone.

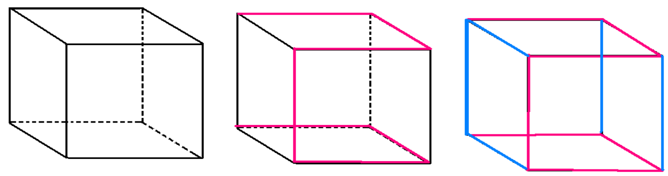
(b) Find the distance r_1 if the surface area of the upper cone equals the orange area between two circles.



▶ **11-18.** Find the surface area and volume created when the following figures are rotated about the z-axis.

Figure	Centroid		Area
	$\bar{x}$	$\bar{z}$	
	$\bar{x} = \frac{b+a\sin\theta}{3}$	$\bar{z} = \frac{a\cos\theta}{3}$	$\frac{1}{2}ba\cos\theta$
	$\bar{x} = \frac{4r}{3\pi}$	$\bar{z} = \frac{4r}{3\pi}$	$\frac{1}{4}\pi r^2$
	$\bar{x} = \frac{1}{2}b$	$\bar{z} = \frac{1}{2}a$	ab

▶ **11-19.** Special case associated with proof of Euler's formula $E = V + F - 2$ for convex polyhedra.



One proof of Euler's formula is a generalization of the following process applied to the special polyhedron which is a cube.

(a) Color any vertex red and then move and color an edge going to another vertex which **is not colored red**. Repeat the process, each time moving to a vertex which **is not colored red** until there are no more vertices to move to.

(b) Let R equal the number of red edges and V the number of vertices. Show that $R = V - 1$

(c) All the edges which are not red are to be colored blue. Let B equal the number of blue edges and F the number of faces. Show that $B = F - 1$.

(d) Add the red and blue edges and show the total number of edges is $E = V + F - 2$.

(e) You try this process on a tetrahedron and some other polyhedron.

▶ **11-20.** Fill in the following table.

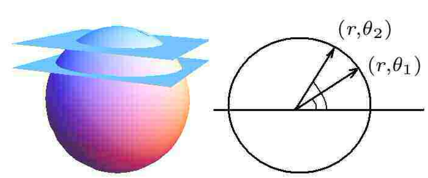
Volume change with edge size change	
Edge size	Volume of cube
s	
2s	
3s	
4s	
5s	

▶ **11-21.** Find the cross-sectional area associated with a plane intersecting with a sphere. Assume the plane is a perpendicular distance of h from the center of the sphere.

▶ **11-22.**
Two parallel planes intersecting a sphere creates a zone.

(a) Find the surface area of the zone.

(b) Find the volume of the zone.



Geometry
Chapter 12
Solid Geometry II

Prismatoid

Given two parallel planes where all the vertices of a polyhedron lie only in these planes. Any solid satisfying these conditions is called a prismatoid. The faces of a prismatoid that lie within a parallel plane are called the bases of the prismatoid. We use the notation B_u, B_ℓ for the area of the upper and lower bases. The altitude h of a prismatoid is the perpendicular distance between the parallel planes. A plane midway between the upper and lower planes cuts the prismatoid in a section called the midsection with area denote by M.

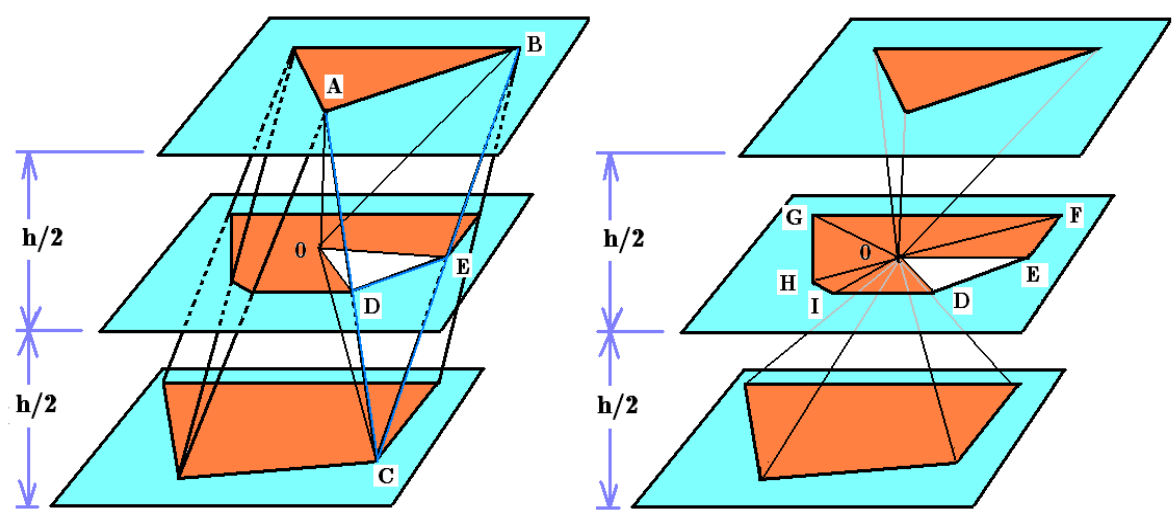


Figure 12-1. Finding the volume of a prismatoid

The figure 12-1 illustrates a prismatoid with height h. Select a point 0 in the middle section and construct straight lines from 0 to the vertices of the upper, middle and lower parallel planes. By passing planes through these lines one can divide the prismatoid into many pyramids all having a common vertex 0. If any pyramid is not triangular, then a construction must be performed to make it into a triangular pyramid.

By joining point 0 to the vertices of the upper figure gives a pyramid with volume

$$V_1 = \frac{1}{3}(base)(height) = \frac{1}{3}B_u\,\frac{h}{2} \tag{12.1}$$

In a similar fashion construct lines from 0 to the vertices of the lower figure to produce a pyramid with volume

$$V_2 = \frac{1}{3}(base)(height) = \frac{1}{3}B_\ell\frac{h}{2} \tag{12.2}$$

Examine the triangle $\triangle 0DE$ which cuts the triangle $\triangle ABC$ in line $\overline{\rm DE}$. Recall that ratio of areas are proportional to the square of their similarity ratios so that one can write

$$\frac{[DEC]}{[ABC]} = \frac{(\frac{h}{2})^2}{h^2} \quad \Rightarrow \quad [DEC] = \frac{1}{4}[ABC] \tag{12.3}$$

One can now examine the volume of the tetrahedron

$$\text{Volume } 0 - DEC = \frac{1}{4}\text{Volume } 0 - ABC = \frac{1}{3}(base)(height) = \frac{1}{3}[ODE]\frac{h}{2} \tag{12.4}$$

Hence, one can write

$$\text{Volume } 0 - ABC = \frac{4}{6}h[0DE] \tag{12.5}$$

As you move around point 0, the fractional part of the total volume can be calculated in a similar fashion as above. Summation of these volumes gives

$$V_3 = \frac{4}{6}h\left([0DE] + [0FE] + [0FG] + [0GH] + [0HI] + [0ID]\right) = \frac{4}{6}\,h\,M \tag{12.6}$$

The total volume of the prismatoid is then

$$V = V_1 + V_2 + V_3 = \frac{1}{6}h\left(B_u + B_\ell + 4M\right) \tag{12.7}$$

where h is the height of the prismatoid, B_u is the area of upper figure, B_ℓ is area of lower figure and M is the area of the midsection.

Plane sections

If a pyramid with a polygon for its base is cut by three planes parallel to the base, one of the planes through the base and another plane through the vertex and the remaining plane cutting all edges of the pyramid, then similar pyramids are created as illustrated in the figure 12-2.

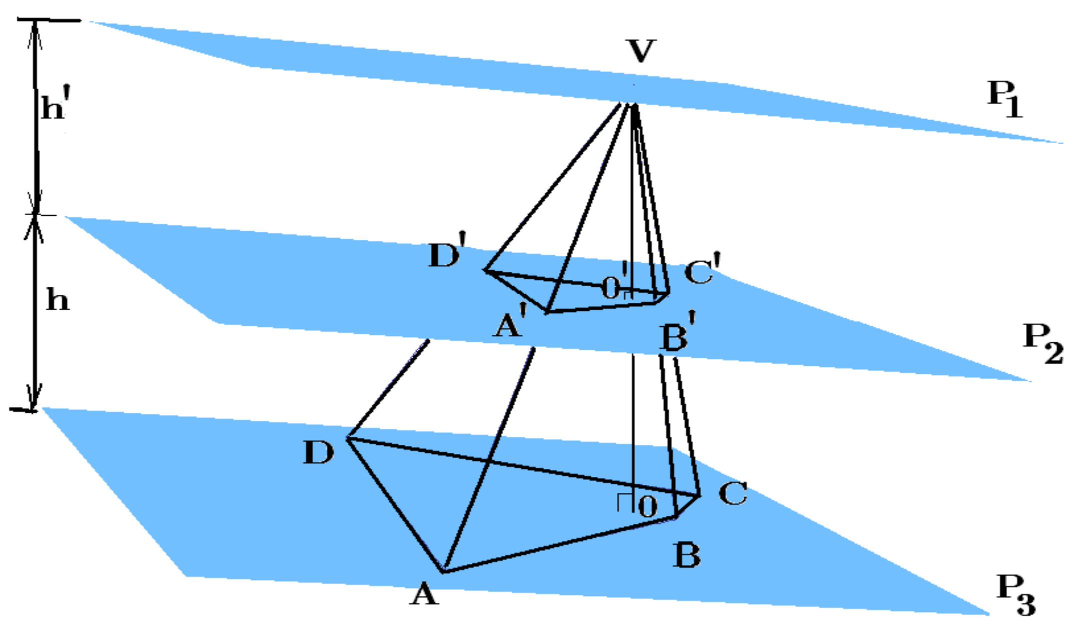


Figure 12-2. Pyramid cut by three planes parallel to base

This can be proven by observing that the face VAB of the pyramid consists of similar triangles so that one can write $\frac{\overline{VA}}{\overline{VA'}} = \frac{\overline{VB}}{\overline{VB'}}$. If one does this for each face of the pyramid one finds

$$\frac{\overline{VA}}{\overline{VA'}} = \frac{\overline{VB}}{\overline{VB'}} = \frac{\overline{VC}}{\overline{VC'}} = \frac{\overline{VD}}{\overline{VD'}} = \frac{h'}{h+h'} \tag{12.8}$$

The last ratio results when the bases of the altitudes are joined to each vertex to form similar right triangles. For example, $\triangle 0VA \sim \triangle 0'VA'$. The equation (12.8) gives the similarity ratio's associated with the similar figures and so one can write pyramid $ABCD \sim A'B'C'D'$.

Recall that the ratio of the pyramid bases B' and B is proportional to the similarity ratio squared giving

$$\frac{B}{B'} = \frac{[ABCD]}{[A'B'C'D']} = \frac{(h+h')^2}{(h')^2} \quad \text{or} \quad B' = \frac{(h')^2}{(h+h')^2}B \tag{12.9}$$

.

The volume of the smaller pyramid is $V' = \frac{1}{3}(\text{area of base})(\text{height}) = \frac{1}{3}B'h'$ and the volume of the larger pyramid is $V = \frac{1}{3}(\text{area of base})(\text{height}) = \frac{1}{3}B(h+h')$. Using equation (12.9) one can show the ratio of the volumes is given by

$$\frac{V}{V'} = \frac{\frac{1}{3}B(h+h')}{\frac{1}{3}B'h'} = \frac{(h+h')^2}{(h')^2}\frac{(h+h')}{h'} = \frac{(h+h')^3}{(h')^3}$$

showing the ratio of the volumes of similar figures is proportional to the similarity ratio cubed.

Each face of the pyramids is a triangle and so the lateral surface area is a summation of these face areas.

The frustum created when V' is subtracted from V has the volume

$$\begin{aligned} V_{frustum} = V - V' =& \frac{1}{3}B(h+h') - \frac{1}{3}B'h' = \frac{1}{3}B(h+h') - \frac{1}{3}B\frac{(h')^2}{(h+h')^2}h' \\ =& \frac{1}{3}B\left[\frac{(h+h')^3 - h'^3}{(h+h')^2}\right] = \frac{1}{3}Bh\left[1 + \frac{h'}{h+h'} + \left(\frac{h'}{h+h'}\right)^2\right] \\ =& \frac{1}{3}Bh\left[1 + \sqrt{\frac{B'}{B}} + \frac{B'}{B}\right] \end{aligned}$$

Each face of the frustum is a trapezoid so that the area of each trapezoid can be calculated to obtain the lateral surface area.

In the special case of a regular square pyramid where $B = b_1^2$ and $B' = b_2^2$ the above equation reduces to

$$V_{frustum} = \frac{1}{3}h[b_1^2 + b_1 b_2 + b_2^2]$$

verifying our previous result.

Plane intersecting a triangular prism

A truncated triangular prism results when a plane intersects a triangular prism. The situation is illustrated in the figure 12-3 where the intersecting plane produces the truncated prism ABCDEF. The truncated triangular prism can be divided into three triangular pyramids

$$E - \triangle ABC, \quad E - \triangle ACD, \quad E - \triangle CFD \tag{12.10}$$

It will be demonstrated that the volume associated with the pyramids in equation (12.10) is the same as the volume associated with the pyramids

$$E - \triangle ABC, \quad F - \triangle ABC, \quad D - \triangle ABC \tag{12.11}$$

which all have the common base $\triangle ABC$ but with different vertices D, E, F at heights h_1, h_2, h_3. One can then use the fact that the volume of any one pyramid is $\frac{1}{3}$(area base)(height) to obtain the total volume of the truncated prism. That is, the volume of the pyramids $D - \triangle ABC + E - \triangle ABC + F - \triangle ABC$ is the same as the volume of the truncated triangular prism.

The above assertions can be demonstrated by construction of the planes ACE and DCE which divides the truncated triangular prism into the three triangular pyramids indicated above. Observe that the line $\overline{\text{EB}}$ is parallel to the lines $\overline{\text{AD}}$ and $\overline{\text{CF}}$ and consequently is parallel to the plane $ACFD$.

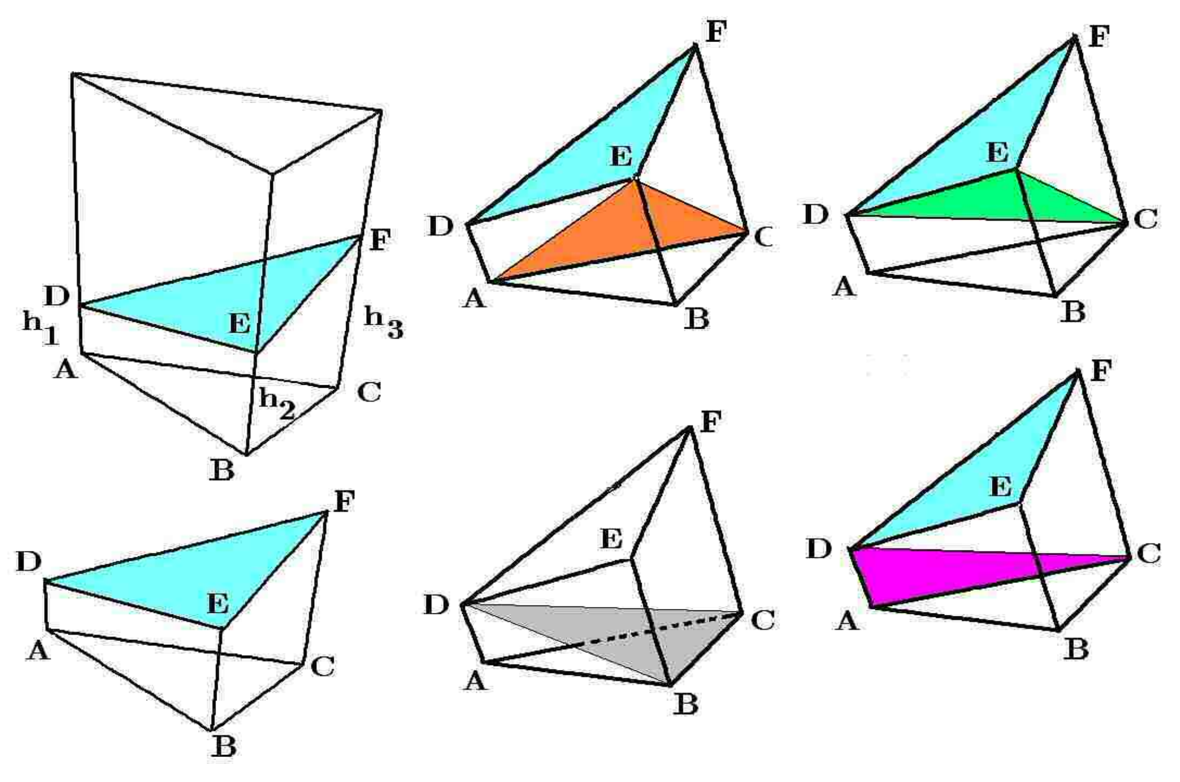


Figure 12-3. Triangular prism cut by a plane

Note the triangular pyramids $E-\triangle ACD$ and $B-\triangle ACD$ have the same altitudes so that these pyramids have the same volume. Two solids which have the same volume are said to be equivalent to each other. This equivalencey is expressed by writing $E-\triangle ACD \equiv B-\triangle ACD$. Observe also that in the pyramid $B-\triangle ACD$ one can take D as a vertex with triangle $\triangle ABC$ as the base and so one has $D-\triangle ABC \equiv E-\triangle ACD$. In a similar fashion one can show $\overline{\text{AD}} \parallel \overline{\text{EB}}$ and that

$$E-\triangle CFD \equiv B-\triangle CFD \equiv D-\triangle BCF \equiv A-\triangle BCF \equiv F-\triangle ABC$$

Consequently, the volume of the truncated triangular prism is given by

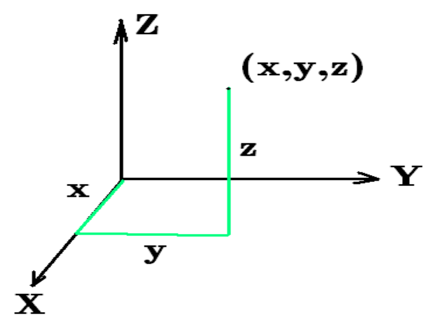
$$V = \frac{1}{3}(h_1 + h_2 + h_3)[ABC] \tag{12.12}$$

where $[ABC]$ is the area of the triangle base $\triangle ABC$.

The total surface area of the truncated triangular prism is the sum of the areas from the top and bottom triangles added to the summation of the face areas where each face is a trapezoid.

Cartesian coordinates

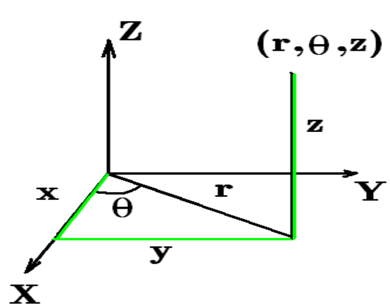
A point (x, y, z) in three dimensional Cartesian coordinates is illustrated in the diagram on the right. For points (x, y) in two dimensions it was found convenient at times to introduce polar coordinates (r, θ) to represent various quantities. In three dimensions it is often convenient to represent points (x, y, z) in cylindrical coordinates or spherical coordinates.



Cylindrical coordinates

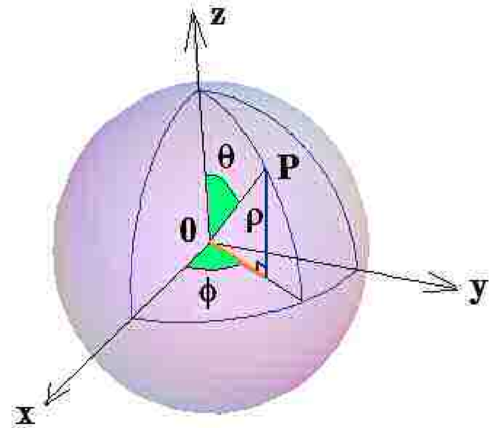
Cylindrical coordinates (r, θ, z) are related to rectangular coordinates (x, y, z) by the equations

$$\begin{aligned} x =& r\cos\theta & x^2+y^2 =& r^2 \\ y =& r\sin\theta & \tan\theta =& \frac{y}{x} \\ z =& z & z =& z \end{aligned} \tag{12.13}$$



Note the radial distance r is projected onto the x and y axes to obtain the above equations.

Spherical coordinates



The relation between Cartesian coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ) can be obtained as follows. First examine the projection of the line segment $\overline{0P} = \rho$ onto the xy-plane to obtain $\rho \sin\theta$ (the orange line). Next examine the projection of the orange line onto the x-axis and its projection onto the y-axis to obtain

$$x = \rho \sin\theta \cos\phi, \qquad y = \rho \sin\theta \sin\phi \tag{12.14}$$

The projection of line $\overline{0P} = \rho$ onto the z-axis gives

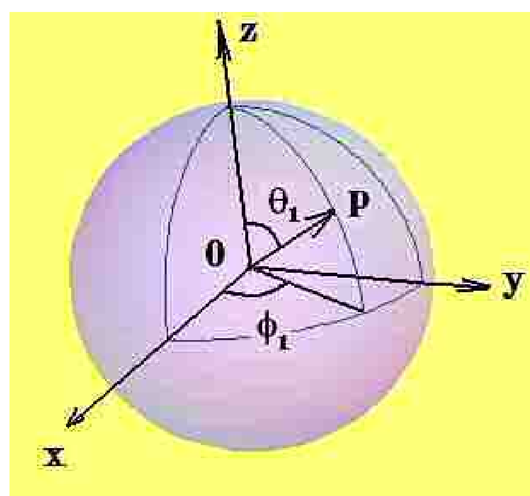
$$z = \rho \cos\theta \tag{12.15}$$

These projections give the coordinate transformations from spherical coordinates (ρ, θ, ϕ) of a point P on the surface to Cartesian coordinates (x, y, z) of the point P as

$$\begin{aligned} x =&\rho \sin\theta \cos\phi, \quad 0 \le \theta \le \pi & \rho^2 =&x^2 + y^2 + z^2 \\ y =&\rho \sin\theta \sin\phi, \quad 0 \le \phi \le 2\pi & \theta =& \arctan\frac{y}{x} \\ z =&\rho \cos\theta & \phi =& \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \end{aligned} \tag{12.16}$$

The angle θ is called the polar angle and the angle ϕ is called the azimuthal angle. Using algebra the parameters θ and ϕ can be eliminated from the equations (12.16) to obtain the Cartesian coordinate equation for the sphere as

$$x^2 + y^2 + z^2 = \rho^2 \tag{12.17}$$



The special case of a unit sphere results when $\rho = 1$. In what follows we will work with unit spheres to make things simpler. One can always add a nonzero value $\rho > 1$ at any time and scale things accordingly. Note that a point P on the surface of a unit sphere with surface coordinates (θ_1, ϕ_1) can be associated with a vector from the origin to point P on the surface of the unit sphere

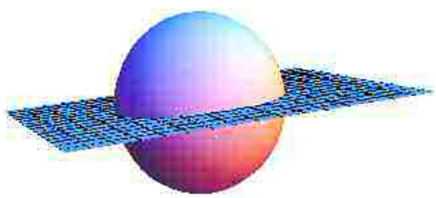
This vector is represented

$$\vec{r}_1 = \overrightarrow{0P} = \sin\theta_1\cos\phi_1\ \hat{\mathbf{e}}_1 + \sin\theta_1\sin\phi_1\ \hat{\mathbf{e}}_2 + \cos\theta_1\ \hat{\mathbf{e}}_3$$

where $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ are unit vectors in the directions of the x, y, z axes.

Make note of the fact that $\vec{r}_1$ is **a unit vector** from the center of the **unit sphere** to point P on the surface of the unit sphere.

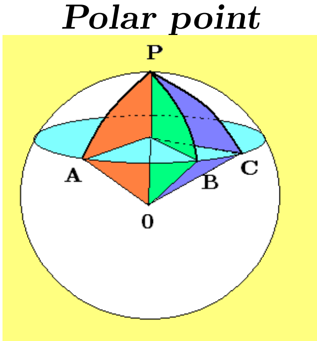
Great circles



The curve of intersection of a sphere with a plane is called a circle of the sphere. The curve of intersection between sphere and plane is called a **great circle** of the sphere **whenever the plane passes through the center of the sphere.** Any plane intersecting a sphere which does not pass through the center of the sphere produces what is called a small circle of the sphere.

Great circles on a sphere are used to define the shortest distance between two points on a sphere. These curves of shortest distance are called geodesics. **The planes of great circles intersect to form the angles associated with spherical geometry.** A line perpendicular to a great circle and passing through the center of the circle intersects the sphere at two points called poles associated with the great circle. For example, the North and South poles in relation to the equator of the Earth (assuming the Earth to be a sphere). The polar distance associated with a circle of the sphere is the spherical distance from any point on the circle to its nearest pole (either pole if circle is equator of the sphere).

Polar point



Every plane and sphere intersection produces a circle. This circle has a polar point P associated with it such that all the arcs $\overset{\frown}{PA} = \overset{\frown}{PB} = \overset{\frown}{PC}$ are equal. That is the planes $0AP$, $0BP$, $0CP$ produce great circles for any points C on the circle of intersection and so all the polar angles are equal.

Parametric equations for representing a circle In an orthogonal two-dimensional system of coordinates the parametric equations for the representation of a circle centered at the origin is given by

$$x = \cos t \qquad y = \sin t \qquad \Rightarrow \quad x^2 + y^2 = 1$$

and a position vector $\vec{r}(t)$ to a point on the circumference of the circle has the representation

$$\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 = \cos t\ \hat{\mathbf{e}}_1 + \sin t\ \hat{\mathbf{e}}_2, \qquad 0 \le t \le 2\pi \tag{12.18}$$

in terms of a parameter t and orthogonal base vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ having unit length.

In a non-orthogonal coordinate system with base vectors $\hat{\mathbf{e}}_1$ and $\hat{E}_2$, which are also unit vectors, we want to represent the above vector $\vec{r}(t)$ giving the position of a point on the unit circle.

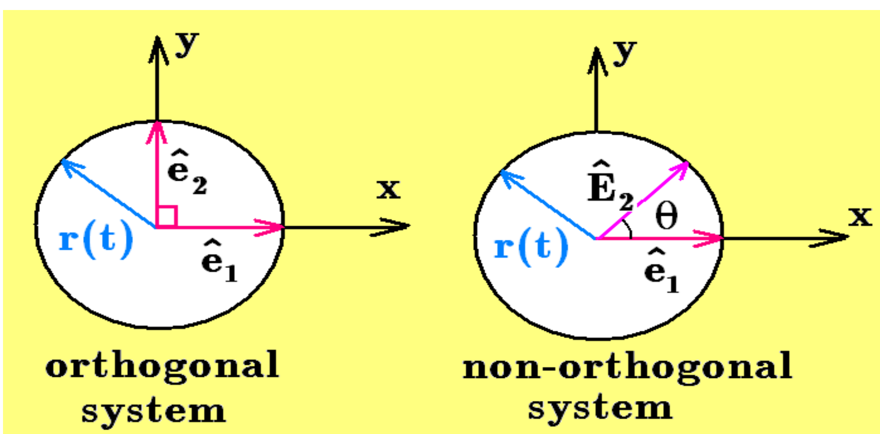


Figure 12-4. Orthogonal and non-orthogonal coordinate systems

Assume that

$$\vec{r}(t) = \cos t\ \hat{\mathbf{e}}_1 + \sin t\ \hat{\mathbf{e}}_2 = f(t)\ \hat{\mathbf{e}}_1 + g(t)\,\hat{E}_2 \tag{12.19}$$

where $f(t)$ and $g(t)$ are to be determined. Take the dot product of both sides of equation (12.19) with the unit vector $\hat{\mathbf{e}}_1$ and show

$$\cos t \;=\; f(t) + g(t)\,\hat{E}_2 \cdot \hat{\mathbf{e}}_1$$

Next take the dot product of both sides of equation (12.19) with the unit vector $\hat{\mathbf{e}}_2$ to obtain

$$\sin t \;=\; g(t)\,\hat{E}_2 \cdot\; \hat{\mathbf{e}}_2$$

Note the dot product relations

$$\hat{E}_2 \cdot\, \hat{\mathbf{e}}_1 = \cos\theta \qquad\qquad \hat{E}_2 \cdot\, \hat{\mathbf{e}}_2 = \cos(\frac{\pi}{2} - \theta) = \sin\theta$$

where θ is the angle between the vectors $\hat{\mathbf{e}}_1$ and $\hat{E}_2$ when their origins coincide. These dot products show

$$\begin{aligned} \cos t \;&= f(t) + g(t)\cos\theta \\ \sin t \;&= g(t)\sin\theta \end{aligned} \quad\Rightarrow\quad \begin{aligned} g(t) =& \frac{\sin t}{\sqrt{1-\cos^2\theta}} \\ f(t) =& \cos t - \sin t\,\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right) \end{aligned} \tag{12.20}$$

The position vector describing the circle in a non-orthogonal coordinate system can therefore be represented

$$\vec{r} = \vec{r}(t) = \left(\cos t - \sin t \frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)\;\hat{\mathbf{e}}_1 + \frac{\sin t}{\sqrt{1-\cos^2\theta}}\,\hat{E}_2 \tag{12.21}$$

where $\hat{\mathbf{e}}_1$ and $\hat{E}_2$ are **non-orthogonal unit vectors**, $\vec{e}_1 \cdot \hat{E}_2 = \cos\theta$ and $0 \le t \le 2\pi$.

Parametric equations for a great circle

Consider two points on the surface of the **unit sphere** with surface coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2). These surface points can also be represented in Cartesian coordinates by vectors from the origin to each of the surface points on the sphere. These vectors are written

$$\begin{aligned} \vec{r}_1 &= \sin\theta_1 \cos\phi_1\;\hat{\mathbf{e}}_1 + \sin\theta_1 \sin\phi_1\;\hat{\mathbf{e}}_2 + \cos\theta_1\;\hat{\mathbf{e}}_3 \\ \vec{r}_2 &= \sin\theta_2 \cos\phi_2\;\hat{\mathbf{e}}_1 + \sin\theta_2 \sin\phi_2\;\hat{\mathbf{e}}_2 + \cos\theta_2\;\hat{\mathbf{e}}_3 \end{aligned}$$

Since the sphere is a unit sphere the above vectors are unit vectors and one can verify that $\vec{r}_1 \cdot \vec{r}_1 = \vec{r}_2 \cdot \vec{r}_2 = 1$.

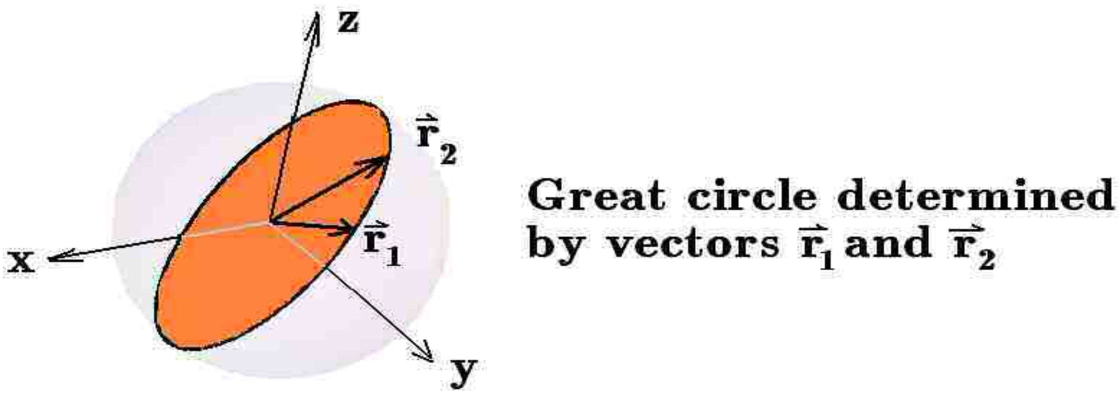


Figure 12-5. Vectors $\vec{r}_1$, $\vec{r}_2$ from origin to surface of sphere.

Note the following

(i) Two known points on the surface of the sphere together with the origin of the sphere gives three points which determine a plane. The intersection of this plane with the sphere produces a great circle.

(ii) The cross product $\vec{r}_1 \times \vec{r}_2 = \vec{N}$ is a normal vector to this plane and can be used to determine the equation of the plane.

(iii) The great circle of the sphere lies in the plane of the vectors $\vec{r}_1$ and $\vec{r}_2$.

(iv) In the plane of the great circle, the vectors $\vec{r}_1$ and $\vec{r}_2$ can be used as a set of non-orthogonal base vectors to represent the vector equation for the great circle as

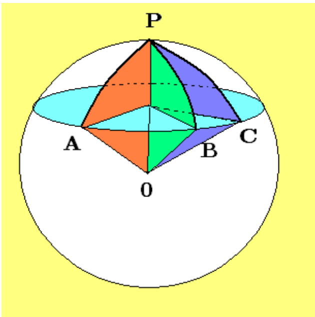
$$\vec{r} = \vec{r}(t) = \left(\cos t - \sin t \frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)\vec{r}_1 + \frac{\sin t}{\sqrt{1-\cos^2\theta}}\vec{r}_2 \tag{12.22}$$

Note that this equation has the same form as equation (12.21) with $\hat{\mathbf{e}}_1$ replaced by $\vec{r}_1$ and $\hat{E}_2$ replaced by $\vec{r}_2$. The equation (12.22) produces the parametric equations

$$\begin{aligned} x = x(t) =& f(t)\sin\theta_1\cos\phi_1 + g(t)\sin\theta_2\cos\phi_2 \\ y = y(t) =& f(t)\sin\theta_1\sin\phi_1 + g(t)\sin\theta_2\sin\phi_2 \\ z = z(t) =& f(t)\cos\theta_1 + g(t)\cos\theta_2 \end{aligned} \tag{12.23}$$

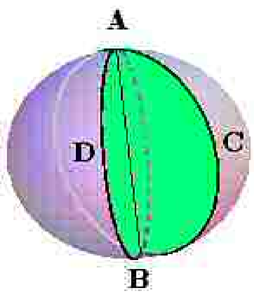
where $f(t) = \cos t - \sin t\left(\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)$ and $g(t) = \dfrac{\sin t}{\sqrt{1-\cos^2\theta}}$ are functions which provide the appropriate scaling associate with the non-orthogonal coordinate system. Note the special case where $\theta = \frac{\pi}{2}$, then the equations (12.22) becomes equations (12.18) and $\vec{r}_1 \cdot \vec{r}_2 = 0$ indicating the vectors are perpendicular.

Polar point



Every plane and sphere intersection produces a circle. This circle has a polar point P associated with it such that all the polar angles and arcs $\widehat{\text{PA}} = \widehat{\text{PB}} = \widehat{\text{PC}}$ associated with the polar angles are equal. That is the planes $0AP$, $0BP$, $0CP$ produce great circles producing equal arcs from P to any points A, B, C on the circle of intersection and so all the polar angles are equal.

The lune



The special case where two great circles intersect, the resulting spherical surface is called a lune. The points A and B are antipodal points[1] and the surface area ADBCA defines the lune. The arc $\text{A}\widehat{\text{C}}\text{B}$ can be thought of as being able to rotate about the axis $\overline{\text{AB}}$ through the antipodal points. The dihedral angle $\theta = \angle C0D$ of the planes defining the two intersecting great circles can vary such that $0 \le \theta \le 2\pi$.

Let A_ℓ denote the area of the lune and let A_s denote the total surface area of the sphere. As θ varies the area of the lune is proportional to the size of the dihedral angle θ so one can write the proportion

$$\frac{A_\ell}{\theta} = \frac{A_s}{2\pi} \quad \Rightarrow \quad \frac{A_\ell}{\theta} = \frac{4\pi r^2}{2\pi} \quad \Rightarrow \quad A_\ell = 2\theta r^2$$

or the surface area of the lune is given by

$$A_\ell = \text{Surface area lune} = 2\theta r^2$$

In the special case the sphere is a unit sphere, then the surface area is given by $A_\ell = 2\theta$.

[1] An antipodal point is an opposite point on a circle or sphere. They lie on the ends of a line through center of circle or sphere.

Spherical triangle Consider three great circles which intersect at surface points A, B and C on the unit sphere as illustrated in the figure 12-6.

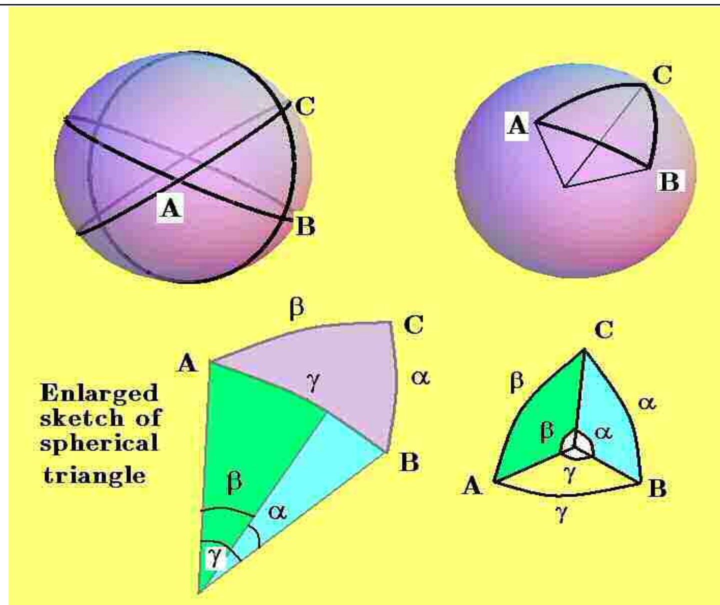
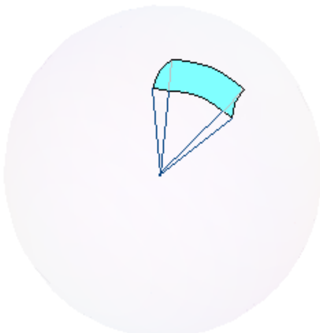


Figure 12-6. Spherical triangle ABC

Whenever the three surface points A, B and C do not lie on the same great circle, then the **intersection of three great circles** produces two spherical triangles. In all future discussions we will only be interested in the smaller spherical triangle. The points A, B, C on the surface are called the vertices of the spherical triangle ABC and the arcs $\widehat{\text{AB}} = \gamma$, $\widehat{\text{AC}} = \beta$, $\widehat{\text{CB}} = \alpha$, representing smaller arcs from great circles and define the sides of the spherical triangle. When the vertices are connected to the center of the sphere a trihedral angle is formed. Note that because the sphere is a unit sphere the arc lengths for the spherical triangle sides are equal to the face angles of the trihedral angle and are measured in radians. The angular measure, in radians, associated with the vertices A, B, C are defined by the dihedral angles associated with the trihedral angle. By convention the length of any side of a spherical triangle on a unit sphere is always less than π radians.

Spherical polygons



Whenever three or more arcs of great circles intersect to form closed convex loops, then a spherical polygon results. The arcs of the great circles are the sides of the spherical polygon. By connecting the vertices of the polygon to the center of the unit sphere one forms a polyhedral angle. The sides of the spherical polygon are equal to the face angles of the polyhedral angles.
The vertex angles of the spherical polygon are defined by the dihedral angles associated with the polyhedral angle faces. All convex spherical polygons can be broken up into connected spherical triangles and so we shall study only spherical triangles.

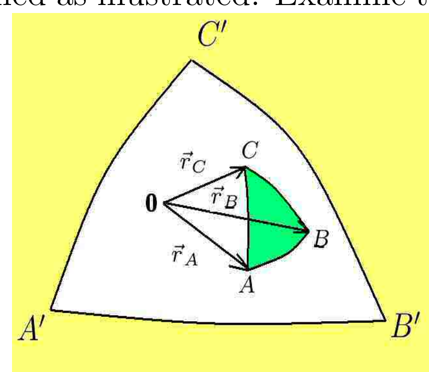
Polar spherical triangles

If $\vec{r}_A$, $\vec{r}_B$, $\vec{r}_C$ are vectors from the origin of the unit sphere to the vertices of the spherical triangle ABC, then a trihedral angle is formed as illustrated. Examine the cyclic rotation of the cross products of these vectors

$\vec{N}_1 = \vec{r}_A \times \vec{r}_B$ is vector $\perp$ to face 0AB.

$\vec{N}_2 = \vec{r}_B \times \vec{r}_C$ is vector $\perp$ to face OBC.

$\vec{N}_3 = \vec{r}_C \times \vec{r}_A$ is vector $\perp$ to face OCA.



Extending these vectors to intersect with the unit sphere, they can be used to determine points A', B', C' on the unit sphere which are vertices of a new spherical triangle $A'B'C'$ called the polar spherical triangle associated with the spherical triangle ABC. Another way of producing the polar spherical triangle is as follows.

(i) Arc $\widehat{\text{AB}}$ is part of a great circle centered at the origin of the unit sphere.

(ii) The line perpendicular to this great circle which passes through the origin intersects the sphere at two opposite points called **antipodal points.** Antipodal points lie on opposite ends of a spheres diameter.

(iii) The antipodal point lying on the same side of the plane as the third vertex C is denoted C'.

(iv) In a similar fashion one can examine the great circles through the arcs $\widehat{\text{BC}}$ and $\widehat{\text{CA}}$ and determine their poles A' and B'.

(v) The spherical triangle $A'B'C'$ is called the polar triangle associated with spherical triangle ABC.

Still another way to view spherical triangle ABC is to assume that each vertex A, B and C is a pole point associated with a great circle.

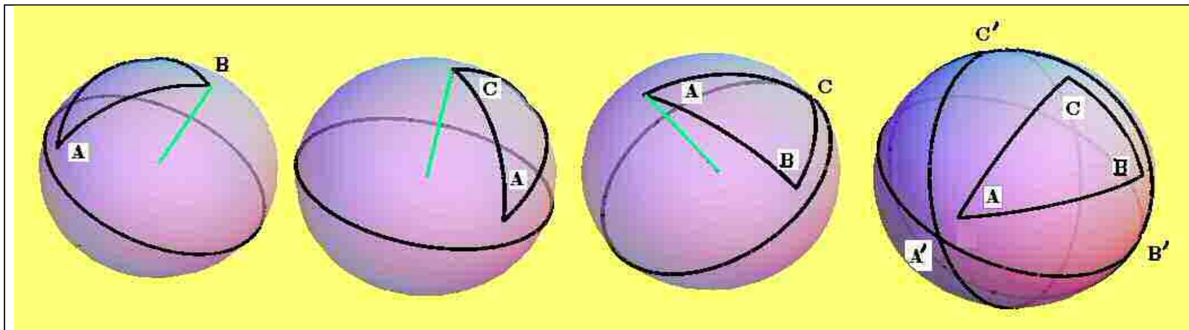
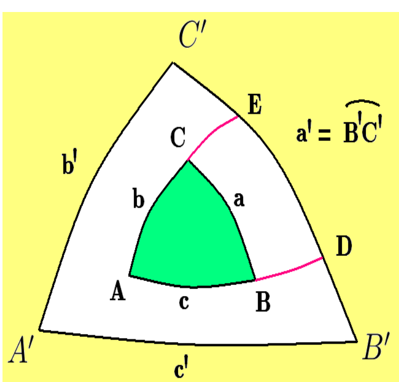


Figure 12-7. Construction of polar spherical triangle $A'B'C'$

Each vertex of the spherical triangle ABC is considered as a pole associated with a great circle. The three great circles created intersect at the points A', B', C' and the shorter geodesics of these great circles form the polar triangle $A'B'C'$.

One can show that if polar triangle $A'B'C'$ results from spherical triangle ABC, then polar triangle ABC results from spherical triangle $A'B'C'$. That is, see figure 12-7, if vertex B is the pole of $\widehat{A'C'}$ and vertex C is the pole of $\widehat{A'B'}$, then vertex A' is a quadrant distance from arc $\widehat{\mathrm{BC}}$, so vertex A' is a pole of arc $\widehat{\mathrm{BC}}$. In a similar fashion one can argue that B' is a pole of arc $\widehat{\mathrm{AC}}$ and C' is a pole of arc $\widehat{\mathrm{AB}}$.

Angle relation between polar triangles



Todhunter[2] showed that the spherical angles and sides of polar triangles are related by the equations

$$\begin{aligned} a' &= \pi - A, \quad & b' &= \pi - B, \quad & c' &= \pi - C \\ A' &= \pi - a, \quad & B' &= \pi - b, \quad & C' &= \pi - c \end{aligned} \tag{12.24}$$

with all measures in radians. The equations (12.24) give a definite relationship between polar triangle sides and angles.

---

[2] Isaac Todhunter (1820-1884) an English mathematician.

Extend the sides $\widehat{AB}$ and $\widehat{AC}$ to intersect the side $\widehat{B'C'}$ at the points D and E as illustrated above. Note that each spherical triangle is a polar triangle of the other and consequently the arcs $\widehat{B'E}$ and $\widehat{DC'}$ are both $\frac{\pi}{2}$ radians because B' is a pole of side $\widehat{ACE}$ and C' is a pole of side $\widehat{ABD}$. Therefore

$$\widehat{B'E} + \widehat{DC'} = \pi \tag{12.25}$$

radians. Examine the above polar triangles and observe that

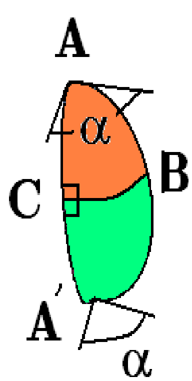
$$\widehat{B'E} = \widehat{B'D} + \widehat{DE} \quad \text{and} \quad \widehat{DC'} = \widehat{DE} + \widehat{DE'} \tag{12.26}$$

which allows one to express equation (12.25) in the form

$$\underbrace{\widehat{B'D} + \widehat{DE} + \widehat{EC'}}_{a'} + \underbrace{\widehat{DE}}_{A} = \pi \tag{12.27}$$

which demonstrates the first of Todhunters results. The other results are derived in a similar fashion.

Special case

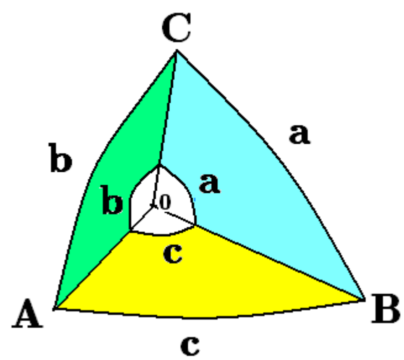


Given a right spherical triangle having minor arcs of a great circle for each side of the triangle, then one can always extend two of the sides of the triangle to have these sides meet to form a lune as illustrated on the left. This lune then is composed of two right spherical triangles $\triangle ABC$ and $A'BC$. If one is given the spherical angle α and side $\widehat{BC}$ and one is asked to find other parts of the right spherical triangle, then an ambiguity arises because the spherical angles $\angle A = \alpha$ and $\angle A' = \alpha$ are the same. Consequently, two spherical triangles can be formed with the given information. If the original spherical triangle is labeled as $\triangle ABC$ and the other spherical triangle of the lune is labeled $\triangle A'B'C'$ where $\widehat{B'C'} = \widehat{BC}$, then the following relations exits between the sides of these two spherical triangles.

$$B' = \pi - B, \quad b' = \pi - b, \quad c' = \pi - c$$

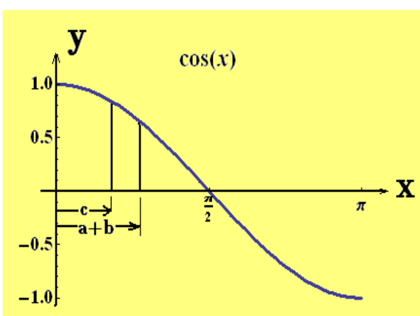
A special case of the above situation arises whenever an isosceles spherical triangle is divided into two symmetric right spherical triangles by constructing an arc from the vertex of the isosceles triangle to the base of the triangle.

Spherical triangles and trihedral angles



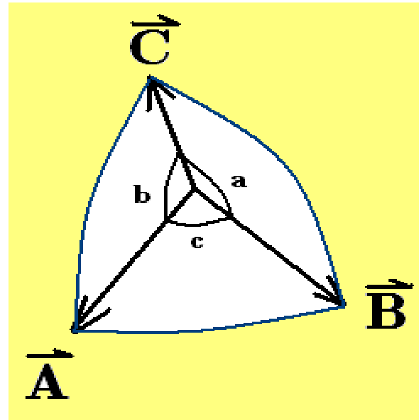
In order to have a proper spherical triangle ABC, the sides of the spherical triangle have to be the smaller arcs of great circles. By connecting the surface vertices of the spherical triangle with lines to the origin of the unit sphere, one creates the trihedral angles $a = \angle C0B,\ b = \angle C0A,\ c = \angle A0B$.

Recall that the arc length $s = \widehat{\text{AB}}$ associated with a circle of radius r is given by $s = r\theta$, where $\theta = \angle A0B$ is the central angle associated with the arc length. In the special case the radius is $r = 1$, then the arc length is $s = c = \theta$. Therefore, each of the trihedral angles associated with the spherical triangle must equal the sides of the spherical triangle so that $\widehat{\text{AB}} = c$, $\widehat{\text{BC}} = a$, $\widehat{\text{CA}} = b$. Each of these angles must be less than π radians.



One of the properties associated with the trihedral angles is that **the sum of any two face angles of a trihedral angle is always greater than the third face angle.** For example, to show $a + b > c$ one can proceed as follows. Examine the graph of a cosine function as illustrated.

Observe that for $0 \le x \le \pi$ the cosine function is continuously decreasing, therefore if $a + b > c$, then $\cos c > \cos(a + b)$. So instead of proving $a + b > c$ for trihedral angles we will prove $\cos c > \cos(a + b)$ over the interval 0 to π, because this is easier to prove.



Use the cosine expansion formula and write what we are trying to prove, namely

$$\cos c > \cos a \cos b - \sin a \sin b$$

and note if the origin of the unit sphere is labeled 0, and $\vec{A}, \vec{B}, \vec{C}$ are labeled as vectors from the origin to the surface points A, B, C, then one can employ the definitions of dot and cross products to write

$$\cos b = \frac{\vec{A}\cdot\vec{C}}{|\vec{A}||\vec{C}|}, \quad \cos c = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}, \quad \cos a = \frac{\vec{B}\cdot\vec{C}}{|\vec{B}||\vec{C}|}, \quad \sin b = \frac{|\vec{A}\times\vec{C}|}{|\vec{A}||\vec{C}|}, \quad \sin a = \frac{|\vec{B}\times\vec{C}|}{|\vec{B}||\vec{C}|}$$

and note the sphere is a unit sphere so that $|\vec{A}| = |\vec{B}| = |\vec{C}| = 1$.

Recall the vector identity from chapter 10

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \tag{12.28}$$

which can be expressed in the form

$$(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) \tag{12.29}$$

Write the identity (12.29) in the special case where one replaces $\vec{D} = \vec{B}$ and $\vec{B} = \vec{C}$ to obtain

$$(\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{C}) = (\vec{A}\cdot\vec{C})(\vec{C}\cdot\vec{B}) - (\vec{A}\times\vec{C})\cdot(\vec{C}\times\vec{B}) \tag{12.30}$$

Using the inequality

$$|a-b| \geq ||a| - |b||$$

one can show

$$\begin{aligned}|(\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{C})| =& |(\vec{A}\cdot\vec{B})(\vec{B}\cdot\vec{D}) - (\vec{A}\times\vec{C})\cdot(\vec{C}\times\vec{B})|\\ \geq& |\vec{A}\cdot\vec{B}||\vec{B}\cdot\vec{C}| - |\vec{A}\times\vec{C}||\vec{C}\times\vec{B}|\end{aligned}$$

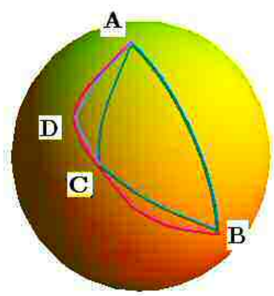
to obtain

$$\cos c \geq \cos b \cos a - \sin b \sin a = \cos(a+b)$$

the inequality holding only in the case the vectors $\vec{A}, \vec{B}, \vec{C}$ are coplaner. Similar arguments apply to all face angles of the trihedral angle and so one can say the sum of any two face angles of a trihedral angle is always greater than the third face angle.

In terms of spherical triangles one can state that –Each side of a spherical triangle is less than the sum of the other two sides.

Shortest distance between points on sphere



The shortest curve between two points on a sphere is along the minor arc of a great circle joining the points. To prove this assertion let A and B denote two points on a sphere with the arc $\widehat{\text{AB}}$ produced by a great circle passing through the points A and B. Select any other path (the red curve) on the surface of the sphere which connects A to B. If the point C is on this path, then one can construct the great circle arcs $\widehat{\text{AC}}$ and $\widehat{\text{CB}}$. This creates the spherical triangle ABC and so one can state that

$$\widehat{\text{AB}} < \widehat{\text{AC}} + \widehat{\text{CB}}$$

since each side of a spherical triangle must be less than the sum of the other two sides. If the red line coincides with the sides of the triangle the assertion is proven. If the red curve does not coincide with the sides of the spherical triangle, then select a point D on the red curve which is not on the arcs $\widehat{\text{AC}}$ and $\widehat{\text{CB}}$. Create the great circle arcs $\widehat{\text{AD}}$ and $\widehat{\text{DC}}$ to form another spherical triangle. One can then write $\widehat{\text{AC}} < \widehat{\text{AD}} + \widehat{\text{DC}}$. Therefore

$$\widehat{\text{AB}} < \widehat{\text{AD}} + \widehat{\text{DC}} + \widehat{\text{CB}}$$

Continuing in this fashion an inequality can be constructed showing $\widehat{\text{AB}}$ is the shortest curve between the given points.

Sum of face angles of convex polyhedral angle

Consider unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ situated at the origin of a three dimensional coordinate system. Let triangle $\triangle A0B$ with angles $\alpha_0, \beta_0, \gamma_0$ lie in the plane of the $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ vectors and let the origin 0 move in the $\hat{\mathbf{e}}_3$ direction a distance $h > 0$ to point C to form triangle $\triangle ABC$ in three dimensional space with angles α, β, γ.

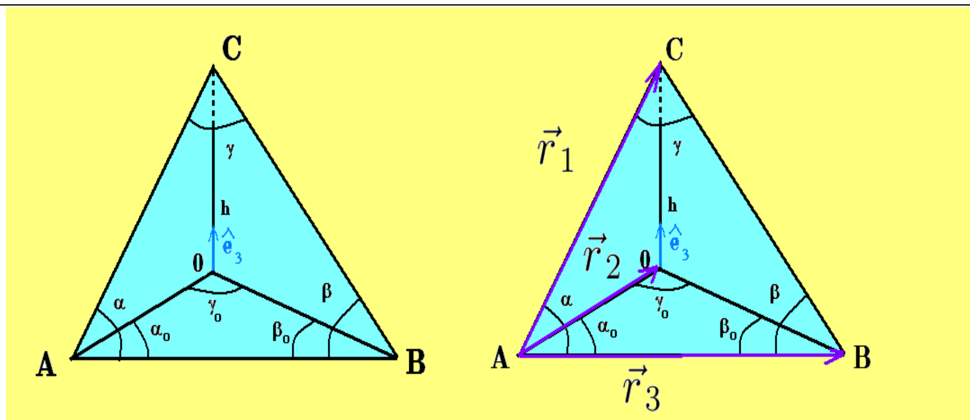


Figure 12-8. Two dimensional triangle changing to three dimensional triangle

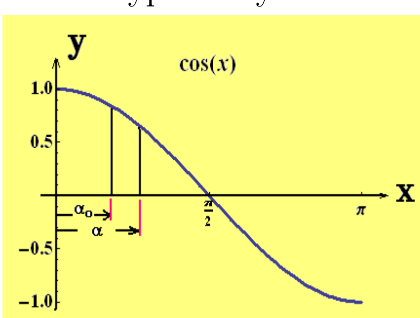
Show $\gamma < \gamma_0$ for all values of $h > 0$. To show this, construct the vectors $\vec{r}_2$ and $\vec{r}_3$ in the $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ plane together with the vector $\vec{r}_1 = \vec{r}_2 + h\,\hat{\mathbf{e}}_3$ in three dimensional space. Calculate the dot products $\vec{r}_2 \cdot \vec{r}_3$ and $\vec{r}_1 \cdot \vec{r}_3$ and show

$$\vec{r}_2 \cdot \vec{r}_3 = |\vec{r}_2||\vec{r}_3|\cos\alpha_0 \quad \text{and} \quad \vec{r}_1 \cdot \vec{r}_3 = |\vec{r}_1||\vec{r}_3|\cos\alpha$$

Note the $\vec{r}_1 \cdot \vec{r}_3 = (\vec{r}_2 + h\,\hat{\mathbf{e}}_3) \cdot \vec{r}_3 = \vec{r}_2 \cdot \vec{r}_3$ so that one can conclude

$$|\vec{r}_2||\vec{r}_3|\cos\alpha_0 = |\vec{r}_1||\vec{r}_3|\cos\alpha \quad \text{or} \quad \cos\alpha = \frac{|\vec{r}_2|}{|\vec{r}_1|}\cos\alpha_0 \tag{12.31}$$

If the vector $\vec{r}_2$ is fixed and $|\vec{r}_1|$ increases, then $\cos\alpha$ decreases and α increases. A similar type analysis shows that as h increases, then the angle β increases.



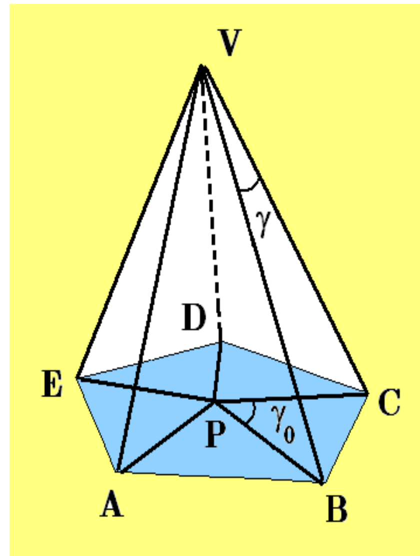
Therefore, for $h > 0$ we have $\alpha > \alpha_0$ and $\beta > \beta_0$. We know that in every planar triangle the sum of the interior angles is π radians and so we can write

$$\alpha_0 + \beta_0 + \gamma_0 = \pi \text{ and } \alpha + \beta + \gamma = \pi$$

If $\alpha + \beta > \alpha_0 + \beta_0 = \pi - \gamma_0$, then $\alpha + \beta + \gamma > \alpha_0 + \beta_0 + \gamma$.

This implies $\alpha_0 + \beta_0 + \gamma < \pi$ or $\pi - \gamma_0 + \gamma < \pi \Rightarrow \gamma < \gamma_0$

We can now use the above result to show that **the sum of the face angles of any convex polyhedral angle is less than 2π radians.** Let P denote the center of the base polygon and construct lines from P to each vertex of the polygon as illustrated in the accompanying figure with only five vertices.



If the polygonal base has n-sides, let the central angles surrounding point P be denoted $\gamma_{10}, \gamma_{20}, \ldots, \gamma_{n0}$ and label the face angles of the convex polyhedral angle as $\gamma_1, \gamma_2, \ldots, \gamma_n$. We have previously demonstrated that the following inequalities must hold for each triangular face

$$\gamma_1 < \gamma_{10}, \ \gamma_2 < \gamma_{20}, \ \ldots \ , \gamma_n < \gamma_{n0}$$

so that by addition

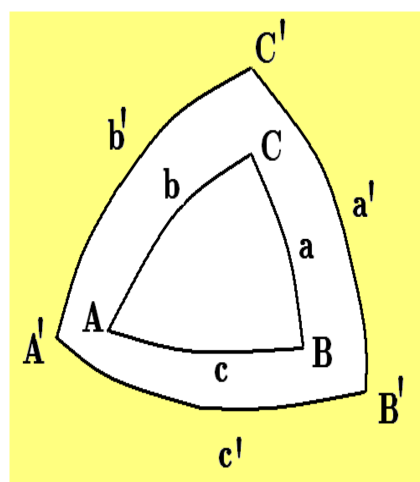
$$\gamma_1 + \gamma_2 + \cdots + \gamma_n < \gamma_{10} + \gamma_{20} + \cdots + \gamma_{n0} = 2\pi$$

since the sum of the angles surrounding point P equals 2π radians.

This demonstrates the fact that **the sum of the face angles of any convex polyhedral angle is less than 2π radians.** An equivalent statement is that the sum of the sides of a spherical polygon is less than 2π radians.

Sum of interior angles of spherical triangle

The two previous results can now be used to show that the sum of the angles of a spherical triangle must be greater than π radians and less than 3π radians. For the spherical triangle illustrated one must show



$$\pi < \angle A + \angle B + \angle C < 3\pi \qquad (12.32)$$

This can be accomplished by first constructing the associated polar triangle and recalling that

$$\angle A + \widehat{B'C'} = \pi, \qquad \angle B + \widehat{A'C'} = \pi, \qquad \angle C + \widehat{A'B'} = \pi$$

so that by addition

$$\angle A + \angle B + \angle C + \widehat{B'C'} + \widehat{A'C'} + \widehat{A'B'} = 3\pi \tag{12.33}$$

Here the arcs $\widehat{B'C'}$, $\widehat{A'C'}$, $\widehat{A'B'}$ are the same as the face angles associated with the spherical triangle $\triangle A'B'C'$ trihedral angle. We have previously shown the sum of these face angles must be less than 2π so one can write

$$\widehat{B'C'} + \widehat{A'C'} + \widehat{A'B'} < 2\pi \tag{12.34}$$

also one can write the inequality

$$\widehat{B'C'} + \widehat{A'C'} + \widehat{A'B'} > 0 \tag{12.35}$$

Recall that when unequals are subtracted from equals, then the results are unequal in the reverse order. Subtraction of (12.34) from (12.33) shows that

$$\angle A + \angle B + \angle C > \pi$$

and subtraction of (12.35) from (12.33) shows that

$$\angle A + \angle B + \angle C < 3\pi \tag{12.36}$$

Therefore one can write

$$\pi < \angle A + \angle B + \angle C < 3\pi \tag{12.37}$$

which shows that the summation of the interior angles of a spherical triangle must lie somewhere between π and 3π radians.

Spherical excess

The quantity ϵ defined by

$$\epsilon = A + B + C - \pi \tag{12.38}$$

is called **the spherical excess** and represents the difference between a summation of the interior angles of a spherical triangle and the summation of the interior angles of a plane triangle. The spherical excess will be used to calculate the area of a spherical triangle.

Surface area of spherical triangle

To demonstrate that the surface area of spherical triangle $\triangle ABC$ is equal to the spherical excess one can define

$$S_{ABC} = \text{Surface area of spherical } \triangle ABC$$

and then show

$$S_{ABC} = (A + B + C - \pi) = \epsilon$$

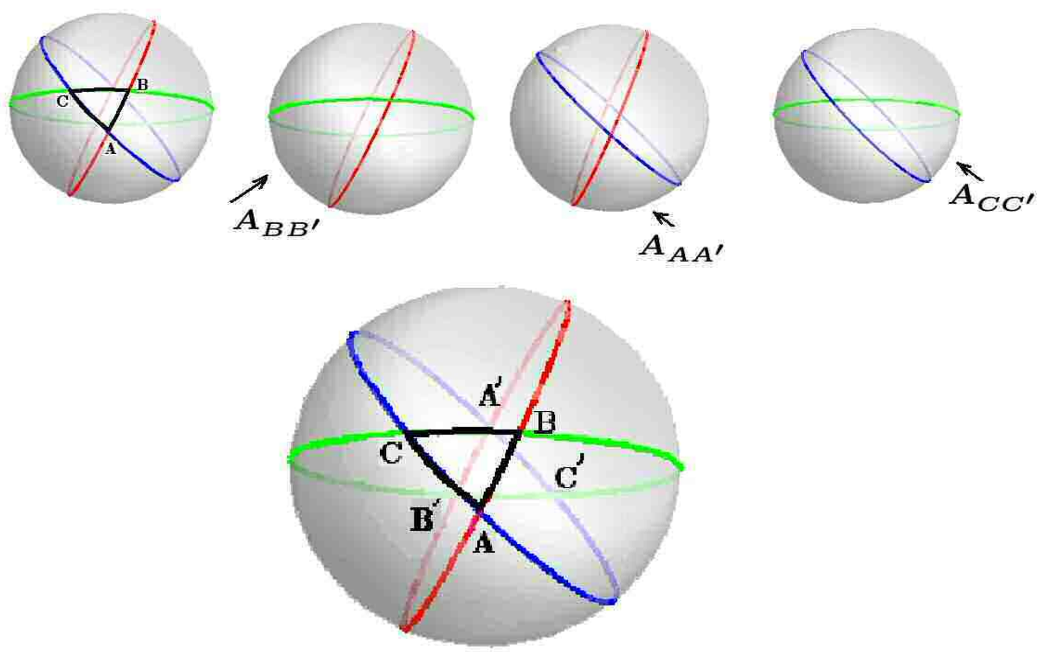


Figure 12-9. Diagram for determining surface area of spherical triangle $\triangle ABC$

Observe in figure 12-9 the total surface area of the sphere is $S_{total} = 4\pi r^2$. In the figure 12-9 think of the great circle through the points B and C (green great circle) as an equator which divides the sphere in half. The area of the lower half of the sphere is then given by $S_{hemi} = 2\pi r^2$. Let the lower hemisphere be divided up into many spherical triangles whose surface area we know will add up to $2\pi r^2$. For example,

$$\begin{aligned} S_{CAB}+S_{CAB'} =&\text{lune with vertex B, where } B,B' \text{ are antipodal points}\\ \text{with area of lune} \quad A_{BB'} =&2Br^2 \end{aligned} \tag{12.39}$$

$$\begin{aligned} S_{ABC}+S_{B'C'A} =&\text{lune with vertex A, where } A,A' \text{ are antipodal points}\\ \text{with area of lune} \quad A_{AA'} =&2Ar^2 \end{aligned} \tag{12.40}$$

$$\begin{aligned} S_{ABC}+S_{BAC'} =&\text{lune with vertex C, where } C,C' \text{ are antipodal points}\\ \text{with area of lune} \quad A_{CC'} =&2Cr^2 \end{aligned} \tag{12.41}$$

where A,B,C are the dihedral angles which define the interior angles of the spherical triangle.

An examination of figure 12-9 shows $S_{ABC}=S_{A'B'C'}$ by symmetry so that addition of the equations (12.39), (12.40), (12.41) gives the result

$$3S_{ABC}+S_{CAB'}+S_{B'C'A}+S_{BAC'}=2(A+B+C)r^2 \tag{12.42}$$

which can be written as

$$2S_{ABC}+\underbrace{S_{A'B'C'}+S_{CAB'}+S_{B'C'A}+S_{BAC'}}_{2\pi r^2}=2(A+B+C)r^2 \tag{12.43}$$

The surface areas $S_{A'B'C'}+S_{CAB'}+S_{B'C'A}+S_{BAC'}=2\pi r^2$ because this surface area covers the lower hemisphere. Consequently the equation (12.43) can be written as

$$2S_{ABC}+2\pi r^2=2(A+B+C)r^2 \quad \text{or} \quad S_{ABC}=(A+B+C-\pi)r^2 \tag{12.44}$$

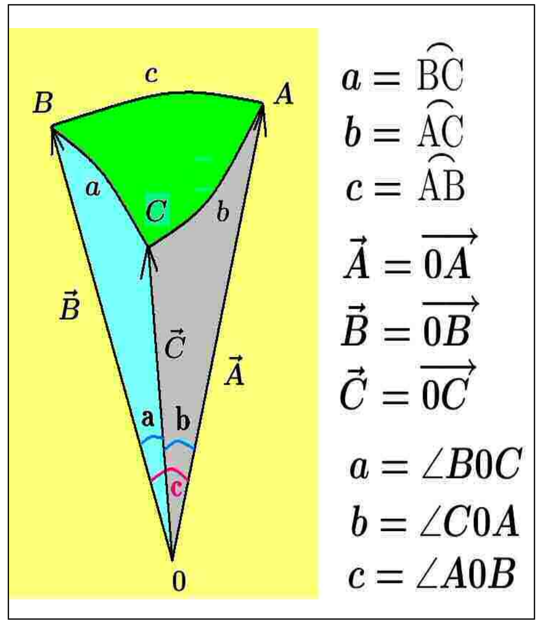
This demonstrates that the surface area of a spherical triangle is equal to the spherical excess times the radius squared. In the **special case the sphere is a unit sphere** the equation (12.44) becomes

$$S_{ABC}=\text{Surface area of spherical } \triangle ABC=(A+B+C-\pi)=\epsilon \tag{12.45}$$

which reads that the surface area is equal to the spherical excess in radians.

Introduction to spherical trigonometry

Spherical trigonometry examines the relations between the angles and sides of a spherical triangle in relation to various trigonometric functions. Let spherical triangle $\triangle ABC$ lie on a unit sphere with $\vec{A}, \vec{B}, \vec{C}$ unit vectors from the origin of the sphere to the respective vertices as illustrated in the accompanying figure. One can make use of the definitions for the vector dot and cross products to show



$$\begin{aligned} \vec{A} \cdot \vec{B} &= \cos c \qquad |\vec{A} \times \vec{B}| = \sin c \\ \vec{B} \cdot \vec{C} &= \cos a \qquad |\vec{B} \times \vec{C}| = \sin a \\ \vec{A} \cdot \vec{C} &= \cos b \qquad |\vec{C} \times \vec{A}| = \sin b \end{aligned} \tag{12.46}$$

since the vectors $\vec{A}, \vec{B}, \vec{C}$ are unit vectors.

Note also that the arc lengths on the great circles are given by $s = r\theta$ and in the special case $r = 1$, then the central angle also equals the arc length in radians. Consequently, one can write $a = \overset{\frown}{BC}$, $b = \overset{\frown}{AC}$, $c = \overset{\frown}{AB}$.

The cosine identities for spherical trigonometry

The equations for the planes defining the dihedral angles, which are used to define the spherical triangle angles, can be calculated using the normal vectors to these planes. The normal vectors can be calculated using the vector cross product notation.

$$\text{normal vector to plane A0C} \quad \vec{N}_1 = \vec{C} \times \vec{A}$$

$$\text{normal vector to plane B0C} \quad \vec{N}_2 = \vec{C} \times \vec{B}$$

so that the dot product $\vec{N}_1 \cdot \vec{N}_2$ defines the dihedral angle C since

$$\vec{N}_1 \cdot \vec{N}_2 = |\vec{N}_1||\vec{N}_2| \cos C = \left(|\vec{C}||\vec{A}| \sin b\right) \left(|\vec{B}||\vec{C}| \sin a\right) \cos C = \sin a \sin b \cos C \tag{12.47}$$

Using the vector identity (10.39) one can verify

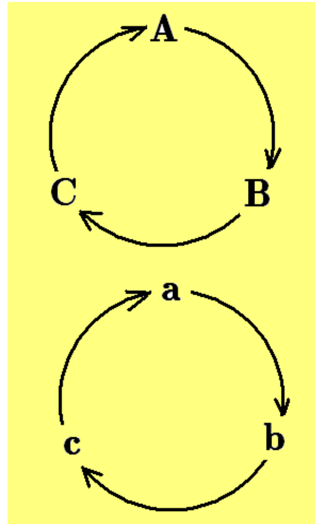
$$\vec{N}_1 \cdot \vec{N}_2 = (\vec{C} \times \vec{B}) \cdot (\vec{C} \times \vec{A}) = (\vec{C} \cdot \vec{C})(\vec{B} \cdot \vec{A}) - (\vec{C} \cdot \vec{A})(\vec{B} \cdot \vec{C}) = \cos c - \cos b \cos a \tag{12.48}$$

Things equal to the same thing are equal to each other so one can equate equations (12.47) and (12.48) to obtain

$$\cos c - \cos b \cos a = \sin a \sin b \cos C \tag{12.49}$$

which relates spherical triangle angle C with the spherical triangle sides a, b, c.

Note that equation (12.49) has the cyclic property of rotating the symbols in the equation to obtain a new equation. This is equivalent to saying "What has been done once can be done again, but with a different set of symbols." Performing a cyclic rotation of the symbols in equation (12.49) one obtains the equations



$$\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C\end{aligned} \tag{12.50}$$

These equations are identities known as the **cosine rules for spherical trigonometry**. They are used to calculate the spherical triangle angles if the sides of the triangle are known. All other trigonometric relations for spherical triangles can be derived from the identities (12.50). The identities in equations (12.50) also apply to the polar triangle $\triangle A'B'C'$ associated with the spherical triangle $\triangle ABC$ so that one can write the equations (12.50) using primed variables. The new equations in the primed variables can then be rewritten using the polar triangle angle relations from the equations (12.24) to obtain another cosine rule for spherical triangles. For example, write the first equation in (12.50) in terms of the polar angle primed variables to show

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A' \tag{12.51}$$

Now use the equations (12.24) to express equation (12.51) in the form

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a)$$

which simplifies to $-\cos A = \cos B \cos C - \sin B \sin C \cos a$ or

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \tag{12.52}$$

Applying a cyclic rotation of symbols to the variables in equation (12.52) one finds

$$\begin{aligned}\cos A &= -\cos B \cos C + \sin B \sin C \cos a \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c\end{aligned} \tag{12.53}$$

giving another set of cosine rules which are used to obtain the spherical triangle sides if the angles are known.

The sine rule for spherical trigonometry

By definition

$$|\vec{N}_1 \times \vec{N}_2| = |(\vec{C} \times \vec{A}) \times (\vec{C} \times \vec{B})| = |\vec{C} \times \vec{A}||\vec{C} \times \vec{B}| \sin C = \sin b \sin a \sin C \tag{12.54}$$

One can employ the vector identities 10.35 to 10.40 to show

$$|(\vec{C} \times \vec{A}) \times (\vec{C} \times \vec{B})| = |\vec{C}[\vec{B} \cdot (\vec{C} \times \vec{A})] - \vec{B}[\vec{C} \cdot (\vec{C} \times \vec{A})| = |\vec{A} \cdot (\vec{B} \times \vec{C})| \tag{12.55}$$

where we have used the cyclic property of the scalar triple product to calculate the dot product $\vec{C} \cdot (\vec{C} \times \vec{A}) = \vec{A} \cdot (\vec{C} \times \vec{C}) = 0$ and also the fact that $|\vec{C}| = 1$ because it is a unit vector. Equating the equations (12.54) and (12.55) one finds

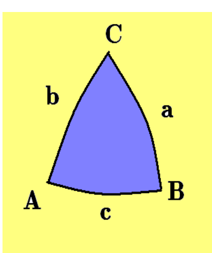
$$\sin C = \frac{|\vec{A} \cdot (\vec{B} \times \vec{C})|}{\sin a \sin b} \quad \Rightarrow \quad \frac{\sin C}{\sin c} = \frac{|\vec{A} \cdot (\vec{B} \times \vec{C})|}{\sin a \sin b \sin c} \tag{12.56}$$

The previous arguments can also be applied to the spherical angles A and B, which is equivalent to performing a cyclic rotation applied to the symbols in equation (12.56) to obtain the sine rule for spherical triangles

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{|\vec{A} \cdot (\vec{B} \times \vec{C})|}{\sin a \sin b \sin c} \tag{12.57}$$

where we have applied the cyclic properties of the triple scalar product to obtain the above result.

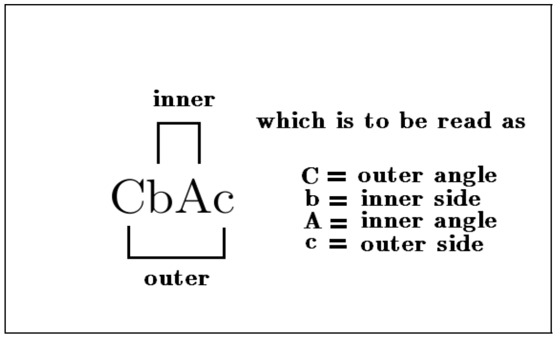
Cotangent Rule for spherical trigonometry



The cotangent formulas, sometimes called the four-part formulas, are based upon one knowing four consecutive parts of a spherical triangle. If one moves counterclockwise about a spherical triangle one can write the appearance of angles and sides in a cyclic fashion as

$$AcBaCbAcBaCbAcBaCb\ldots \tag{12.58}$$

Select **any four consecutive terms** from the pattern given by equation (12.58) for example, say one selects CbAc. The four consecutive symbols selected have **inner terms** and **outer terms** as illustrated.



The cotangent rule for writing out the cotangent formulas is given by[3]

$$\cos\begin{pmatrix} inner \\ side \end{pmatrix} \cos\begin{pmatrix} inner \\ angle \end{pmatrix} = \cot\begin{pmatrix} outer \\ side \end{pmatrix} \sin\begin{pmatrix} inner \\ side \end{pmatrix} - \cot\begin{pmatrix} outer \\ angle \end{pmatrix} \sin\begin{pmatrix} inner \\ angle \end{pmatrix} \tag{12.59}$$

This produces the cotangent four-part formulas

$$\begin{aligned}
&CbAc \qquad \cos b \cos A = \cot c \sin b - \cot C \sin A\\
&bAcB \qquad \cos c \cos A = \cot b \sin c - \cot B \sin A\\
&AcBa \qquad \cos c \cos B = \cot a \sin c - \cot A \sin B\\
&cBaC \qquad \cos a \cos B = \cot c \sin a - \cot C \sin B\\
&BaCb \qquad \cos a \cos C = \cot b \sin a - \cot B \sin C\\
&aCbA \qquad \cos b \cos C = \cot a \sin b - \cot A \sin C
\end{aligned} \tag{12.60}$$

The $bAcB$ formula from equations (12.60) is derived as follows. Start with the cosine identities from equations (12.53) and the sine identities from equations (12.57) and write

$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$

and substitute for $\cos a$ to show

$$\cos b = \cos c\,(\cos b \cos c + \sin b \sin c \cos A) + \sin a \frac{\cos B}{\sin B}[\sin c \sin B]$$

and then simplify this equation to the form

$$\cos b \sin^2 c = \sin b \sin c \cos c \cos A + \sin a \cot B[\sin b \sin C]$$

[3] From Issac Todhunter (1820-1884), text on *Spherical Trigonometry*, available from the internet.

Divide both sides by $\sin b \sin c$ and simplify to obtain the desired result

$$\cot b \sin c = \cos c \cos A + \cot B \sin A$$

The other equations are derived in a similar manner.

Half-angle formula

In solving for the angles and sides of a spherical triangle associated with a spherical triangle, the following half-angle formulas are often used.

Sides known

If the sides a, b, c of a spherical triangle are known, then one can define $s = \dfrac{a+b+c}{2}$ as the semiperimeter of the spherical triangle and then show that

$$\begin{aligned}
\sin^2\left(\frac{A}{2}\right) =& \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\\
\cos^2\left(\frac{A}{2}\right) =& \frac{\sin s \sin(s-a)}{\sin b \sin c}\\
\tan^2\left(\frac{A}{2}\right) =& \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}
\end{aligned} \tag{12.61}$$

The above results can be derived from the trigonometric relations

$$\begin{aligned}
\cos^2 A + \sin^2 A =& 1 \qquad & 2\sin^2 A =& 1 - \cos 2A\\
\cos^2 A - \sin^2 A =& \cos 2A \qquad & 2\cos^2 A =& 1 + \cos 2A
\end{aligned} \tag{12.62}$$

The equations on the right-hand side of (12.62) are obtained by subtraction and addition of the equations on the left-hand side of (12.62). In the equations on the right-hand side of (12.62) replace A by $\frac{A}{2}$ and then substitute for $\cos A$ from the equations (12.50) to show

$$\begin{aligned}
2\sin^2\left(\frac{A}{2}\right) =& 1 - \left[\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right] \qquad & 2\cos^2\left(\frac{A}{2}\right) =& 1 + \left[\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right]\\
=& \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c} \qquad & =& \frac{\cos a - \cos b \cos c + \sin b \sin c}{\sin b \sin c}\\
=& \frac{\cos(b-c) - \cos a}{\sin b \sin c} \qquad & =& \frac{\cos a - \cos(b+c)}{\sin b \sin c}
\end{aligned} \tag{12.63}$$

Using the cosine formula

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha \sin\beta \tag{12.64}$$

together with the relations

$$\begin{aligned} 2\alpha &= a + b - c \\ 2\beta &= a - b + c \\ s - b &= \frac{a + c - b}{2} = \beta \\ s - c &= \frac{a + b - c}{2} = \alpha \end{aligned}$$

the left-hand side of equation (12.63) becomes

$$2\sin^2\left(\frac{A}{2}\right) = 2\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \tag{12.65}$$

Using the same formula for the difference of two cosine functions together with the relations

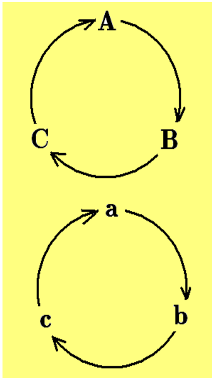
$$\begin{aligned} 2\alpha &= a + b + c \\ 2\beta &= b + c - a \\ s - a &= \frac{b + c - a}{2} \\ s &= \frac{a + b + c}{2} \end{aligned} \tag{12.66}$$

the right-hand side of equation (12.63) becomes

$$2\cos^2\left(\frac{A}{2}\right) = 2\frac{\sin s \sin(s-a)}{\sin b \sin c} \tag{12.67}$$

The tangent function is the sine function divided by the cosine function and so the equations (12.65) and (12.67) can be used to calculate a result for $\tan^2\left(\frac{A}{2}\right)$. These half-angle formulas can be summarized as

$$\begin{aligned} \sin^2\left(\frac{A}{2}\right) &= \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \\ \cos^2\left(\frac{A}{2}\right) &= \frac{\sin s \sin(s-a)}{\sin b \sin c} \\ \tan^2\left(\frac{A}{2}\right) &= \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)} \end{aligned} \tag{12.68}$$



A cyclic rotation of symbols will produce similar equations for

$$\sin^2\left(\frac{B}{2}\right),\ \cos^2\left(\frac{B}{2}\right),\ \tan^2\left(\frac{B}{2}\right)$$
$$\sin^2\left(\frac{C}{2}\right),\ \cos^2\left(\frac{C}{2}\right),\ \tan^2\left(\frac{C}{2}\right)$$

Angles known

If the angles A, B, C of the spherical triangle are known, then one can define $S = \dfrac{A+B+C}{2}$ and proceed as before to derive the half-angle relations

$$\begin{aligned} \sin^2\left(\frac{a}{2}\right) =& \frac{-\cos(S)\cos(S-A)}{\sin B \sin C}, \quad \cos^2\left(\frac{a}{2}\right) = \frac{\cos(S-B)\cos(S-C)}{\sin B \sin C} \\ \tan^2\left(\frac{a}{2}\right) =& \frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)} \end{aligned} \tag{12.69}$$

For example, in the relation $2\sin^2\left(\frac{a}{2}\right) = 1 - \cos a$ substitute from equation (12.53) the relation

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

and show

$$\begin{aligned} 2\sin^2\left(\frac{a}{2}\right) =& \frac{\sin B \sin C - \cos A - \cos B \cos C}{\sin B \sin C} \\ =& \frac{-\cos(B+C) - \cos A}{\sin B \sin C} \end{aligned} \tag{12.70}$$

Using the formula for the sum of two cosine terms $\cos(\alpha-\beta) + \cos(\alpha+\beta) = 2\cos\alpha\cos\beta$ with $\alpha+\beta = B+C$ and $\alpha-\beta = A$ one can verify $\alpha = S$ and $\beta = S-A$ so that equation (12.70) can be expressed in the form

$$2\sin^2\left(\frac{a}{2}\right) = -2\frac{\cos S \cos(S-A)}{\sin B \sin C}$$

which simplifies to the first equation in (12.69). The remaining equations of (12.69) are derived in a similar fashion.

A cyclic rotation of the symbols in the equations (12.69) above will produce equations for

$$\sin^2\left(\frac{b}{2}\right),\ \cos^2\left(\frac{b}{2}\right),\ \tan^2\left(\frac{b}{2}\right)$$
$$\sin^2\left(\frac{c}{2}\right),\ \cos^2\left(\frac{c}{2}\right),\ \tan^2\left(\frac{c}{2}\right)$$

Special right spherical triangles

In a spherical triangle there can be one, two or three right angles. In the accompanying figure the triangle $\triangle ABC$ has three right angles and is called a trirectangular spherical triangle. The spherical triangle $\triangle DBC$ has two right angles and is called a birectangular spherical triangle. If the vertex C moves off the z-axis, there results a spherical triangle with only one right angle which is called a right spherical triangle.

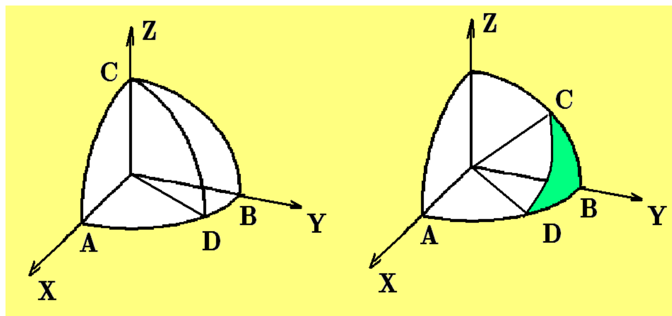
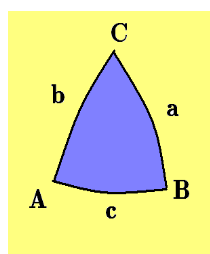


Figure 12-10. Right spherical triangles

One can sketch tangent lines to the spherical triangle sides to better illustrate the dihedral angles which define the angles of the spherical triangle.

A spherical triangle has six parts, $AcBaCb$ consisting of angles and sides. The spherical triangle can be completely solved if any three of the six parts are known. A right spherical triangle needs only two known parts, in addition to the right angle, to completely determine the remaining parts.



In the special case angle C is a right angle in any of the equations (12.50), (12.53), (12.57), then one can employ trigonometry and algebra to simplify these equations to the following ten equations which are used to determine the remaining parts of a right spherical triangle when two parts and the right angle are known.

$$\begin{aligned}
\cos c &= \cos a \cos b & \tan b &= \cos A \tan c \\
\sin a &= \sin A \sin c & \tan a &= \cos B \tan c \\
\sin b &= \sin B \sin c & \cos A &= \sin B \cos a \\
\tan a &= \tan A \sin b & \cos B &= \sin A \cos b \\
\tan b &= \tan B \sin a & \cos c &= \cot A \cot B
\end{aligned} \tag{12.71}$$

Napier's rules

John Napier[4] felt that it was unnecessary to memorize the set of equations (12.71) used for the solution of right spherical triangles. He devised the following method for obtaining these 10 equations.

Given a spherical **right triangle** one can label the sides with 5 labels as illustrated in the figure on the right. Select any one of these 5 labels and call it the middle label. The labels on each side of the middle label are then called the adjacent labels and the remaining labels are called the opposite labels.

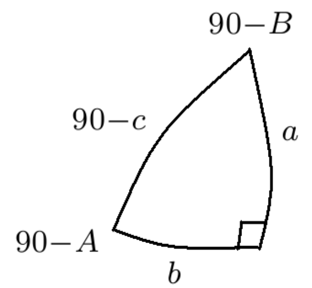


Figure 12-11. Napier's labeling

Napier's rule 1
The sine of any middle label is equal to the product of the tangents of the adjacent labels.

Napier's rule 2
The sine of any middle label is equal to the product of the cosines of the opposite labels.

For example, using rule 1 one can write

$$\sin(90-B) = \tan a \tan(90-c) \quad \Rightarrow \quad \cos B = \tan a \cot c \text{ or } \tan a = \cos B \tan c$$

$$\sin a = \tan b \cot B \quad \Rightarrow \quad \tan b = \tan B \sin a$$

Using rule 2 one can write

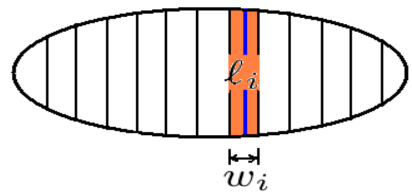
$$\sin b = \cos(90-c)\cos(90-B) \text{ or } \sin b = \sin c \sin B$$

$$\sin(90-A) = \cos(90-B)\cos a \text{ or } \cos A = \cos a \sin B$$

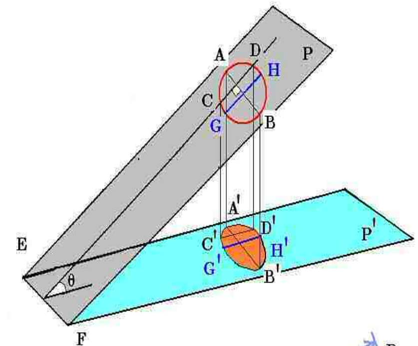
[4] John Napier (1550-1617) A Scottish mathematician well known for development of logarithms.

Projection of area

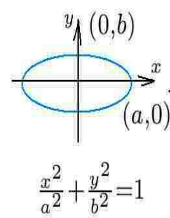
Recall that one can approximate an area by slicing it up into many rectangles and writing the approximation $A \approx \sum_{i=1}^{n} \ell_i w_i$ where ℓ_i is the length and w_i is the width of the ith rectangle. The limiting sum $A = \lim_{n\to\infty} \sum_{i=1}^{n} \ell_i w_i$ then gives the exact area.

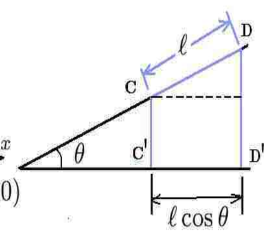


Consider the situation when two planes P, P' intersect to produce the line $\overline{\text{EF}}$ and dihedral angle θ. A figure in plane P is then projected onto the plane P'. Note that lines in plane P which are perpendicular to $\overline{\text{EF}}$ have the projection $\ell \cos\theta$ while lines parallel to $\overline{\text{EF}}$ have the same length when projected to the other plane. For example, in the figure illustrated one finds $\overline{\text{AB}} = \overline{A'B'}$ and $\overline{C'D'} = \overline{\text{CD}} \cos\theta$.



Let A denote the area of the figure in plane P which becomes the area A' when projected onto plane P'. One can then show $A' = A\cos\theta$. To prove this assertion make parallel slices where each slice is perpendicular to the line of intersection $\overline{\text{EF}}$ of the two planes. These parallel lines slice the area to produce rectangles so that one can write $A = \lim_{n\to\infty} \sum_{i=1}^{n} \ell_i w_i$ Each line ℓ_i when projected to the other plane becomes $\ell_i \cos\theta$ and the projected area is $A' = \lim_{n\to\infty} \sum_{i=1}^{n} \ell_i w_i \cos\theta = A\cos\theta$.





Example 12-1. If the area in plane P is the circle πa^2, then the distance one-half $\overline{\text{AB}}$ is the radius a of the circle. When this distance is projected onto the plane P' the projected distance also has length a. The other diameter $\overline{\text{GH}}$ is perpendicular to $\overline{\text{EF}}$ and so when projected onto the plane P' becomes $\overline{\text{GH}} \cos\theta$ or the radius $a = \frac{\overline{\text{GH}}}{2}$ becomes $\frac{\overline{\text{GH}}}{2} \cos\theta$. Define $b = \frac{\overline{\text{GH}}}{2} \cos\theta = a\cos\theta$, then the area of the circle πa^2 when projected onto plane P' becomes the ellipse with area $\pi a^2 \cos\theta$. But $a\cos\theta = b$ so the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

■

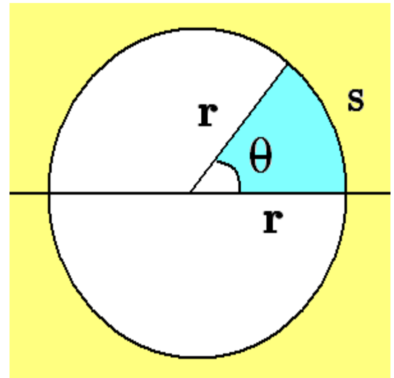
Solid Angle

In the study of plane geometry an angle θ, in radians, was defined by using the dimension-less ratio $\frac{s}{r}$ where

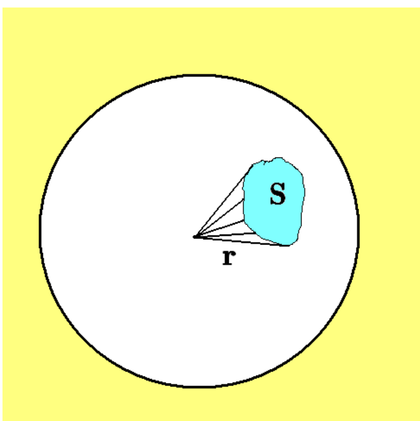
$s =$arc length on circumference of a circle

$r =$radius of circle

$\theta = \frac{s}{r}$ in radians



where one radian is defined as the angle resulting when $s = r$. Observe that as the arc length s varies from 0 to $2\pi r$, then the angle θ varies from 0 to 2π radians.



In solid geometry a solid angle Ω in units called steradians (sr), is defined using the dimension-less ratio $\frac{S}{r^2}$, where

$S =$Surface area cut by a general cone with vertex at the center of sphere.

$r =$radius of sphere

$\Omega = \frac{S}{r^2}$

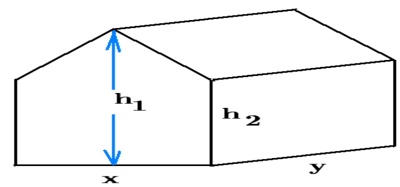
As the surface area S varies from 0 to $4\pi r^2$, the solid angle Ω varies from 0 to 4π steradians. Here one steradian is defined as the solid angle formed when the spherical surface area S equals the sphere radius squared. The solid angle Ω is used in many areas of science, physics and engineering. It arises in deriving many of the basic equations used in physics. It is used in radiation therapy for determining radiation in a given direction measured in steradians. It occurs in calculating cross sections used in radiation scattering. It arises in the study of flux of electric and magnetic fields, weather radar, light intensity and many other areas of study where angular measure in three dimensions is required.

Note that if one knows the value of Ω, then the surface area of the sphere cut by a general cone is given by $S = \Omega\, r^2$.

Exercises

► **12-1.**

Find the volume of the barn illustrated.



► **12-2.** Name three special cases of a prismatoid which have previously been investigated. Does the formula

$$V = \frac{1}{6}h\,(B_u + B_\ell + 4M)$$

hold in these special cases?

► **12-3.** Use the prismatoid formula and find the volume of a prism and a pyramid.

► **12-4.** Use the prismatoid formula and find the volume of a cone and a cylinder.

► **12-5.** Assume a prismatoid with regular polygon upper base B_u and side s_u having a lower base B_ℓ which is a similar regular polygon to the upper base with corresponding sides s_ℓ.

(a) Show the corresponding sides of the middle figure M is $\frac{1}{2}(s_u + s_\ell)$

(b) Show $\dfrac{B_u}{M} = \dfrac{s_u^2}{[\frac{1}{2}(s_u + s_\ell)]^2}$

(c) Show $\dfrac{B_\ell}{M} = \dfrac{s_\ell^2}{[\frac{1}{2}(s_u + s_\ell)]^2}$

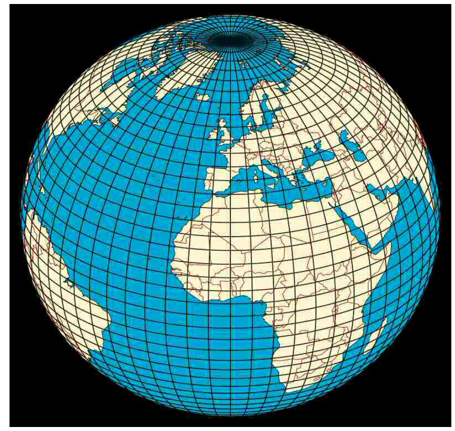
(d) Calculate $4M$ in terms of $\sqrt{B_u B_\ell}$

(e) Use the prismatoid formula to calculate the volume associated with the frustum of a pyramid.

► **12-6.** Use the prismatoid formula to find the volume of a sphere with radius r.

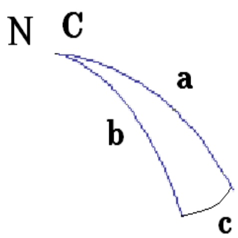
► **12-7.** Use Napier's rules to verify the 10 equations (12.71).

▶ 12-8. Spherical trigonometry and navigation



Even though the earth is not an exact sphere, one can use spherical trigonometry to find the distance between two points on the earths surface. For example, to find the distance between London $(51.507^\circ N,\ 0.128^\circ W)$ and New York City $(40.713^\circ N,\ 74.001^\circ W)$ one can use the spherical trigonometry formula

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$



where

angular difference between north pole and London is

$$a = 90^\circ - 51.507^\circ = 38.493^\circ$$

angular difference between north pole and New York City

$$b = 90^\circ - 40.713^\circ = 49.287^\circ$$

difference in longitude between cities

$$\angle C = 74.001^\circ - 0.128^\circ = 73.873^\circ$$

so that

$$\cos c = \cos\left(38.493 * \frac{\pi}{180}\right)\cos\left(49.287 * \frac{\pi}{180}\right) + \sin\left(38.493 * \frac{\pi}{180}\right)\sin\left(49.287 * \frac{\pi}{180}\right)\cos\left(73.873 * \frac{\pi}{180}\right)$$

which computes to $\cos c = 0.641568$ which implies $c = \arccos(0.641568) = 50.091^\circ$. On the earths surface the conversion factor between degrees and miles is approximately[5] $1^\circ = 69$ miles. Using this conversion factor one finds c corresponds to the distance $50.091^\circ * 69\,\frac{\text{miles}}{\circ} = 3456.279\,\text{miles}$ as the approximate distance between London and New York City.

[5] The distance of 1° latitude varies over the surface of the earth. The distance of 1° longitude depends upon what latitude you are at. The approximate conversion is

$$1^\circ \text{ longitude} = \cos(\text{degree latitude})\text{times}(\text{length of } 1^\circ \text{ at the equator})$$

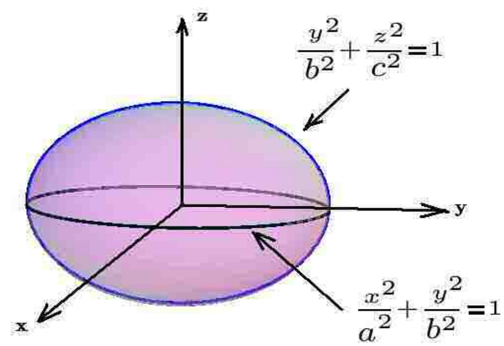
At the equator $1^\circ \approx 69.172$ miles.

The following are latitude and longitude locations for various cities. Pick two cities (different from the example above) and use spherical trigonometry to find the distance between them.

City	Latitude	Longitude
Los Angeles	34.052° N	118.244° W
Tokyo	35.690° N	139.692° E
Sydney	33.869° S	151.209° E
Calcutta	22.573° N	88.364° E

City	Latitude	Longitude
New York City	40.713° N	74.001° W
London	51.507° N	0.128° W
Moscow	55.756° N	37.617° E
Rio de Janeiro	22.907° S	43.173° W

▶ **12-9.**
Given the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Use the prismatoid formula to find the volume of an ellipsoid. Hint: See the example 12-1.

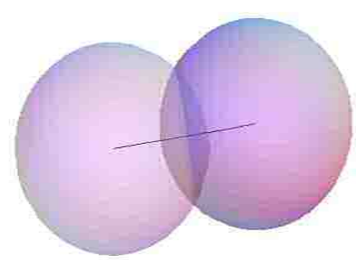


▶ **12-10.**
(a) Construct a cylinder of height $2r$ enclosing a sphere of radius r. Find volume of sphere in terms of the volume of the cylinder and then find the area of the sphere in terms of the lateral area of the cylinder
(b) Construct a cone of height $2r$ inside a cylinder with base πr^2 and height $2r$. Find volume of cone in terms of the volume of the cylinder. Find volume of cone in terms of the volume of a sphere.
(c) If a sphere is inscribed within a cube, then find the ratio of volume of sphere divided by volume of cube. Show this ratio is approximately one half.

▶ **12-11.** If V is the volume of a sphere and S denotes its surface area, then show that $S=\sqrt[3]{36\pi V^2}$

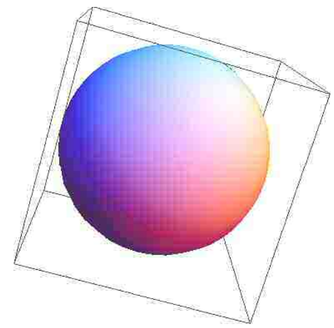
▶ **12-12.**

Prove that if two same size spheres intersect, then the plane of intersection is perpendicular to the line connecting the sphere centers.



▶ **12-13.**

A bowling ball is inserted into a watertight box where all sides of the box touch the bowling ball. The space inside the box will hold 1 gallon of water. What is the volume of the bowling ball and what is the size of the box?
Hint: 1 US Gallon = 231 cubic inches.



▶ **12-14.**
An optimization problem in geometry is to find either the maximum number of geometric objects that can be packed into a given container or to find how to pack a given number of geometric objects into as few containers as possible.

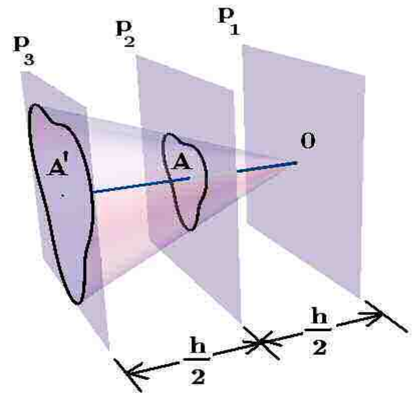
(a) Find the smallest square that will hold 6 unit circles.
(b) Find the smallest circle that will hold 6 unit squares.
(c) Find the smallest equilateral triangle that will hold 5 unit circles.
(d) Find the smallest cube that will hold 5 unit spheres.

▶ **12-15.** Find the smallest circle which will hold n unit squares where n has the values

Number of squares	Radius of smallest circle
1	
2	
3	
4	
5	

▶ **12-16.** Let r denote the radius of a sphere with a spherical triangle having angles of $60^\circ, 80^\circ$ and 85°. Find the area of the triangle.

► **12-17.**



Given three parallel planes p_1, p_2, p_3 with the distance $\frac{h}{2}$ apart as illustrated in the accompanying figure. Any line from point 0 in plane p_1 to a point in plane p_2, when extended will intersect plane p_3. If you move the point in plane p_2 around to form a simple closed curve with area A one generates a conical surface between planes p_1 and p_2. By extending all lines from 0 to plane p_2 onto the plane p_3 a dilatation is performed creating a simple closed curve in plane p_3 with area A'. The areas A and A' are similar. You can imagine the areas A and A' to be similar shaped figures such as circles, triangles, squares, etc.

(a) Show the plane areas satisfy $\dfrac{A}{A'} = \dfrac{1}{4}$

(b) Show the large general cone has volume given by $V = \frac{1}{3}A'h$

(c) Show the volume of the general frustum between planes p_2 and p_3 is given by $V_{frustum} = \frac{7}{24}A'h$

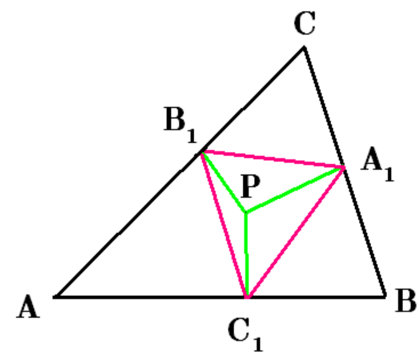
(d) Check the above results in the special case A and A' are area of circles.

(e) Check the above results in the special case A and A' are area of squares.

Geometry
Chapter 13
Additional Topics

One simply geometric concept that can be generalized to create a more complicated topic is the following.

Pedal triangle



Select a point P inside or outside of a given triangle $\triangle ABC$ and from the point P drop perpendicular lines to each side of the given triangle $\triangle ABC$ as illustrated in the figure on the right. Let the perpendicular lines intersect the given triangle sides at the points A_1, B_1, C_1 as illustrated. These intersection points define a new triangle $\triangle A_1B_1C_1$ called the pedal triangle of point P. A circle circumscribed about the pedal triangle is called the pedal circle.

The original triangle $\triangle ABC$ has a pedal triangle $\triangle A_1B_1C_1$ associated with the point P called a first generation pedal triangle associated with point P. If the process is repeated, one can form a pedal triangle associated with point P and triangle $\triangle A_1B_1C_1$. This would produce a pedal triangle $\triangle A_2B_2C_2$ called a second generation pedal triangle associated with the point P. By repeating this simple process one can form second, third, fourth,... generation pedal triangles associated with the selected point P.

In general the interior angles associated with each generation of pedal triangles will all be different in comparison with the original triangle $\triangle ABC$. However, the following theorem has been discovered

Theorem

The third generation pedal triangle associated with the point P and the original triangle are similar triangles.

This implies that the original, third, sixth,... generation pedal triangles are all similar to one another. Similarly, one can state that the second, fifth, eighth,... generation pedal triangles will all be similar to one another and so forth.

The same thing can be done with quadrilaterals. Select a point P inside the quadrilateral and drop perpendiculars to the sides to obtain intersection points or vertices which define a first generation pedal quadrilateral. Repeating this process one can construct second, third, fourth,... generation pedal quadrilaterals associated with the point P.

Theorem

The fourth generation pedal quadrilateral associated with a point P and the original quadrilateral will be similar quadrilaterals.

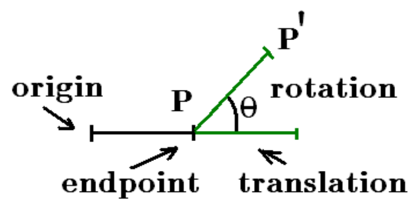
It turns out that the procedure of selecting a point P and then dropping perpendiculars to the sides of triangles and quadrilaterals can also be performed with n-gons and one can show

Theorem

The nth generation pedal n-gon associated with a point P is similar to the original n-gon.

Polygons and star polygons

One rule of construction for polygons is the transformation of vertices from $P \to P'$ defined by a translation followed by a rotation through an angle θ. That is, one can start with a **unit line segment** having an origin and an endpoint. The line segment can be translated one unit and then rotated counterclockwise through an angle $\theta = \frac{2\pi}{n}$ where n is a positive integer. This produces a new starting point where the process can be repeated. This repeated translation and rotation is repeated until the process starts to repeat itself. The result is a polygon having n sides.



If the integer n is replace by a rational number greater than 2, say $n = \frac{p}{d}$ where p, d are coprime with $d > 1$, then the angle of rotation is $\theta = \frac{2\pi}{n} = \frac{2\pi d}{p}$ and the sides of the polygon begin crossing one another. The resulting polygons are called star polygons. Familiar star polygons are the pentagram, hexagram, heptagram, octagram, enneagram, decagram,.... The following are some selected star polygons.

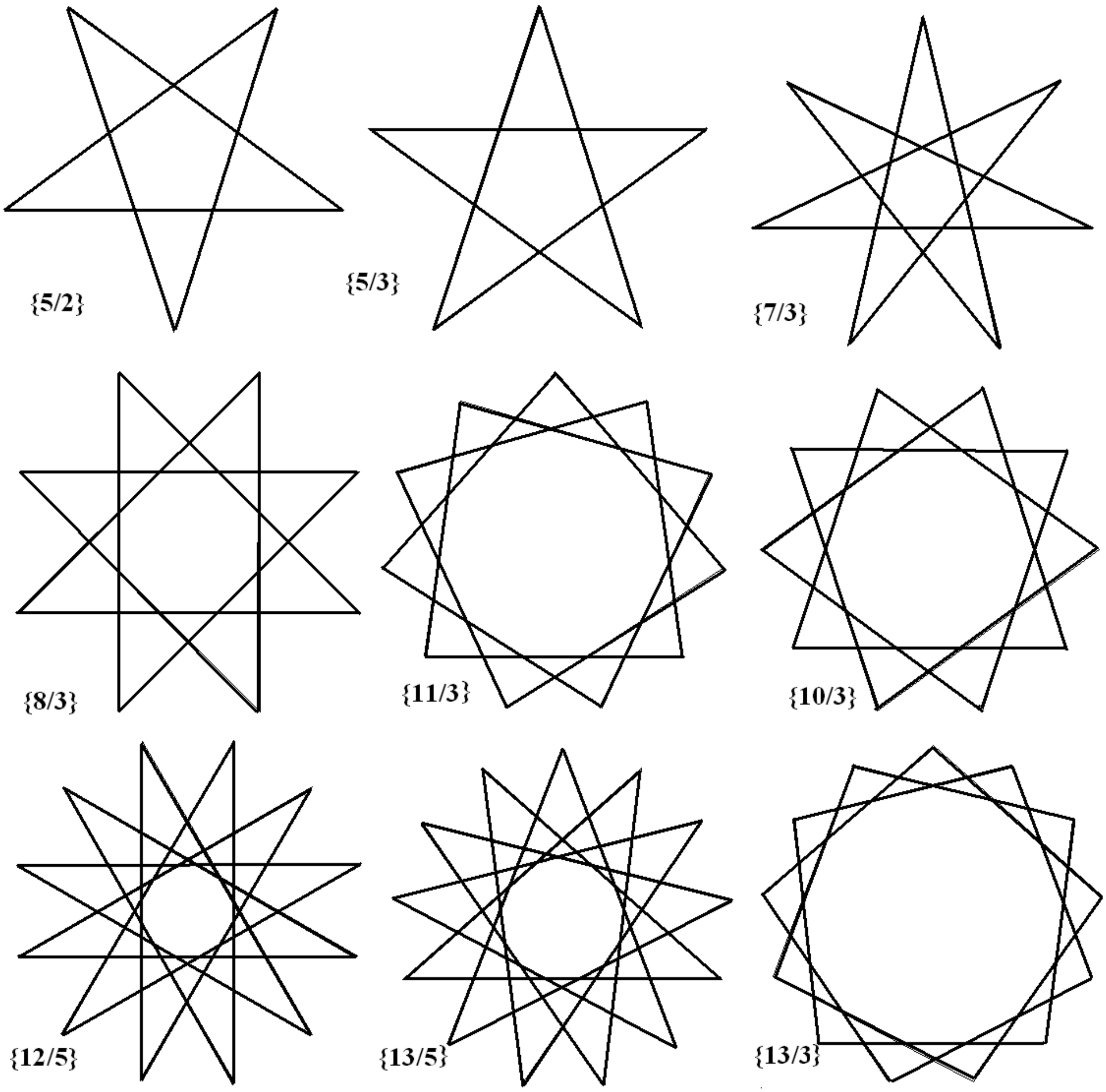
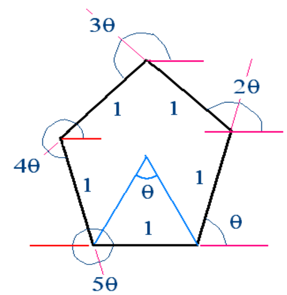


Figure 13-1. Selected star polygons

Schläfi[1] introduced the notation $\{p/d\}$ for representing star polygons where p represents the number of lines used and $d > 1$ represents the number of rotations before line comes back to its starting position. The integer d is called the density of the polygon.

Assume one starts with the unit line segment from $(0,0)$ to $(1,0)$ and applies the translation rotation construction rule for polygons. For $n = 5$ one would obtain the pentagon of density 1 by connecting the points



$$
\begin{aligned}
&(1,0)\\
&(1+\cos\theta,\ \sin\theta)\\
&(1+\cos\theta+\cos 2\theta,\ \sin\theta+\sin 2\theta)\\
&(1+\cos\theta+\cos 2\theta+\cos 3\theta,\ \sin\theta+\sin 2\theta+\sin 3\theta)\\
&(1+\cos\theta+\cos 2\theta+\cos 3\theta+\cos 4\theta,\ \sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta)
\end{aligned}
\tag{13.1}
$$

where $\theta = \frac{2\pi}{5}$.

In the case of star polygons one would connect the points

$$
\begin{aligned}
&(1,0)\\
&(1+\cos\theta,\ \sin\theta)\\
&(1+\cos\theta+\cos 2\theta,\ \sin\theta+\sin 2\theta)\\
&(1+\cos\theta+\cos 2\theta+\cos 3\theta,\ \sin\theta+\sin 2\theta+\sin 3\theta)\\
&\qquad\vdots\\
&(1+\sum_{i=1}^{p-1}\cos i\theta,\ \sum_{i=1}^{p-1}\sin i\theta)
\end{aligned}
\tag{13.2}
$$

where $\theta = 2\pi\frac{d}{p}$.

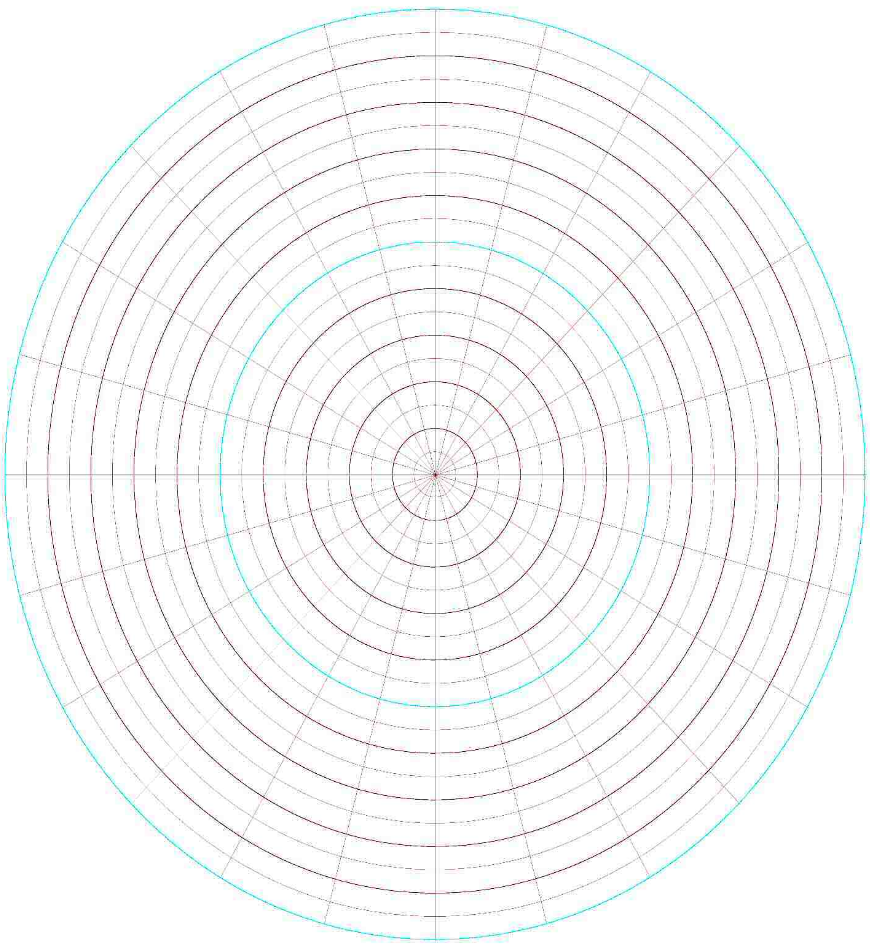
Many polygonal and star polygonal patterns can be found in the study of two dimensional crystallography, x-ray diffraction, architecture, biology, chemistry, physics, atomic physics and lattice structure.

Graph paper

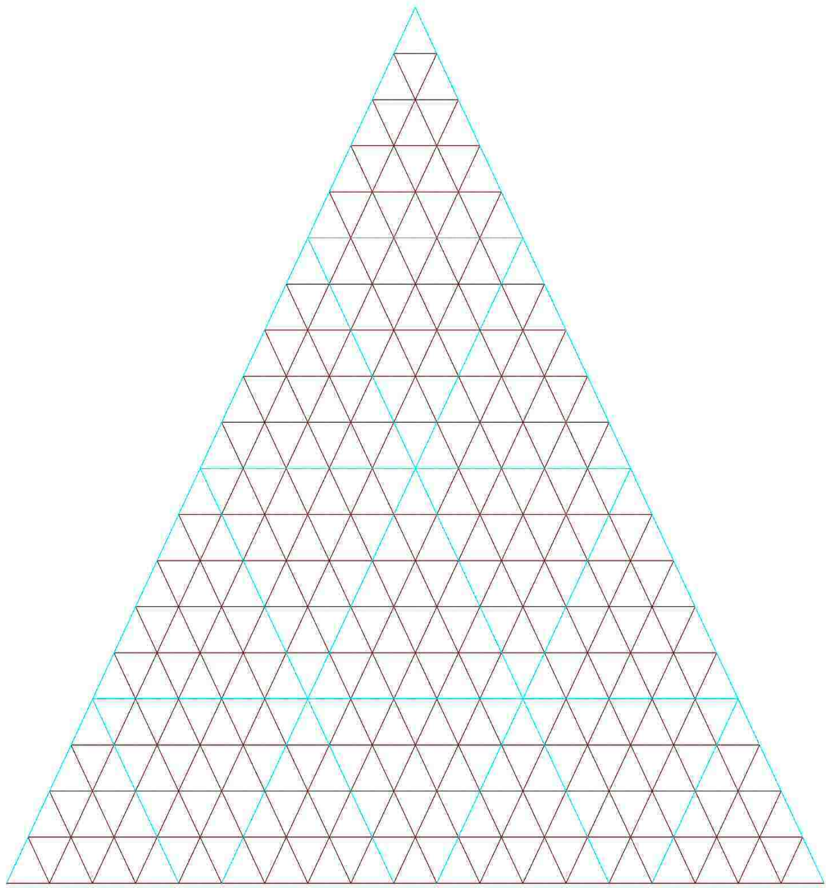
The following is a presentation of four different types of graph paper for representing lines and curves in two-dimensions.

[1] Ludwig Schläfi (1814-1895) A swiss mathematician who studied geometric figures.

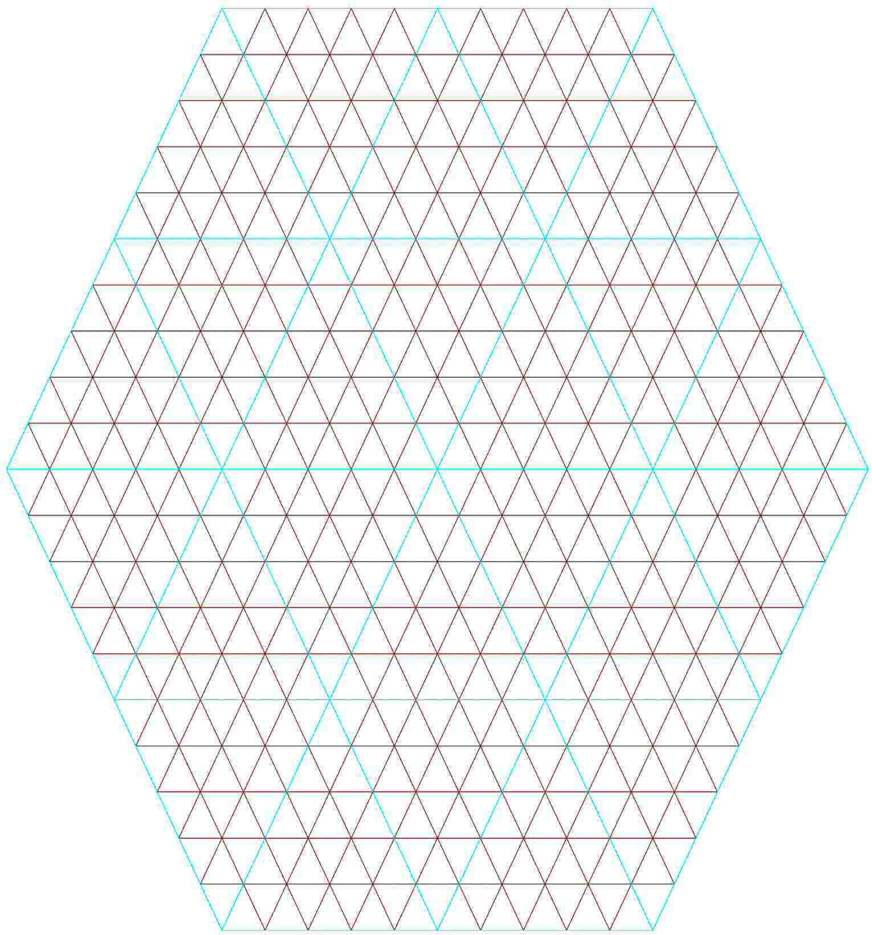
Cartesian equal spaced graph paper



Polar graph paper



Triangular or ternary graph paper



Hexagonal graph paper

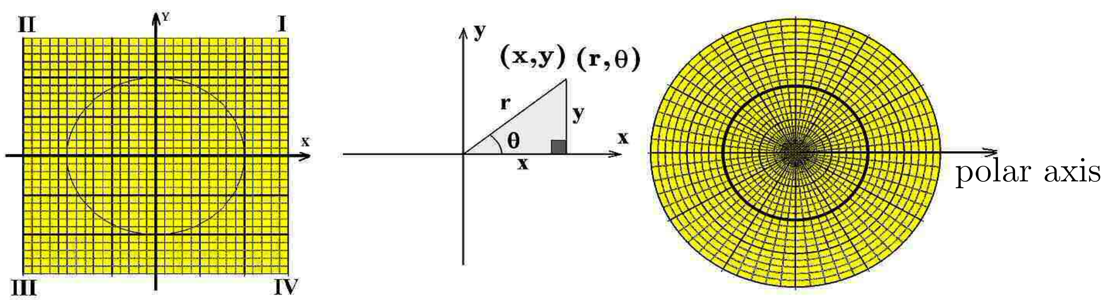
There are many types of graph paper used in the sciences and engineering. The previous examples are just a very small sampling from the large variety of graph paper that is available.

Note that the polar coordinate graph paper has rays emanating from the origin at equally spaced angular distances around the origin and then concentric circles are constructed representing constant distances from the origin. In many cases the types of graph paper one finds have been derived in order that one can use straight lines in certain special circumstances.

Polar coordinates

Cartesian coordinates uses distances from two perpendicular lines to locate a point. Another method to locate a point is to specify its distance r and direction θ from a fixed point. This alternative representation (r, θ) for locating a point is called a polar coordinate representation.

rectangular coordinates (x, y) polar coordinates (r, θ)



$x^2 + y^2 = \rho^2$ $\quad x = r\cos\theta,\quad \tan\theta = \frac{y}{x}$ $\quad y = r\sin\theta,\quad \sqrt{x^2+y^2} = r$ $\quad r = \rho$

Figure 13-3. Rectangular and polar coordinate systems

To construct a **polar coordinate system** one selects an origin for the polar coordinates and labels it 0. Next construct a half-line $\overline{0X}$ similar to the x-axis of the rectangular coordinate system. This half-line is called the **polar axis** or **initial ray** and the origin is called the **pole** of the polar coordinate system. By placing another line on top of the polar axis and rotating this line counterclockwise about the pole through a positive angle θ, measured in radians, one can create a ray emanating from the origin at an angle θ as illustrated in the figure 13-4.

On polar coordinate graph paper rays are illustrated emanating from the origin at equally spaced angular distances around the origin and then concentric circles are constructed representing constant distances from the origin. A point in polar coordinates is then denoted by the number pair (r, θ) where θ is the angle of rotation associated with the ray and r is a distance outward from the origin along the ray. The polar origin or pole has the coordinates $(0, \theta)$ for any angle θ. All points having the polar coordinates $(\rho, 0)$, with $\rho \geq 0$, lie on the polar axis.

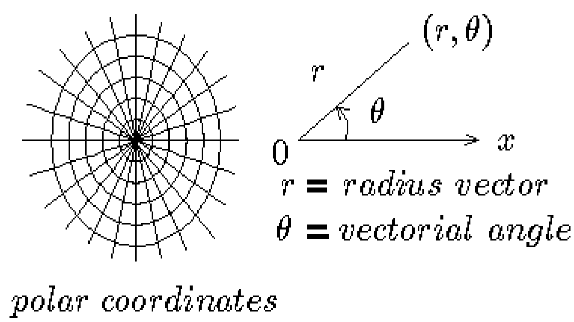


Figure 13-4.
Construction of polar axes

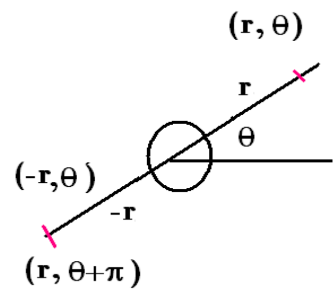


Figure 13-5. Points $(-r, \theta)$ and (r, θ)

For a point (r, θ) in polar coordinates the quantity r is called the radius vector and the quantity θ is called the vectorial angle. Counterclockwise rotations for θ are considered as positive and clockwise rotations are considered as negative. Make note of the following three facts.

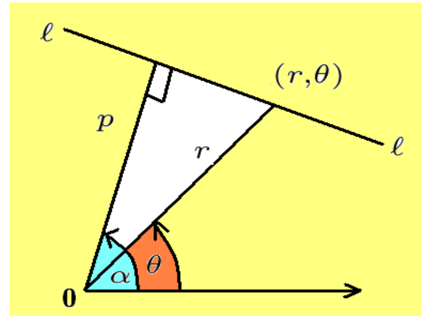
1 The angle θ can be increased or decreased any multiple of 2π without changing the point (r, θ).
2 Positions $(-r, \theta)$ and $(r, \theta + \pi)$ are different representations for the same point.
3 One can verify that the point (r, θ) in polar coordinates has multiple representations. For example

$$(r, \theta) = [(-1)^n r, \theta \pm 2n\pi] \quad \text{for } n = 0, 1, 2, 3, \ldots \tag{13.3}$$

Straight lines in polar coordinates

A straight line in polar coordinates having the slope-intercept form $y = mx + b$, with $b \neq 0$, can be converted to polar coordinates as follows. Make the transformation $x = r\cos\theta$ and $y = r\sin\theta$ and then solve for r to obtain the polar form for the straight line $r = \dfrac{b}{\sin\theta - m\cos\theta}$ with $b \neq 0$.

A more general form for the straight line in polar coordinates is to first find p the perpendicular distance from the origin to the line ℓ. This perpendicular distance is inclined at an angle α. The general point (r,θ) on the line can then be determined by using trigonometry on a right triangle to show $\cos(\alpha-\theta)=\frac{p}{r}$. The equation of the line in polar coordinates is then



$$r = p\sec(\alpha-\theta) \tag{13.4}$$

Note that

(i) If the line passes through the origin, then it polar form is $\theta=c$ a constant, $-\infty<r<\infty$.

(ii) The Cartesian line $y=c$ a constant, has the polar form $r=c\sec(\theta-\pi/2)$

(iii) The Cartesian line $x=c$ a constant, has the polar form $r=c\sec\theta$

Curves in polar coordinates

A polar curve having the form r as a function of θ written $r=f(\theta)$, may be sketched by assigning values to θ and then calculating the value for r defined by the function $f(\theta)$.

Example 13-1. Sketch the curve $r=f(\theta)=3\sin\theta-2$.

Solution Construct a table of values for the function of θ.

		r			r
θ degrees	θ radians	$3\sin\theta-2$	θ degrees	θ radians	$3\sin\theta-2$
0°	0	-2.00000	180°	π	-2.00000
15°	$\frac{\pi}{12}$	-1.22354	195°	$\frac{13\pi}{12}$	-2.77646
30°	$\frac{\pi}{6}$	-0.50000	210°	$\frac{7\pi}{6}$	-3.50000
45°	$\frac{\pi}{4}$	0.12132	225°	$\frac{5\pi}{4}$	-4.12132
60°	$\frac{\pi}{3}$	0.59808	240○	$\frac{4\pi}{3}$	-4.59808
75°	$\frac{5\pi}{12}$	0.89778	255°	$\frac{17\pi}{12}$	-4.89778
90°	$\frac{\pi}{2}$	1.00000	270°	$\frac{3\pi}{2}$	-5.00000
105°	$\frac{7\pi}{12}$	0.89778	285°	$\frac{19\pi}{12}$	-4.89778
120°	$\frac{2\pi}{3}$	0.59808	300°	$\frac{5\pi}{3}$	-4.59808
135°	$\frac{3\pi}{4}$	0.12132	315°	$\frac{7\pi}{4}$	-4.12132
150°	$\frac{5\pi}{5}$	-0.50000	330°	$\frac{11\pi}{6}$	-3.50000
165°	$\frac{11\pi}{12}$	-1.22354	345°	$\frac{23\pi}{12}$	-2.77646
180°	π	-2.00000	360°	π	-2.00000

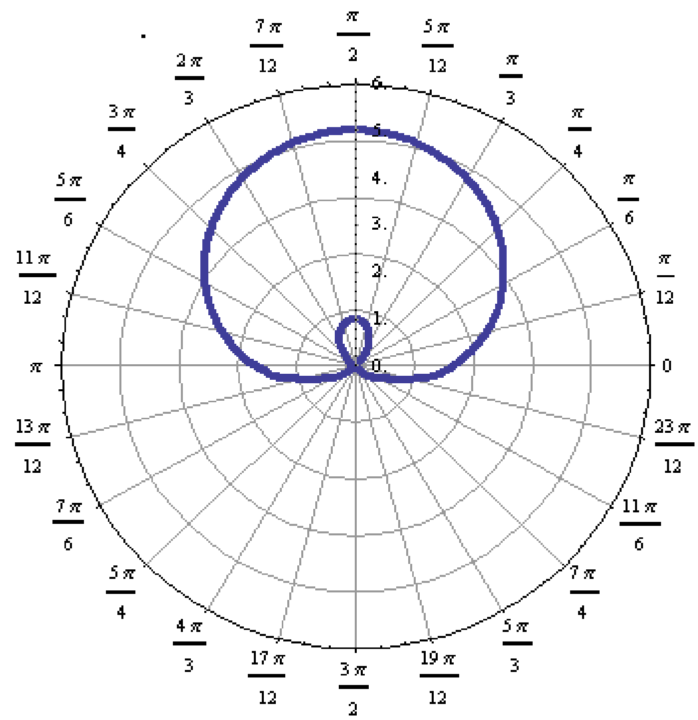
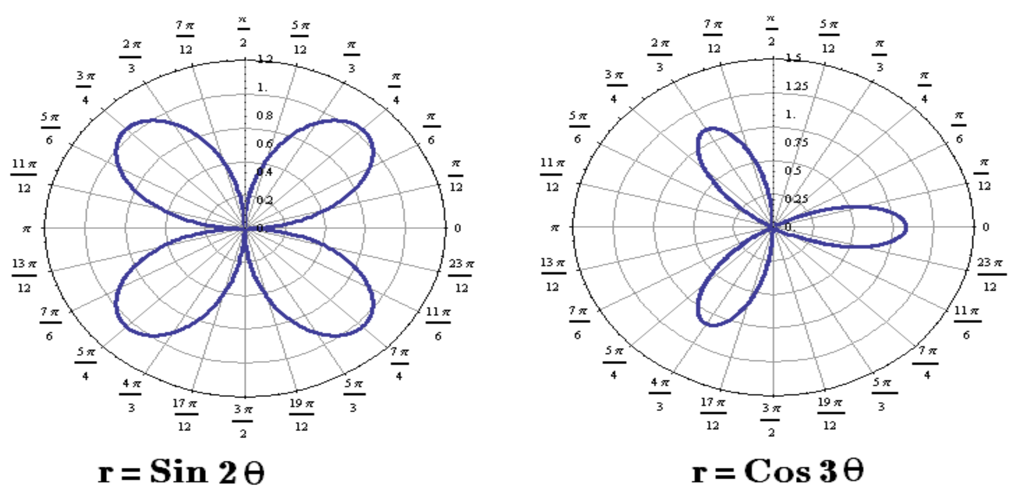


Figure 13-6. Graph of $3\sin\theta - 2$

As an exercise verify the following polar plots.



Points in polar coordinates

A point in polar coordinates is denoted by the number pair (r, θ) where θ is the counterclockwise angle of rotation from the polar axis and a positive value for r is a distance outward from the origin along the ray. The polar origin or pole has the coordinates $(0, \theta)$ for any angle θ. All points having the polar coordinates $(\rho, 0)$, with

$\rho \geq 0$, lie on the polar axis. Also note that a ray at angle θ can be extended to represent negative distances along the ray. Points $(-r, \theta)$ can also be represented by the number pair $(r, \theta + \pi)$ for positive r values.

Alternatively, one can think of a rectangular point (x, y) and the corresponding polar point (r, θ) as being related by the equations

$$\begin{aligned} \theta = \arctan(y/x), \qquad & x = r\cos\theta \\ r = \sqrt{x^2 + y^2}, \qquad & y = r\sin\theta \end{aligned} \tag{13.5}$$

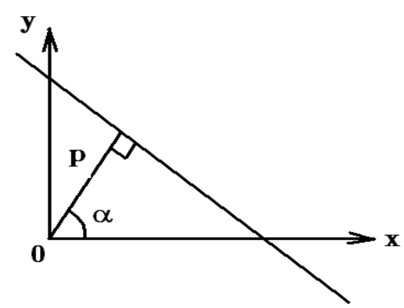
Example 13-2.

To transform an equation from Cartesian coordinates to polar coordinates make the substitutions $x = r\cos\theta,\ y = r\sin\theta$.
The equation of a circle centered at the origin with radius ρ is $x^2 + y^2 = \rho^2$. In polar coordinates this equation becomes $r = \rho$ for all values of θ.

The general equation of a line is $Ax + By = C$. This equation becomes $Ar\cos\theta + Br\sin\theta = C$ or $r = \dfrac{C}{A\cos\theta + B\sin\theta}$

Recall the normal form for equation of a straight line in Cartesian coordinates is $x\cos\alpha + y\sin\alpha = p$. In polar coordinates the equation of the line is $r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$ which simplifies to

$$r\cos(\theta - \alpha) = p \quad \text{or} \quad r = p\sec(\theta - \alpha) \tag{13.6}$$



The equation $x^2 + y^2 - 2ax - 2by = 0$ is a circle. Recall this can be written $(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) = a^2 + b^2$ or $(x-a)^2 + (y-b)^2 = a^2 + b^2$. In polar coordinates this equation is written $r = 2a\cos\theta + 2b\sin\theta$.

■

Distance Between Two Points in the Plane

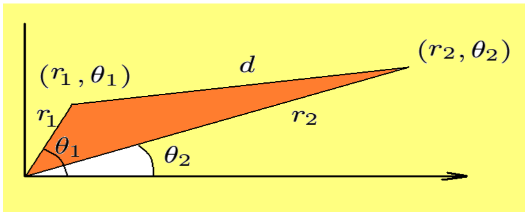


Figure 13-7.
Distance between points in polar coordinates.

If two points are given in polar coordinates as (r_1, θ_1) and (r_2, θ_2), as illustrated in the figure 13-7, then one can use the law of cosines to calculate the distance d between the points since

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) \tag{13.7}$$

Conic Sections

A general equation of the second degree has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{13.8}$$

where A, B, C, D, E, F are constants. All curves which have the form of equation (13.8) can be obtained by cutting a right circular cone with a plane.

Recall that a right circular cone can be generated by two parallel congruent circles with their centers aligned one over the other and a point V half-way between the circles on the line connecting the circle centers. A line from the upper circle circumference through point V to a point on the circumference of the lower circle is called a **generator of the cone.** The set of all generators produces the right circular cone.

The figure 13-8(b) illustrates a horizontal plane intersecting the cone in a **circle**. The figure 13-8(c) illustrates a nonhorizontal plane section which cuts two opposite generators. The resulting curve of intersection is called an **ellipse**. Figure 13-8(d) illustrates a plane parallel to a generator of the cone which also intersects the cone. The resulting curve of intersection is called a **parabola**. Any plane cutting both the upper and lower parts of a cone will intersect the cone in a curve called a **hyperbola** which is illustrated in the figure 13-8(e).

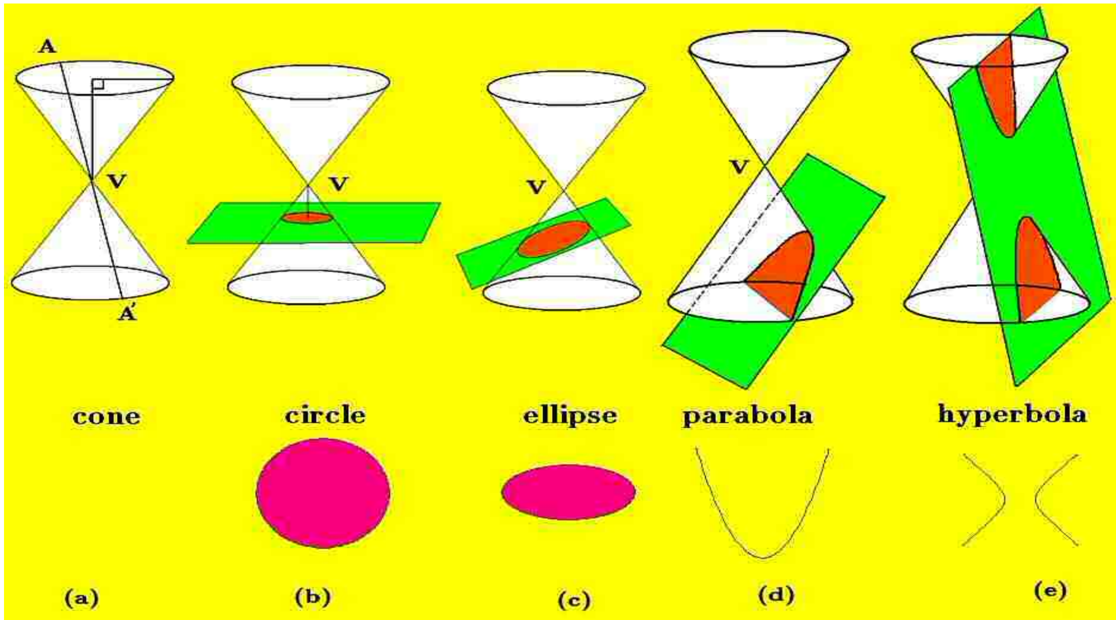


Figure 13-8. The intersection of right circular cone with a plane.

Conic sections were studied by the early Greeks. Euclid[2] supposedly wrote four books on conic sections. The Greek geometer Appollonius[3] wrote eight books on conic sections which summarized Greek knowledge of conic sections and his work has survived the passage of time. Appollonius is also responsible for the names parabola, ellipse and hyperbola.

Conic sections can be defined using either polar coordinates or Cartesian coordinates as follows. Select a point f, called the **focus**, and a line D not through f. This line is called the **directrix**. The set of points P satisfying the condition that the distance from f to P, call it $r = \overline{Pf}$, is some multiple e times the perpendicular distance from P to the directrix. Let $d = \overline{PP'}$ represents the perpendicular distance from the point P to the directrix D, then the resulting equation for a conic section in polar coordinates is obtained from the equation $r = ed$ with the geometric interpretation of this equation illustrated in the figure 13-9.

2 Euclid of Alexandria (325-265 BCE)
3 Appollonius of Perga (262-190 BCE)

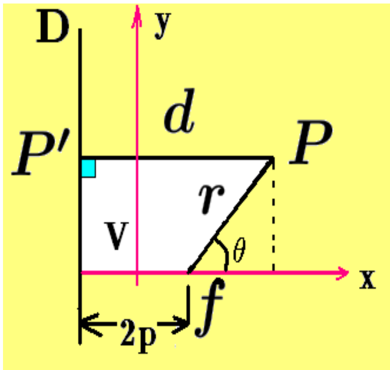


Figure 13-9.
Defining a conic section.

The plane curve resulting from the polar equation

$$\frac{r}{d} = e \tag{13.9}$$

is called a **conic section with eccentricity** e**, focus** f **and directrix** D and if the eccentricity e satisfies

$0 < e < 1$, the conic section is an ellipse.

$e = 1$, the conic section is a parabola.

$e > 1$, the conic section is a hyperbola.

$e = 0$, the conic section is a circle.

Note that in addition to the **focus** and **directrix** there is associated with each conic section the following quantities.

The vertex *V*

The vertex V of a conic section is the midpoint of the line from the focus perpendicular to the directrix.

Axis of symmetry

The line through the focus and perpendicular to the directrix is called an axis of symmetry.

Focal parameter distance 2*p*

This is the perpendicular distance from the focus to the directrix, where p is the distance from the focus to the vertex or distance from vertex to directrix.

Latus rectum 2*ℓ*

This is a chord parallel to a directrix and perpendicular to a focus which passes between two points on the conic section. The latus rectum is used as a measure associated with **the spread of a conic section**.

If ℓ is the semi-latus rectum intersecting the conic section at the point where $x = p = \frac{d}{2}$, one finds $r = \ell = ed = 2ep$. Note that in general $d = 2p + r\cos\theta$ so that the ratio $\frac{r}{d} = e$ becomes $\dfrac{r}{2p + r\cos\theta} = e$ or

$$r = \frac{2ep}{1 - e\,\cos\theta} = \frac{\ell}{1 - e\,\cos\theta} \tag{13.32}$$

is the polar form for the equation of a conic section.

Representing conic section curves

The conic section curves can be thought of as a collection of point (x, y) where x and y are related in some fashion. The mathematical notation for representing a set of points satisfying some condition is

$$C = \{\ (x, y) \mid \text{Condition satisfied by } x \text{ and } y\ \}$$

which is read " C is the set of points (x,y) such that the following condition is satisfied." These (x, y) values are plotted on Cartesian graph paper and the points are connected by a smooth curve.

Parabola

The parabola can be defined as the locus of points (x, y) in a plane, such that (x, y) moves to **remain equidistant from a fixed point** (x_0, y_0) **and fixed line** ℓ. The fixed point is called **the focus of the parabola** and the fixed line is called **the directrix of the parabola**. The midpoint of the perpendicular line from the focus to the directrix is called **the vertex of the parabola**.

In figure 13-10(b), let the point $(0, p)$ denote the focus of the parabola symmetric about the y-axis and let the line $y = -p$ denote the directrix of the parabola. If (x, y) is a general point on the parabola, then

$$\begin{aligned} d_1 =& \text{distance from } (x, y) \text{ to focus} = \sqrt{x^2 + (y-p)^2} \\ d_2 =& \text{distance from } (x, y) \text{ to directrix} = y + p \end{aligned}$$

If $d_1 = d_2$ for all values of x and y, then

$$\sqrt{x^2 + (y-p)^2} = y + p \quad \text{or} \quad x^2 = 4py, \quad p \neq 0 \tag{13.11}$$

This parabola has its vertex at the origin, an eccentricity of 1, a semi-latus rectum of length $2p$, latus rectum of $2\ell = 4p$ and focal parameter of $2p$.

The parabola is the set of points

$$C = \{\ (x, y) \mid x^2 = 4py\ \}$$

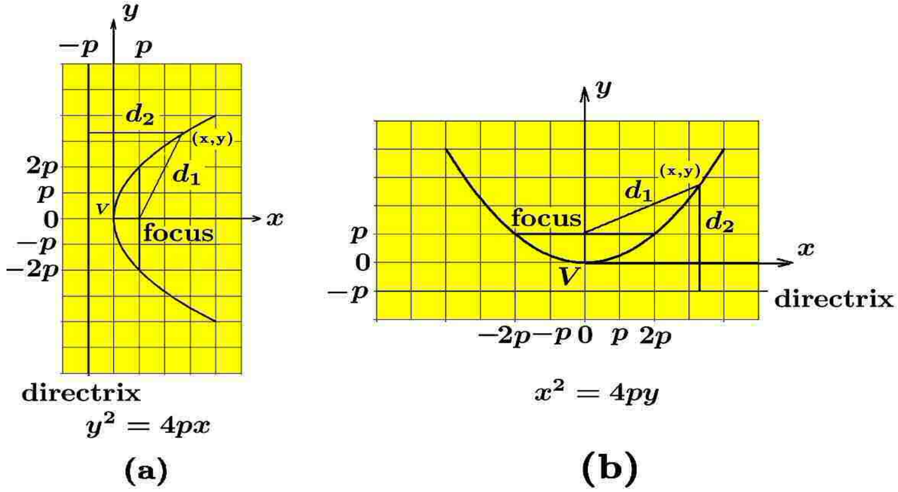


Figure 13-10.

Parabolas symmetric about the x and y axes.

Other forms for the equation of a parabola are obtained by replacing p by $-p$ and interchanging the variables x and y. For $p > 0$, other standard forms for the equation of a parabola are illustrated in the figure 13-11. In the figure 13-11 observe the upward/downward and left/right opening of the parabola depend upon the sign before the parameter p, where $p > 0$ represents the distance from the origin to the focus. By replacing x by $-x$ and y by $-y$ one can verify the various symmetries associated with these shapes.

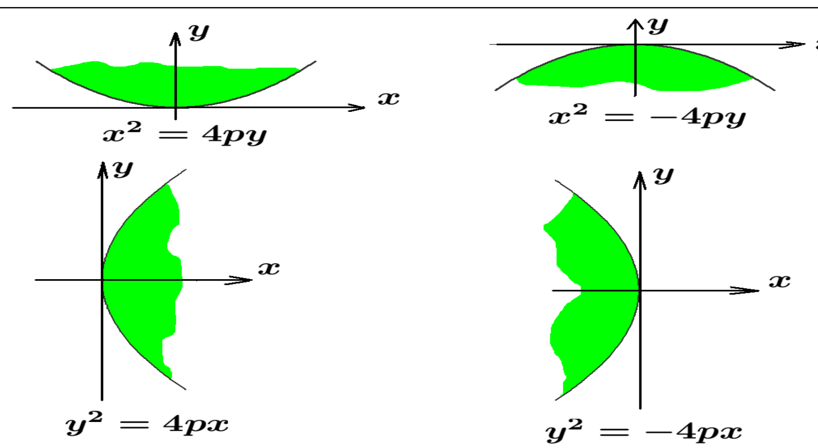


Figure 13-11.

Other forms for representing a parabola.

Using the translation of axes equations, the vertex of the parabolas in the figure 13-11 can be translated to a point (h,k). These translated equations have the representations

$$\begin{aligned} (x-h)^2 =& 4p(y-k) \qquad & (y-k)^2 =& 4p(x-h) \\ (x-h)^2 =& -4p(y-k) \qquad & (y-k)^2 =& -4p(x-h) \end{aligned} \tag{13.12}$$

Also note that the lines of symmetry are also shifted.

One form for the parametric representation of the parabola $(x-h)^2 = 4p(y-k)$ is given by

$$P = \{\, (x,y) \mid \; x = h+t,\; y = k + t^2/4p,\; -\infty < t < \infty \,\} \tag{13.13}$$

with similar parametric representations for the other parabolas.

Use of determinants

The equation of the parabola passing through the three distinct points (x_1,y_1), (x_2,y_2) and (x_3,y_3) can be determined by evaluating the determinant[4]

$$\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix} = 0$$

provided the following determinants are different from zero.

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} \neq 0, \qquad \text{and} \qquad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \neq 0$$

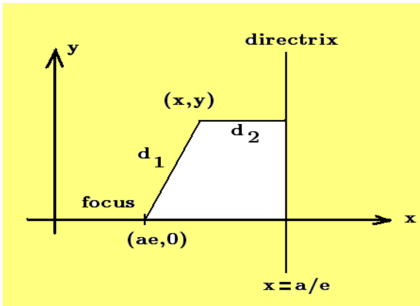
Ellipse

The eccentricity e of an ellipse satisfies $0 < e < 1$ so that for any given positive number a one can state that

$$ae < \frac{a}{e}, \qquad 0 < e < 1 \tag{13.14}$$

Consequently, if the point $(ae, 0)$ is selected as the focus of an ellipse and the line $x = a/e$ is selected as the directrix of the ellipse, then in relation to this fixed focus and fixed line a general point (x,y) will satisfy

[4] Determinants and their properties are discussed in chapter 6.



$d_1 =$ distance of (x, y) to focus $= \sqrt{(x - ae)^2 + y^2}$

$d_2 = \perp$ distance of (x, y) to directrix $= |x - a/e|$

The ellipse can then be defined as the set of points (x, y) satisfying the constraint condition $d_1 = e\,d_2$ which can be expressed as the set of points

$$E_1 = \{\,(x, y) \mid \sqrt{(x - ae)^2 + y^2} = e|x - a/e|,\ 0 < e < 1\,\} \tag{13.15}$$

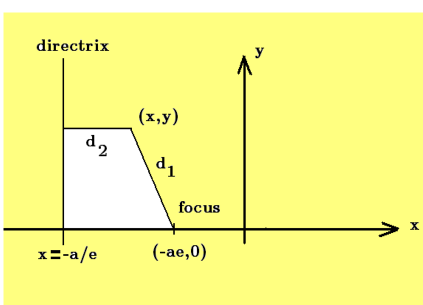
Applying some algebra to the constraint condition on the points (x, y), the ellipse can be expressed in a different form. Observe that if $d_1 = e\,d_2$, then

$$\begin{aligned} (x - ae)^2 + y^2 &= e^2(x - a/e)^2 \\ \text{or} \qquad x^2 - 2aex + a^2e^2 + y^2 &= e^2x^2 - 2aex + a^2 \end{aligned}$$

which simplifies to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2(1 - e^2) \tag{13.16}$$

where the eccentricity satisfies $0 < e < 1$. In the case where the focus is selected as $(-ae, 0)$ and the directrix is selected as the line $x = -a/e$, there results the following situation



$d_1 =$ distance of (x, y) to focus $= \sqrt{(x + ae)^2 + y^2}$

$d_2 = \perp$ distance of (x, y) to directrix $= |x + a/e|$

The condition that $d_1 = e\,d_2$ can be represented as the set of points

$$E_2 = \{\,(x, y) \mid \sqrt{(x + ae)^2 + y^2} = e|x + a/e|,\ 0 < e < 1\,\} \tag{13.17}$$

As an exercise, show that the simplification of the constraint condition for the set of points E_2 also produces the equation (13.16).

Examine the limiting case where $e \to 0$. In this case the foci of the ellipse approach each other and the ellipse becomes a circle with foci going to the origin $(0,0)$ and the directrix $x = \frac{a}{e}$ going to the line at infinity. The circle being a limiting case of the ellipse when the eccentricity is zero. Note that the equation (13.9) becomes

$$r = \lim_{e\to 0} e\, d = \lim_{e\to 0} e\, \frac{a}{e} \qquad \Rightarrow \qquad r = a$$

which is the equation of a circle in polar coordinates.

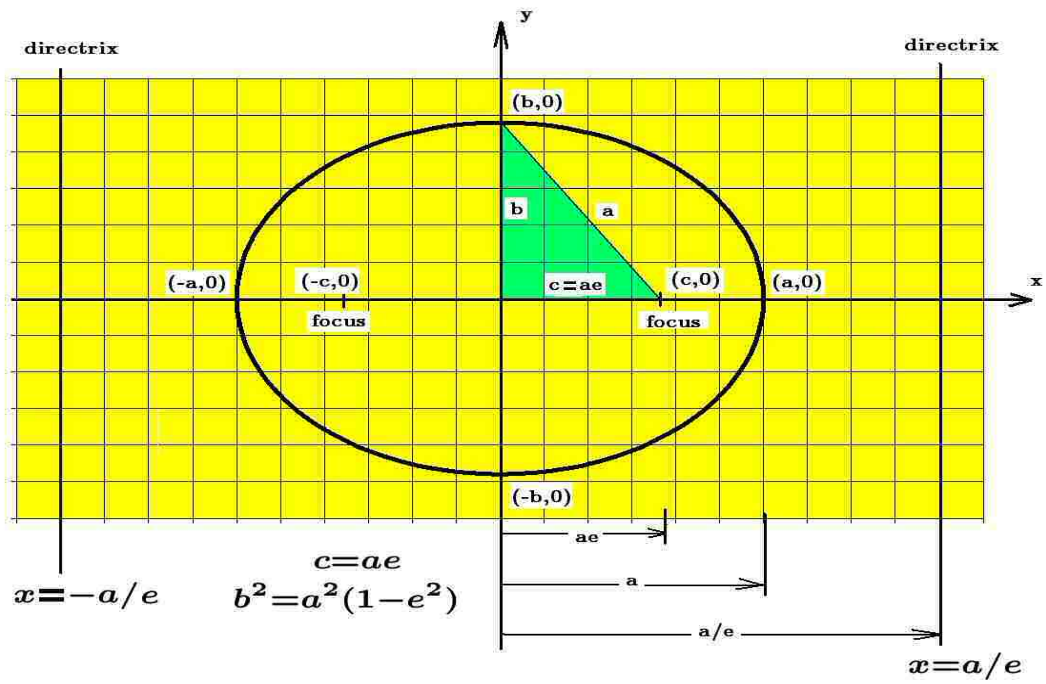


Figure 13-12. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Define the constants

$$c = ae \quad \text{and} \quad b^2 = a^2(1-e^2) = a^2 - c^2 \tag{13.18}$$

and note that $b^2 < a^2$, then from the above discussion one can conclude that an ellipse is defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad 0 < e < 1, \quad b^2 = a^2(1-e^2), \quad c = ae \tag{13.19}$$

and has the points $(ae, 0)$ and $(-ae, 0)$ as foci and the lines $x = -a/e$ and $x = a/e$ as directrices. The resulting graph for the ellipse is illustrated in the figure 13-12. This ellipse has vertices at $(-a, 0)$ and $(a, 0)$, a latus rectum of length $2b^2/a$ and eccentricity given by $\sqrt{1 - b^2/a^2}$.

In the figure 13-12 a right triangle has been constructed as a mnemonic device to help remember the relations given by the equations (13.18). The distance $2a$ between $(-a, 0)$ and $(a, 0)$ is called the **major axis of the ellipse** and the distance $2b$ from $(0, -b)$ to $(0, b)$ is called the **minor axis of the ellipse**. The origin $(0, 0)$ is called the **center of the ellipse**.

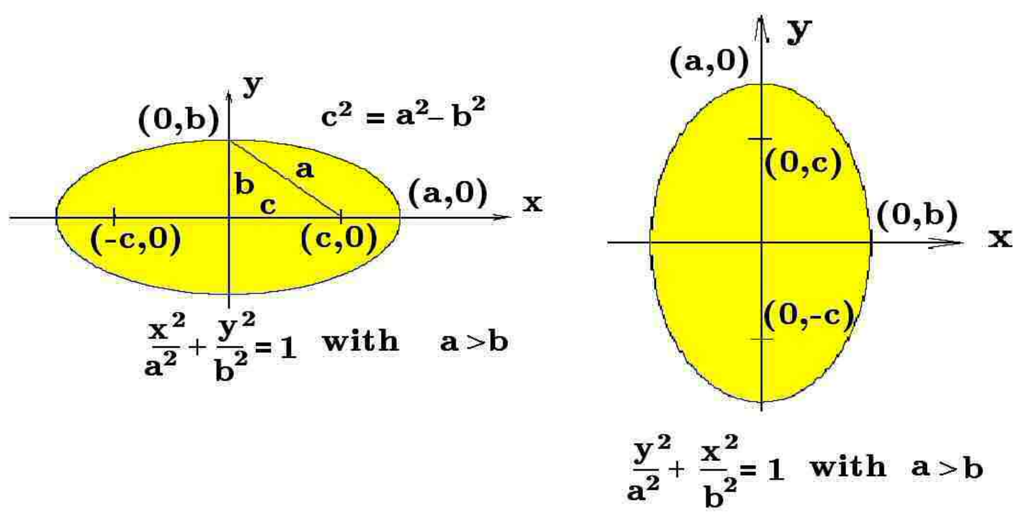


Figure 13-13. Symmetry of the ellipse.

Some algebra can verify the following property satisfied by a general point (x, y) on the ellipse. Construct the distances

$$\begin{aligned} d_3 =& \text{distance of } (x,y) \text{ to focus } (c,0) = \sqrt{(x-c)^2 + y^2} \\ d_4 =& \text{distance of } (x,y) \text{ to focus } (-c,0) = \sqrt{(x+c)^2 + y^2} \end{aligned} \tag{13.20}$$

and show

$$d_3 + d_4 = \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \tag{13.21}$$

One can use this property to define the ellipse as the locus of points (x, y) such that the sum of its distances from two fixed points equals a constant.

The figure 13-13 illustrates that when the roles of x and y are interchanged, then the major axis and minor axis of the ellipse are reversed. A shifting of the axes so that the point (x_0, y_0) is the center of the ellipse produces the equations

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \qquad \text{or} \qquad \frac{(y-y_0)^2}{a^2} + \frac{(x-x_0)^2}{b^2} = 1 \tag{13.22}$$

These equations represent the ellipses illustrated in the figure 13-13 where the centers are shifted to the point (x_0, y_0).

The ellipse given by $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ which is centered at the point (h, k) can be represented in a parametric form[5]. One parametric form is to represent the ellipse as the set of points

$$E = \{\, (x, y) \mid\; x = h + a\cos\theta,\; y = k + b\sin\theta, \quad 0 \le \theta \le 2\pi \;\} \tag{13.23}$$

involving the parameter θ which varies from 0 to 2π.

Circle

A circle is the locus of points (x, y) in a plane equidistant from a fixed point called the center of the circle. Note that no real locus occurs if the radius r is negative or imaginary. It has been previously demonstrated how to calculate the equation of a circle. The figure 13-14 is a summary of these previous results. The circle $x^2 + y^2 = r^2$ has eccentricity zero and latus rectum of $2r$. Parametric equations for the circle $(x-x_0)^2 + (y-y_0)^2 = r^2$, centered at (x_0, y_0), are

$$x = x_0 + r\cos t, \qquad y = y_0 + r\sin t, \quad 0 \le t \le 2\pi$$

[5] The parametric representation of a curve or part of a curve is not unique.

When dealing with second degree equations of the form $x^2 + y^2 + \alpha x + \beta y = \gamma$, where α, β and γ are constants, it is customary to **complete the square** on the x and y terms to obtain

$$(x^2 + \alpha x + \frac{\alpha^2}{4}) + (y^2 + \beta y + \frac{\beta^2}{4}) = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4} \quad \rightarrow \quad (x + \frac{\alpha}{2})^2 + (y + \frac{\beta}{2})^2 = r^2$$

where it is assumed that $r^2 = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4} > 0$. This produces the equation of a circle with radius r which is centered at the point $\left(-\frac{\alpha}{2}, -\frac{\beta}{2}\right)$.

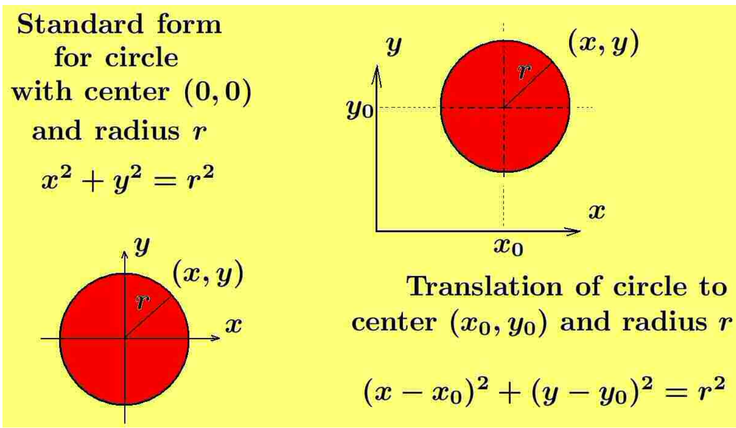
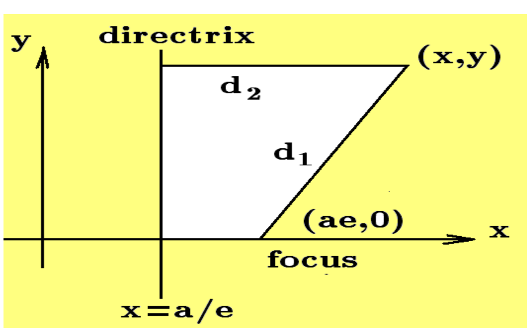


Figure 13-14.

Circle about origin and circle translated to point (x_0, y_0)

Hyperbola

Let $e > 1$ denote the eccentricity of a hyperbola. Again let $(ae, 0)$ denote the focus of the hyperbola and let the line $x = a/e$ denote the directrix of the hyperbola. The hyperbola is defined such that points (x, y) on the hyperbola satisfy $d_1 = ed_2$ where d_1 is the distance from (x, y) to the focus and d_2 is the perpendicular distance from the point (x, y) to the directrix. The hyperbola can then be represented by the set of points

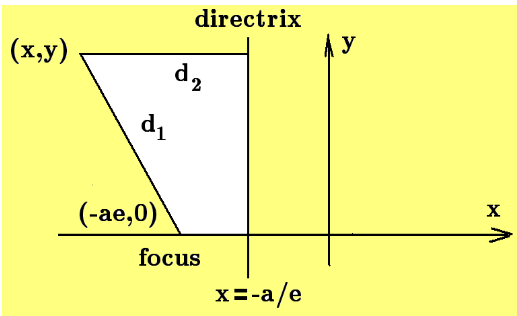


$$H_1 = \{\, (x, y) \mid \sqrt{(x - ae)^2 + y^2} = e|x - a/e|,\ e > 1 \,\}$$

A simplification of the constraint condition on the set of points (x,y) produces the alternative representation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1, \qquad e>1 \tag{13.24}$$

Placing the focus at the point $(-ae,0)$ and using as the directrix the line $x=-a/e$, one can verify that the hyperbola is represented by the set of points



$$H_2 = \{\,(x,y) \mid \sqrt{(x+ae)^2+y^2} = e|x+a/e|,\ e>1\,\}$$

and it can be verified that the constraint condition on the points (x,y) also simplifies to the equation (13.24).

Define $c=ae$ and $b^2=a^2(e^2-1)=c^2-a^2>0$ and note that for an eccentricity $e>1$ there results the inequality $c>a$. The hyperbola can then be described as having the foci $(c,0)$ and $(-c,0)$ and directrices $x=a/e$ and $x=-a/e$. The hyperbola represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad b^2=a^2(e^2-1)=c^2-a^2 \tag{13.25}$$

is illustrated in the figure 13-15.

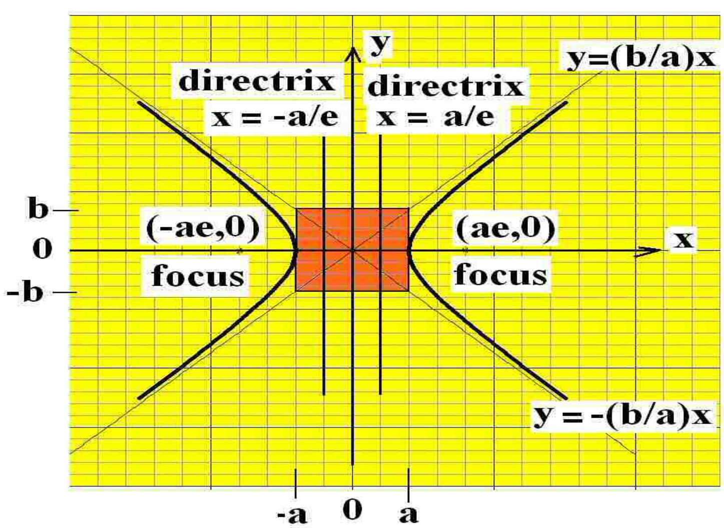


Figure 13-15. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

This hyperbola has vertices at $(-a,0)$ and $(a,0)$, a latus rectum of length $2b^2/a$ and eccentricity of $\sqrt{1+b^2/a^2}$. The origin is called the center of the hyperbola. The line containing the two foci of the hyperbola is called the principal axis of the hyperbola. Setting $y=0$ and solving for x one can determine that the hyperbola intersects the principal axis at the points $(-a,0)$ and $(a,0)$ which are called the vertices of the hyperbola. The line segment between the vertices is called the **major axis of the hyperbola** or **transverse axis of the hyperbola**. The distance between the points $(b,0)$ and $(-b,0)$ is called the **conjugate axis of the hyperbola**. The chord through either focus which is perpendicular to the transverse axis is called a **latus rectum**. One can verify that the latus rectum intersects the hyperbola at the points $(c, b^2/a)$ and $(c,-b^2/a)$.

Write the equation (13.25) in the form

$$y = \pm\frac{b}{a}x\sqrt{1-\frac{a^2}{x^2}} \tag{13.26}$$

and note that for very large values of x the right-hand side of this equation approaches 1. Consequently, for large values of x the equation (13.26) becomes the lines

$$y = \frac{b}{a}x \qquad \text{and} \qquad y = -\frac{b}{a}x \tag{13.27}$$

These lines are called the asymptotic lines associated with the hyperbola and are illustrated in the figure 13-15. Note that the hyperbola has two branches with each branch approaching the asymptotic lines for large values of x.

Let (x,y) denote a general point on the above hyperbola and construct the distances

$$\begin{aligned} d_3 =& \text{distance from } (x,y) \text{ to the focus } (c,0) = \sqrt{(x-c)^2+y^2} \\ d_4 =& \text{distance from } (x,y) \text{ to the focus } (-c,0) = \sqrt{(x+c)^2+y^2} \end{aligned} \tag{13.28}$$

Use some algebra to verify that

$$d_4 - d_3 = 2a \tag{13.29}$$

This property of the hyperbola is sometimes used to define the hyperbola as the locus of points (x,y) in the plane such that the difference of its distances from two fixed points is a constant.

The hyperbola with transverse axis on the x-axis have the asymptotic lines $y=+\frac{b}{a}x$ and $y=-\frac{b}{a}x$. Any hyperbola with the property that the conjugate axis has the same length as the transverse axis is called a rectangular or equilateral hyperbola.

Rectangular hyperbola are such that the asymptotic lines are perpendicular to each other.

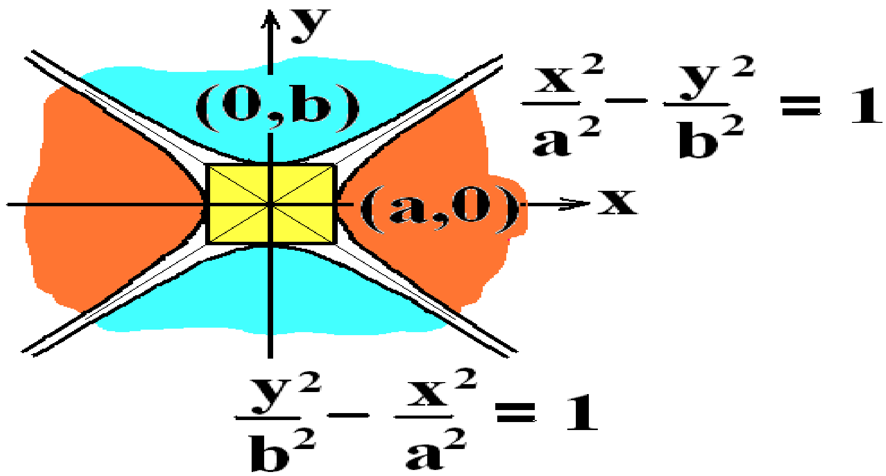


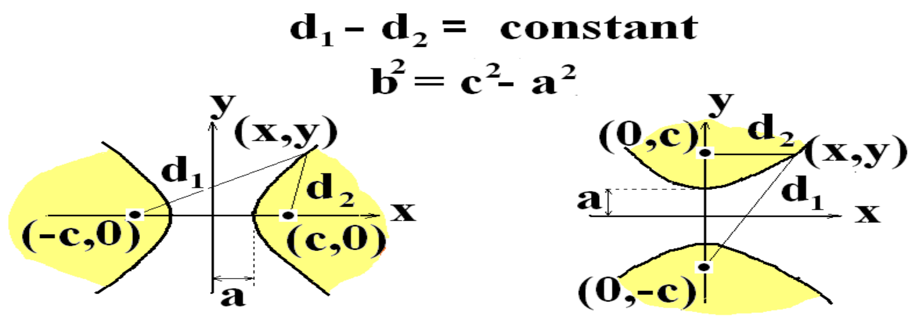
Figure 13-16. Conjugate hyperbola.

If two hyperbola are such that the transverse axis of either is the conjugate axis of the other, then they are called conjugate hyperbola. Conjugate hyperbola will have the same asymptotic lines. Conjugate hyperbola are illustrated in the figure 13-16.

The figure 13-17 illustrates that when the roles of x and y are interchanged, then the transverse axis and conjugate axis of the hyperbola are reversed. A shifting of the axes so that the point (x_0, y_0) is the center of the hyperbola produces the equations

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1 \qquad \text{or} \qquad \frac{(y-y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1 \tag{13.30}$$

The figure 13-17 illustrates what happens to the hyperbola when the values of x and y are interchanged.



If the foci are on the x-axis at $(c,0)$ and $(-c,0)$, then $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

If the foci are on the y-axis at $(0,c)$ and $(0,-c)$, then $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

Figure 13-17.
Symmetry of the hyperbola .

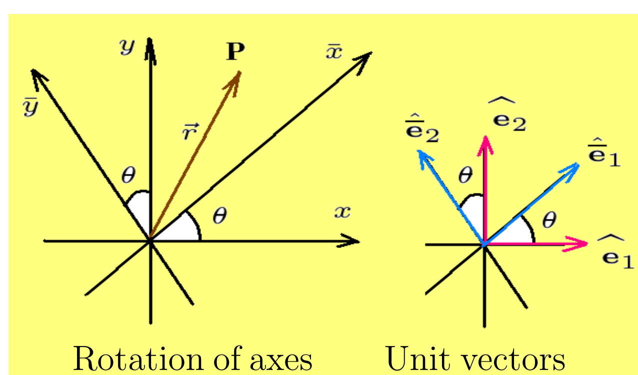
The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ can also be represented in a parametric form as the set of points

$$
\begin{aligned}
&H = H_1 \cup H_2 \quad \text{where} \\
&\quad H_1 = \{\, (x,y) \mid x = a\cosh t,\ y = b\sinh t, \quad -\infty < t < \infty \,\} \qquad (13.31)\\
\text{and} \quad &\quad H_2 = \{\, (x,y) \mid x = -a\cosh t,\ y = b\sinh t, \quad -\infty < t < \infty \,\}
\end{aligned}
$$

which represents a union of the right-branch and left-branch of the hyperbola. Similar parametric representations can be constructed for those hyperbola which undergo a translation or rotation of axes. Remember that the parametric representation of a curve is not unique.

Rotation of axes

If the (x,y)-axes are rotated about the origin to form a new set of barred axes $(\bar{x},\bar{y})$, then the relation between the old and new coordinate systems is obtained by using projections. Note the the vector $\vec{r}$ from the origin to a point P remains invariant under a rotation of axes and so one use this distance under the conditions



(i) $\vec{r}$ is projected onto the $\bar{x}$ and $\bar{y}$ axes.
(ii) $\vec{r}$ is projected onto the x and y axes.

If $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2$ are unit vectors in the (x, y)-coordinates and $\hat{\bar{\mathbf{e}}}_1, \hat{\bar{\mathbf{e}}}_2$ are unit vectors in the barred coordinates, then one can write

$$\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 = \bar{x}\,\hat{\bar{\mathbf{e}}}_1 + \bar{y}\,\hat{\bar{\mathbf{e}}}_2 \tag{13.32}$$

since $\vec{r}$ is to be an invariant.

One can verify the following dot product relations between the unit vectors in the two systems

$$\begin{aligned} &\widehat{\mathbf{e}}_1 \cdot \hat{\bar{\mathbf{e}}}_1 = \cos\theta & &\widehat{\mathbf{e}}_2 \cdot \hat{\bar{\mathbf{e}}}_1 = \cos(\frac{\pi}{2} - \theta) = \sin\theta \\ &\widehat{\mathbf{e}}_1 \cdot \hat{\bar{\mathbf{e}}}_2 = \cos(\frac{\pi}{2} + \theta) = -\sin\theta & &\widehat{\mathbf{e}}_2 \cdot \hat{\bar{\mathbf{e}}}_2 = \cos\theta \end{aligned} \tag{13.33}$$

Taking the dot product of equation (13.32) with the vectors $\widehat{\mathbf{e}}_1$ and $\widehat{\mathbf{e}}_2$ one finds

$$\begin{aligned} \vec{r} \cdot \widehat{\mathbf{e}}_1 =& \text{projection of } \vec{r} \text{ onto } x\text{-axis} = x = \bar{x}\,\hat{\bar{\mathbf{e}}}_1 \cdot \widehat{\mathbf{e}}_1 + \bar{y}\,\hat{\bar{\mathbf{e}}}_2 \cdot \widehat{\mathbf{e}}_1 \\ \vec{r} \cdot \widehat{\mathbf{e}}_2 =& \text{projection of } \vec{r} \text{ onto } y\text{-axis} = y = \bar{x}\,\hat{\bar{\mathbf{e}}}_1 \cdot \widehat{\mathbf{e}}_2 + \bar{y}\,\hat{\bar{\mathbf{e}}}_2 \cdot \widehat{\mathbf{e}}_2 \end{aligned} \tag{13.34}$$

or the old system of coordinates (x, y) in terms of the new rotated sytem of coordinates $(\bar{x}, \bar{y})$ is given by

$$\begin{aligned} x =& \bar{x}\,\cos\theta - \bar{y}\,\sin\theta \\ y =& \bar{x}\,\cos\theta + \bar{y}\,\cos\theta \end{aligned} \tag{13.35}$$

Taking the dot product of equation (13.32) with the vectors $\hat{\bar{\mathbf{e}}}_1$ and $\hat{\bar{\mathbf{e}}}_2$ one finds

$$\begin{aligned} \vec{r} \cdot \hat{\bar{\mathbf{e}}}_1 =& \text{projection of } \vec{r} \text{ onto } \bar{x} \text{ axis} = \bar{x} = x\,\widehat{\mathbf{e}}_1 \cdot \hat{\bar{\mathbf{e}}}_1 + y\,\widehat{\mathbf{e}}_2 \cdot \hat{\bar{\mathbf{e}}}_1 \\ \vec{r} \cdot \hat{\bar{\mathbf{e}}}_2 =& \text{projection of } \vec{r} \text{ onto } \bar{y} \text{ axis} = \bar{y} = x\,\widehat{\mathbf{e}}_1 \cdot \hat{\bar{\mathbf{e}}}_2 + y\,\widehat{\mathbf{e}}_2 \cdot \hat{\bar{\mathbf{e}}}_2 \end{aligned} \tag{13.36}$$

or the new system of coordinates $(\bar{x}, \bar{y})$ in terms of the old system of coordinates (x, y) is given by

$$\begin{aligned} \bar{x} =& \quad x\,\cos\theta + y\,\sin\theta \\ \bar{y} =& -x\,\sin\theta + y\,\cos\theta \end{aligned} \tag{13.37}$$

There are occasions in mathematics where equations in an (x, y) system of coordinates become simplified if they are expressed in a rotated $(\bar{x}, \bar{y})$ coordinate system. To achieve this simplification the equations (13.35) and (13.37) will be needed.

General Equation of the Second Degree

Consider the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{13.38}$$

where a, b, c, d, e, f are constants, which is **a general equation of the second degree**. If one performs a rotation of axes by substituting the rotation equations

$$x = \bar{x}\cos\theta - \bar{y}\sin\theta \quad \text{and} \quad y = \bar{x}\sin\theta + \bar{y}\cos\theta \tag{13.39}$$

into the equation (13.38), one obtains the new equation

$$\bar{a}\bar{x}^2 + \bar{b}\bar{x}\bar{y} + \bar{c}\bar{y}^2 + \bar{d}\bar{x} + \bar{e}\bar{y} + \bar{f} = 0 \tag{13.40}$$

with new coefficients $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ defined by the equations

$$\begin{aligned}
\bar{a} =& a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta & \bar{d} =& d\cos\theta + e\sin\theta \\
\bar{b} =& b(\cos^2\theta - \sin^2\theta) + 2(c-a)\sin\theta\cos\theta & \bar{e} =& -d\sin\theta + e\cos\theta \\
\bar{c} =& a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta & \bar{f} =& f
\end{aligned} \tag{13.41}$$

As an exercise one can show the quantity $b^2 - 4ac$, called **the discriminant**, is **an invariant** under a rotation of axes. One can show $b^2 - 4ac = \bar{b}^2 - 4\bar{a}\bar{c}$. The discriminant is used to predict the conic section from the equation (13.38). For example, if $b^2 - 4ac < 0$, then an ellipse results, if $b^2 - 4ac = 0$, then a parabola results, if $b^2 - 4ac > 0$, then a hyperbola results.

In the case where the original equation (13.38) has a cross product term xy, so that $b \neq 0$, then **one can always find a rotation angle θ such that in the new equations (13.40) and (13.41) the term $\bar{b} = 0$.** If the cross product term $\bar{b}$ is made zero, then one can complete the square on the $\bar{x}$ and $\bar{y}$ terms which remain. This completing the square operation converts the new equation (13.39) into one of the standard forms associated with a conic section. By setting the $\bar{b}$ term in equation (13.41) equal to zero one can determine the angle θ such that $\bar{b}$ vanishes. Using the trigonometric identities

$$\cos^2\theta - \sin^2\theta = \cos 2\theta \qquad \text{and} \qquad 2\sin\theta\cos\theta = \sin 2\theta \tag{13.42}$$

one can determine the angle θ which makes the cross product term vanish by solving the equation

$$\bar{b} = b\cos 2\theta + (c-a)\sin 2\theta = 0 \tag{13.43}$$

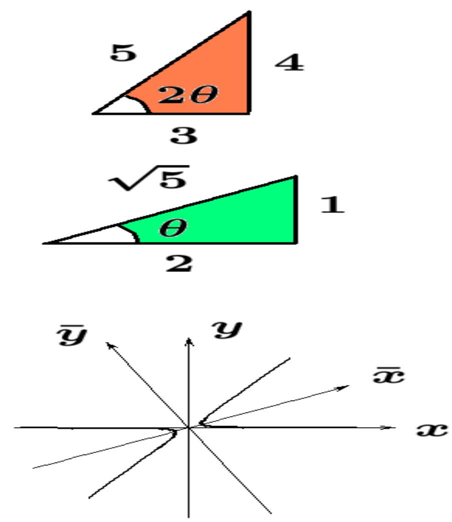
for the angle θ. One finds the new term $\bar{b}$ is zero if θ is selected to satisfy

$$\cot 2\theta = \frac{a-c}{b} \quad \text{(recall our hypothesis that } b \neq 0) \tag{13.44}$$

Example 13-3. **(Conic Section)** Sketch the curve $4xy - 3y^2 = 64$

Solution

To remove the product term xy from the general equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ of a conic, the axes must be rotated through an angle θ determined by the equation $\cot 2\theta = \frac{a-c}{b} = \frac{3}{4}$. For the given conic $a = 0,\ b = 4,\ c = -3.$



This implies that $\cos 2\theta = 3/5 = 1 - 2\sin^2\theta$ or $2\sin^2\theta = 2/5$ giving $\sin\theta = 1/\sqrt{5}$ and $\cos\theta = 2/\sqrt{5}$. The rotation equations (13.39) become

$$x = \frac{1}{\sqrt{5}}(2\bar{x} - \bar{y}) \qquad \text{and} \qquad y = \frac{1}{\sqrt{5}}(\bar{x} + 2\bar{y})$$

The given equation then becomes

$$4\left(\frac{2\bar{x} - \bar{y}}{\sqrt{5}}\right)\left(\frac{\bar{x} + 2\bar{y}}{\sqrt{5}}\right) - 3\left(\frac{\bar{x} + 2\bar{y}}{\sqrt{5}}\right)^2 = 64$$

which simplifies to the hyperbola $\dfrac{\bar{x}^2}{8^2} - \dfrac{\bar{y}^2}{4^2} = 1$ with respect to the $\bar{x}$ and $\bar{y}$ axes.

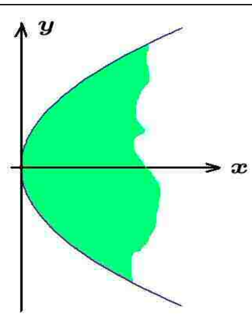
■

Example 13-4.

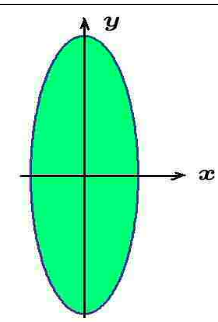
The parametric forms for representing conic sections are not unique. For a, b constants and θ, t used as parameters, the following are some representative parametric equations which produce conic sections.

Parametric form for conic sections			
Conic Section	x	y	parameter
Circle	$a\cos\theta$	$a\sin\theta$	θ
Parabola	at^2	$2at$	t
Ellipse	$a\cos\theta$	$b\sin\theta$	θ
Hyperbola	$a\sec\theta$	$b\tan\theta$	θ
Rectangular Hyperbola	at	a/t	t
The symbols $a, b > 0$ denote nonzero constants.			

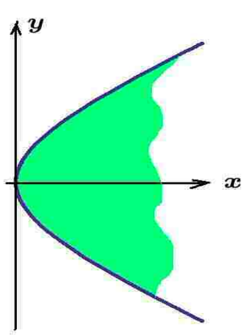
The shape of the curves depends upon the range of values assigned to the parameters representing the curve. Because of this restriction, the parametric representation usually only gives a portion of the total curve. Sample graphs using the parameter values indicated are given below.



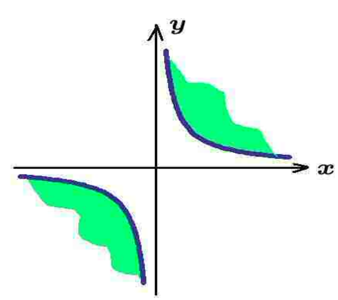
$\{ (x,y) \mid x=t^2,\ y=2t,\ -2\le t\le 2 \}$



$\{ (x,y) \mid x=\cos\theta,\ y=3\sin\theta,\ 0\le\theta\le 2\pi \}$



$\{ (x,y) \mid x=\sec\theta,\ y=\frac{1}{2}\tan\theta,\ -\frac{\pi}{3}\le\theta\le\frac{\pi}{3} \}$



$\{ (x,y) \mid x=3t,\ y=3/t$ where $t\in N_0 \}$
$N_0=\{ x \mid 0<|x|\le 2 \}$

Three dimensional graphics

In the following discussions it will be demonstrated how to construct curves and surfaces in three dimensional space. In order to examine a curve or surface in three dimensions one will need a computer program capable of taking equations and plotting them.

In addition to a graphics program one will need to know about the exponential functions and hyperbolic functions. We will use the exponential functions $y = e^x$ and $y = e^{-x}$ where $e = 2.71828182845904523 54\ldots$ is an irrational number known as Euler's number. Graphs of these functions are illustrated in the figure below.

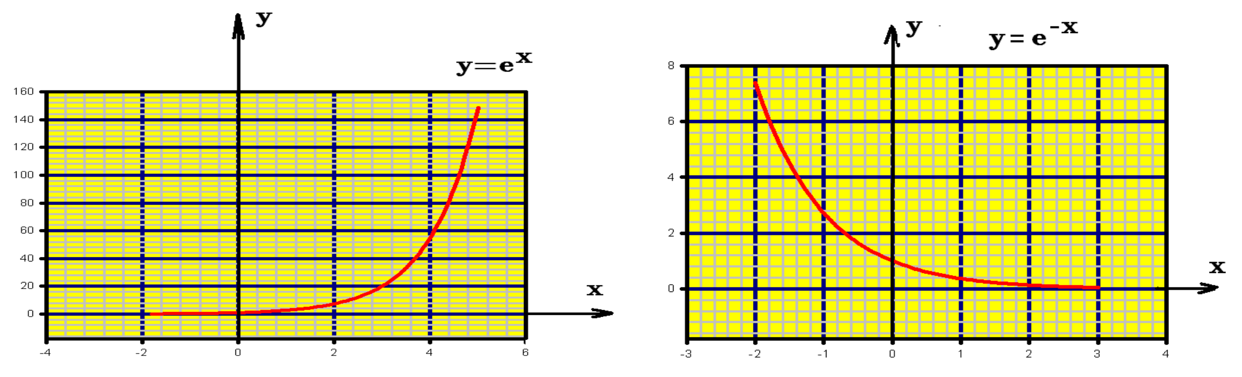


Figure 13-18. The exponential functions e^x and e^{-x}

The irrational number $2.71828182845904523 54\ldots$ occurs so often in mathematics it is given the symbol e just like $3.14159265358979323 85\ldots$ was given the symbol π. The irrational number e can be calculated from any of the limits

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e, \qquad \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}, \qquad e = \sum_{m=0}^{\infty}\frac{1}{m!} = 1 + 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2\cdot 1} + \cdots$$

where $0!$ is defined to be 1.

Associated with exponential functions e^x and e^{-x} are the hyperbolic functions defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

and the reciprocal functions

$$\operatorname{csch} x = \frac{1}{\sinh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}$$

Graphs of these functions are illustrated below.

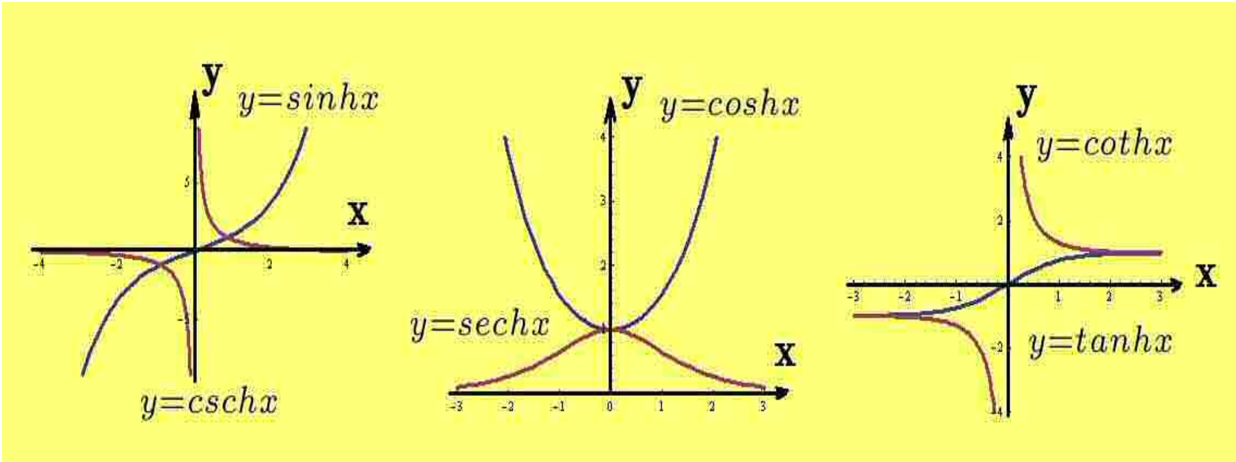


Figure 13-19. The hyperbolic functions

The hyperbolic functions have the properties

$$\cosh^2 x - \sinh^2 x = 1, \qquad \operatorname{sech}^2 x + \tanh^2 x = 1, \qquad \coth^2 x - \operatorname{csch}^2 x = 1$$

Surfaces and curves

A three dimensional surface can be represented by the implicit equation of the form $f(x,y,z) = 0$. If one can solve for z, then an explicit form for representing the surface is $z = F(x,y)$. For example, the unit sphere is given by $x^2 + y^2 + z^2 - 1 = 0$ which is an implicit equation. By solving for z the unit sphere can be represented by the upper hemisphere $z = +\sqrt{1 - x^2 - y^2}$ and the lower hemisphere $z = -\sqrt{1 - x^2 - y^2}$ which are explicit equations. If $z = 0$, then a curve in the xy-plane results. Curves in xyz space can be defined as the intersection of surfaces or by defining a set of parametric equations.

Curves

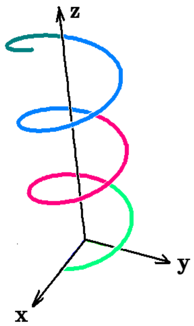
The parametric form for the representation of a curve in space is to specify the coordinates (x,y,z) as a function of a **single variable**, say t, and write

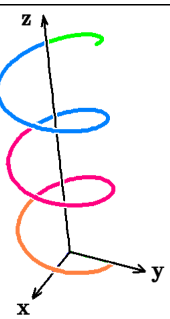
$$x = x(t), \qquad y = y(t), \qquad z = z(t)$$

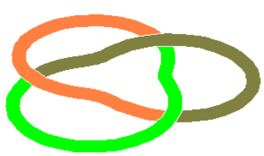
The vector form for the curve is then

$$\vec{r} = \vec{r}(t) = x(t)\, \widehat{\mathbf{e}}_1 + y(t)\, \widehat{\mathbf{e}}_2 + z(t)\, \widehat{\mathbf{e}}_3$$

Some example three dimensional curves are illustrated below.

Right-handed Helix	Parametric Equations	Parameters
	$x = r\cos\theta$ $y = r\sin\theta$ $z = \alpha\theta$	$0 \le \theta \le 2\pi$

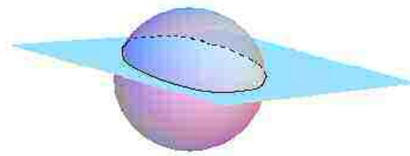
Left-handed Helix	Parametric Equations	Parameters
	$x = r\sin\theta$ $y = r\cos\theta$ $z = \alpha\theta$	$0 \le \theta \le 2\pi$

(p,q) Torus knot	Parametric Equations	Parameters
 **(2,3)**	$x = (a\sin(q\theta) + b)\sin(p\theta)$ $y = (a\sin(q\theta) + b)\cos(p\theta)$ $z = a\cos(q\theta)$ $b > a$	$0 \le \theta \le 2\pi$

Example 13-5.

A plane intersects a sphere to form a circle. Find the equation of the circle.

Solution



Let (x_s, y_s, z_s) denote the center of the sphere having a radius R. The equation of the sphere can then be expressed in the scalar form

$$(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2 = R^2$$

or the vector form

$$|\vec{r} - \vec{r}_s| = R, \quad \vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3, \quad \vec{r}_s = x_s\,\widehat{\mathbf{e}}_1 + y_s\,\widehat{\mathbf{e}}_2 + z_s\,\widehat{\mathbf{e}}_3$$

The equation of the plane intersecting the sphere can be expressed in the scalar form

$$N_1x + N_2y + N_3z + D = 0 \tag{13.45}$$

or the vector form

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = 0, \quad \vec{r}_0 = x_0\,\widehat{\mathbf{e}}_1 + y_0\,\widehat{\mathbf{e}}_2 + z_0\,\widehat{\mathbf{e}}_3, \quad \vec{N} = N_1\,\widehat{\mathbf{e}}_1 + N_2\,\widehat{\mathbf{e}}_2 + N_3\,\widehat{\mathbf{e}}_3$$

where $\vec{r}_0$ is the position vector to a point in the plane and $\vec{N}$ is a normal vector to the plane and $D = -\vec{r}_0 \cdot \vec{N}$ is a constant.

When the normal vector $\vec{N}$ is placed at the origin of the sphere, then a line through the center of the sphere (x_s, y_s, z_s) and the center of the circle (x_c, y_c, z_c) has the form

$$\vec{r} = \vec{r}(\lambda) = \vec{r}_s + \lambda\,\vec{N}$$

where λ is a parameter. The line can also be represented by the parametric equations

$$x = x_s + \lambda N_1, \quad y = y_s + \lambda N_2, \quad z = z_s + \lambda N_3 \tag{13.46}$$

This line intersects the plane when the equations (13.46) are substituted into the equation (13.45) to obtain

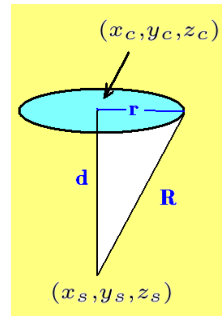
$$N_1(x_s + \lambda N_1) + N_2(y_s + \lambda N_2) + N_3(z_s + \lambda N_3) + D = 0 \tag{13.47}$$

which implies λ must have the value

$$\lambda^* = \frac{-(N_1x_s + N_2y_s + N_3z_s + D)}{N_1^2 + N_2^2 + N_3^2} = \frac{-\vec{N} \cdot \vec{r}_s - D}{\vec{N} \cdot \vec{N}} \tag{13.48}$$

The line from the center of the sphere is normal to the plane of the circle and intersects the center of circle at the point (x_c, y_c, z_c) given by

$$x_c = x_s + \lambda^* N_1, \quad y_c = y_s + \lambda^* N_2, \quad z_c = z_s + \lambda^* N_3$$



We know the radius of the sphere and can calculated the distance d between the center of the sphere and center of the circle. This distance is

$$d = |\vec{r}_c - \vec{r}_s| = \sqrt{(x_c - x_s)^2 + (y_c - y_s)^2 + (z_c - z_s)^2}$$

and using the Pythagorean theorem one can determine the radius of the circle as

$$r = \sqrt{R^2 - d^2} \tag{13.49}$$

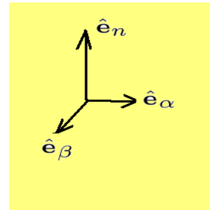
A vector $\vec{a} = a_1 \,\widehat{\mathbf{e}}_1 + a_2 \,\widehat{\mathbf{e}}_2 + a_3 \,\widehat{\mathbf{e}}_3$ is perpendicular to the vector $\vec{N}$ if

$$\vec{N} \cdot \vec{a} = N_1 a_1 + N_2 a_2 + N_3 a_3 = 0 \tag{13.50}$$

Assume $N_3 \neq 0$ and

(i) Select positive values for a_1 and a_2
(ii) Select $a_3 = \dfrac{-N_1 a_1 - N_2 a_2}{N_3}$
The vector $\vec{a}$ will then be perpendicular to the normal vector $\vec{N}$.

Using the above information one can construct the following unit vectors.



$$\begin{aligned}\hat{\mathbf{e}}_\alpha =&\frac{\vec{a}}{|\vec{a}|} = \frac{a_1\,\hat{\mathbf{e}}_1 + a_2\,\hat{\mathbf{e}}_2 + a_3\,\hat{\mathbf{e}}_3}{\sqrt{a_1^2+a_2^2+a_3^2}} = \alpha_1\,\hat{\mathbf{e}}_1 + \alpha_2\,\hat{\mathbf{e}}_2 + \alpha_3\,\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_n =&\frac{\vec{N}}{|\vec{N}|} = \frac{N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3}{\sqrt{N_1^2+N_2^2+N_3^2}} = n_1\,\hat{\mathbf{e}}_1 + n_2\,\hat{\mathbf{e}}_2 + n_3\,\hat{\mathbf{e}}_3\end{aligned}$$

and then use the vector cross product to define the unit vector

$$\hat{\mathbf{e}}_\beta = \hat{\mathbf{e}}_\alpha \times \hat{\mathbf{e}}_n = \beta_1\,\hat{\mathbf{e}}_1 + \beta_2\,\hat{\mathbf{e}}_2 + \beta_3\,\hat{\mathbf{e}}_3$$

Note the unit vectors $\hat{\mathbf{e}}_n$, $\hat{\mathbf{e}}_\beta$, $\hat{\mathbf{e}}_\alpha$ form a right-handed system and the unit vectors $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\beta$ can be used to define the circle created by the plane and sphere intersecting. The equation of the circle is given by

$$\vec{r} = \vec{r}_c + r\cos\theta\;\hat{\mathbf{e}}_\alpha + r\sin\theta\;\hat{\mathbf{e}}_\beta, \qquad 0 \le \theta \le 2\pi$$

This circle can also be represented using the parametric form

$$\begin{aligned}x =&x_c + r\cos\theta\,\alpha_1 + r\sin\theta\,\beta_1\\ y =&y_c + r\cos\theta\,\alpha_2 + r\sin\theta\,\beta_2\\ z =&z_c + r\cos\theta\,\alpha_3 + r\sin\theta\,\beta_3\end{aligned} \tag{13.51}$$

■

Surfaces

The parametric form for the representation of a surface is to have three equations

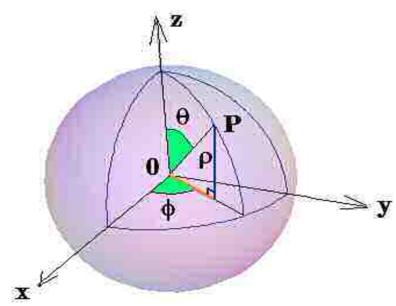
$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$$

in terms of **two parameters**, say u, v, which define the points (x, y, z) on the surface as u and v range over a set of specified values. The vector form for representing the surface is

$$\vec{r} = \vec{r}(u,v) = x(u,v)\;\hat{\mathbf{e}}_1 + y(u,v)\;\hat{\mathbf{e}}_2 + z(u,v)\;\hat{\mathbf{e}}_3$$

Sphere

To find the parametric equations for the representation of a sphere having a radius ρ. Let $u = \theta$ and $v = \phi$ from spherical coordinates be the parameters for the representation of the sphere. For P an arbitrary point on the surface of the sphere one finds the projection of the line segment $\overline{0P}$ onto the xy-plane is $\rho \sin\theta$.



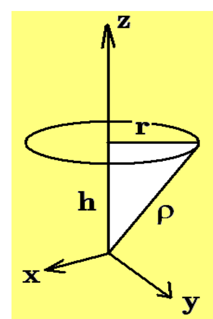
When $\rho \sin\theta$ is projected onto first the x-axis and then the y-axis one finds

$$x = \rho \sin\theta \cos\phi, \qquad y = \rho \sin\theta \sin\phi$$

The projection of $\overline{0P}$ onto the z-axis gives $z = \rho \cos\theta$. This give the parametric equations defining the sphere as

$$x = \rho \sin\theta \cos\phi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho \cos\theta \tag{13.52}$$

where θ is restricted so that $0 \le \theta \le \pi$ and ϕ is restricted so that $0 \le \phi \le 2\pi$.



Note that parametric representations of surfaces are not unique. For example, a plane parallel to the xy-plane will cut the sphere in a circle with radius r. The parametric equation for a circle is

$$x = r \cos t, \quad y = r \sin t$$

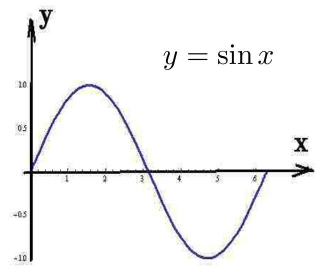
with t a parameter ranging from 0 to 2π. The radius r of the circle associated with a plane intersecting a sphere with radius ρ at distance h above the xy-plane is determined by the Pythagorean theorem as $r = \sqrt{\rho^2 - h^2}$. This gives an alternative parametric representation for the sphere as

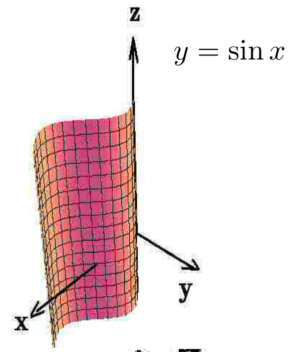
$$x = \sqrt{\rho^2 - h^2}\,\cos t, \quad y = \sqrt{\rho^2 - h^2}\,\sin t, \quad z = h$$

for $-\rho \le h \le \rho$ and $0 \le t \le 2\pi$.

Cylinders

An equation $y = f(x)$ is a curve in the two dimensional plane of the xy-axes. In three dimensions $y = f(x)$ is a cylindrical surface for $a \le z \le b$. For example, the curve given by $y = \sin x$ in two dimensions is a curve, but in three dimensions it is a cylindrical surface with $a \le z \le b$. If the range for z is not specified it is an infinite cylindrical surface in the z-direction. A circle $x^2 + y^2 = r^2$ becomes an infinite cylinder in the z direction when moving the equation from two dimensions to three dimensions.

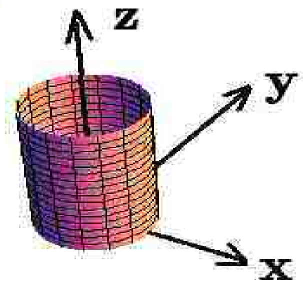




The parametric equation for a cylinder in three dimensions is

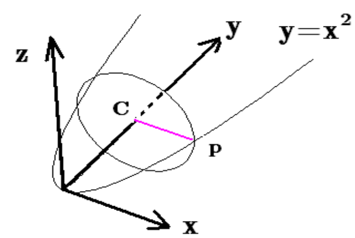
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = h \tag{13.53}$$

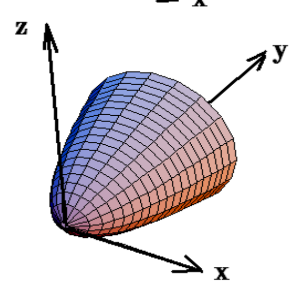
for $0 \le \theta \le 2\pi$ and $0 \le h \le H$.



Surfaces of revolution

An implicit curve $F(x, y) = 0$ or an explicit curve $y = f(x)$ in two dimensions can be rotated about an axis to produce a three dimensional surface. For example, when the parabola $y = x^2$ is revolved about the y-axis every point P on the curve $y = x^2$ moves in a circle with radius $\overline{\mathrm{CP}} = x = \sqrt{y} = r$ and having the equation $x^2 + z^2 = r^2 = y$. The surface generated by a rotation of the parabola is called a paraboloid of revolution. One parametric representation of this surface is





$$x = \sqrt{v}\cos u, \quad y = v, \quad z = \sqrt{v}\sin u, \quad 0 \le u \le 2\pi, \quad 0 \le v \le H \tag{13.54}$$

Example 13-6.

Examine the curves (a) the line segment $y = H$= a constant, (b) the line $y = x$ and (c) the curve $y = \sqrt{x - x_0}$ which are rotated about the x-axis to create a surface of revolution.

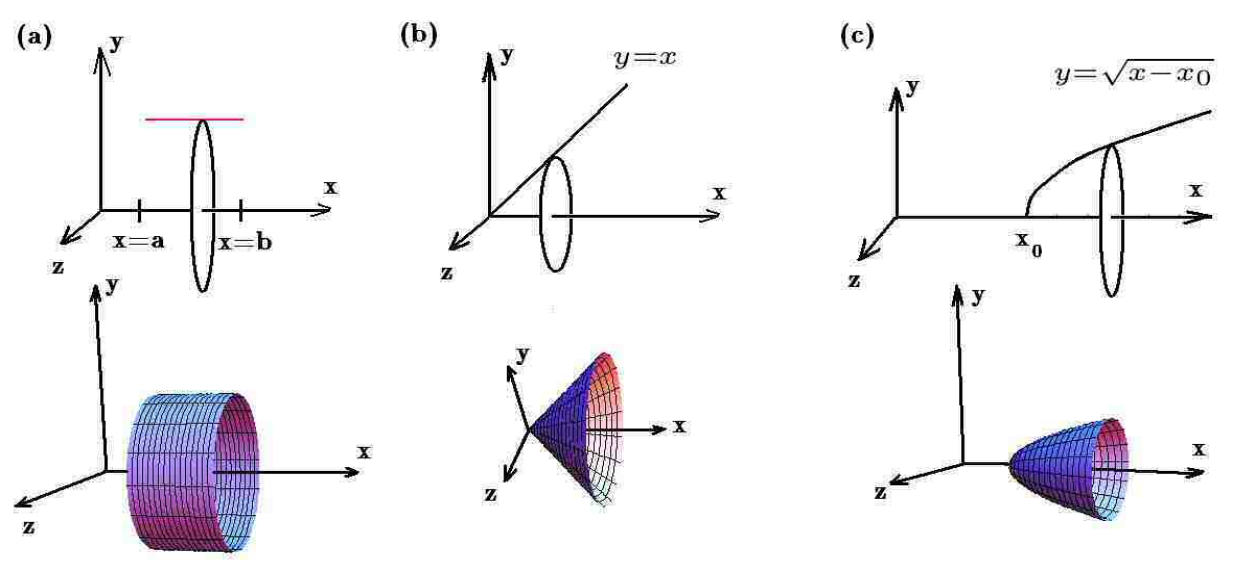


Figure 13-20. Surfaces of revolution

(a) When the line segment is rotated about the x-axis a cylinder results having the equation $y^2+z^2 = H^2$ for $a \le x \le b$. The parametric representation for this surface is

$$x = u, \qquad y = H\cos\theta, \qquad z = H\sin\theta$$

for $a \le u \le b$ and $0 \le \theta \le 2\pi$.

(b) When the line $y = x$ is rotated about the x-axis a right circular cone results having the equation $y^2 + z^2 = x^2$. The parametric equation for this surface is

$$x = u, \qquad y = u\cos t, \qquad z = u\sin t$$

for $0 \le t \le 2\pi$ and $0 \le u \le H$.

(c) When the curve $y = f(x) = \sqrt{x - x_0}$ is rotated about the x-axis there results the paraboloid $y^2 + z^2 = [f(x)]^2$ or $y^2 + z^2 = x - x_0$. The parametric equation for the representation of this surface is

$$x = u, \qquad y = \sqrt{u - x_0}\cos t, \qquad z = \sqrt{u - x_0}\sin t$$

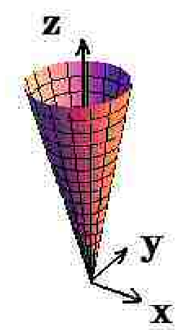
for $0 \le t \le 2\pi$ and $x_0 \le u \le H$.

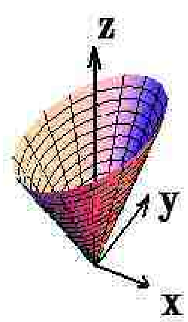
Note that the parametric equations are given in terms of two parameters. If one uses algebra to eliminate the parameters, then the equation of the surface in terms of x, y and z results.

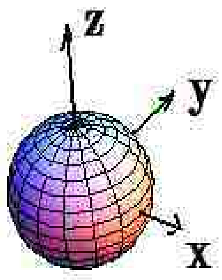
■

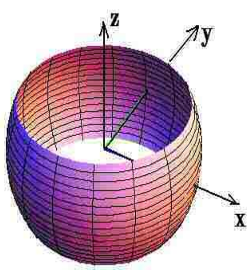
Surfaces

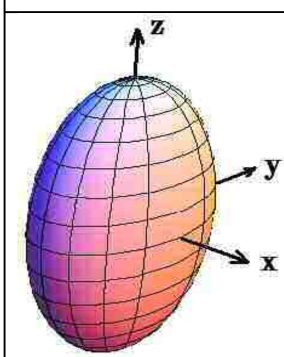
The following are some well known surfaces along with equations which can be used to construct the surfaces. Some of these surfaces are surfaces of revolution. In most of these equations the lower case letters at the beginning of the alphabet such as $a, b, c, \ldots$ are to be considered as constants.

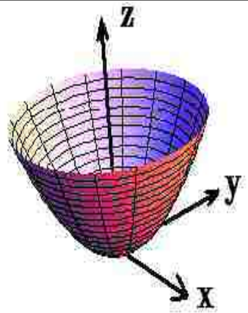
Cone	Equation	Parametric Equations	Parameters
	$x^2 + y^2 = \frac{z^2}{c^2}$	$x = u \cos v$ $y = u \sin v$ $z = cu$	$0 \le v \le 2\pi$ $0 \le u \le 1$

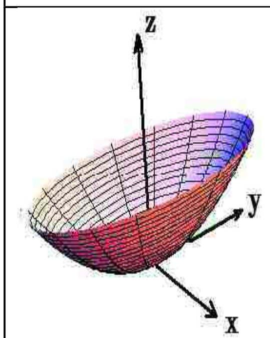
Elliptic Cone	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	$x = \frac{au}{c} \cos v$ $y = \frac{bu}{c} \sin v$ $z = u$	$0 \le v \le 2\pi$ $0 \le u \le 1$

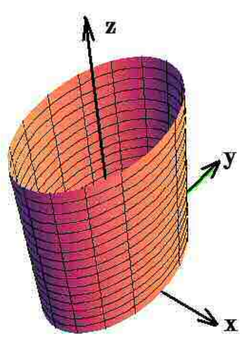
Sphere	Equation	Parametric Equations	Parameters
	$x^2 + y^2 + z^2 = \rho^2$	$x = \rho\cos\phi\sin\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\theta$	$0 \le \theta \le \pi$ $0 \le \phi \le 2\pi$

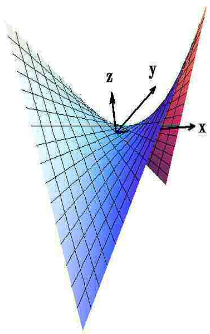
Zone of Sphere	Equation	Parametric Equations	Parameters
	$x^2 + y^2 + z^2 = \rho^2$	$x = \rho\cos\phi\sin\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\theta$	$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$ $0 \le \phi \le 2\pi$

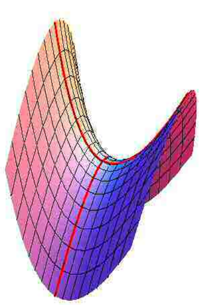
Ellipsoid	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	$x = a\cos\phi\sin\theta$ $y = b\sin\phi\sin\theta$ $z = c\cos\theta$	$0 \le \theta \le \pi$ $0 \le \phi \le 2\pi$

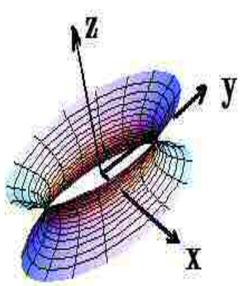
Paraboloid	Equation	Parametric Equations	Parameters
	$x^2 + y^2 = cz$	$x = \sqrt{cv}\cos\phi$ $y = \sqrt{cv}\sin\phi$ $z = v$	$0 \le v \le H$ $0 \le \phi \le 2\pi$

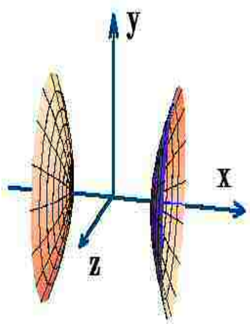
Elliptic Paraboloid	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$	$x = a\sqrt{cv}\cos\phi$ $y = b\sqrt{cv}\sin\phi$ $z = v$	$0 \le v \le H$ $0 \le \phi \le 2\pi$

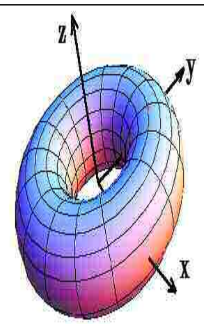
Elliptic Cylinder	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos\phi$ $y = b\sin\phi$ $z = v$	$0 \le \phi \le 2\pi$ $0 \le v \le H$

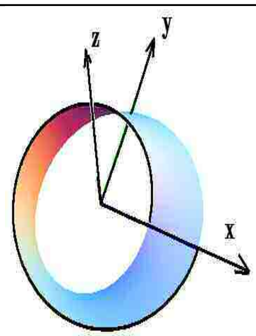
Hyperbolic Paraboloid	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$	$x = \frac{a}{2}(u+v)\sqrt{c}$ $y = \frac{b}{2}(u-v)\sqrt{c}$ $z = uv$	$-d \le u \le d$ $-d \le v \le d$

Hyperbolic Paraboloid	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$	$x = u$ $y = v$ $z = \frac{1}{c}\left(\frac{u^2}{a^2} - \frac{v^2}{b^2}\right)$	$-0.5 \le u \le 0.5$ $-0.5 \le v \le 0.5$ $a = b = 1,\ c = \frac{1}{3}$

Hyperboloid of one sheet	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	$x = a\cos u \cosh v$ $y = b\sin u \cosh v$ $z = c\sinh v$	$0 \le u \le 2\pi$ $-1 \le v \le 1$

Hyperboloid of two sheets	Equation	Parametric Equations	Parameters
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	$x = \pm\, a \cosh v$ $y = b \cos u \sinh v$ $z = c \sin u \sinh v$	$0 \le u \le 2\pi$ $-1 \le v \le 1$

Torus	Equation	Parametric Equations	Parameters
	$(R - \sqrt{x^2 + y^2})^2 + z^2 = r^2,$ $R > r$	$x = (R + r \cos v) \cos u$ $y = (R + r \cos v) \sin u$ $z = r \sin v$	$0 \le u \le 2\pi$ $0 \le v \le 2\pi$

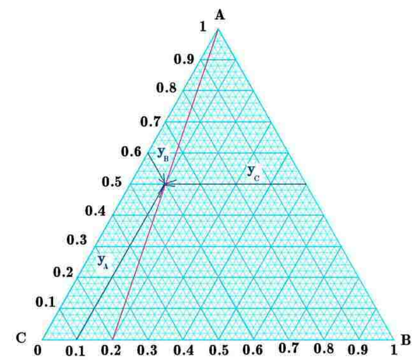
Möbius Strip	Equation	Parametric Equations	Parameters
	$F(x, y, z) = 0^*$	$x = (R - v \cos \frac{\theta}{2}) \cos \theta$ $y = (R - v \cos \frac{\theta}{2} v) \sin \theta$ $z = v \sin \frac{\theta}{2}$	$0 \le \theta \le 2\pi$ $-a \le v \le a$

* $F(x, y, z) = -R^2 y + x^2 y + y^3 - 2Rxz - 2x^2 z - 2y^2 z + yz^2 = 0.$ The Möbius strip is a one-sided surface. It is known as a cubic surface with boundaries.

Exercises

▶ **13-1.** Given triangle $\triangle ABC$ with vertices $A(0,0)$, $B(50,100)$, $C(100,-50)$. Find the vertices of the pedal triangle associated with the point $P(26,22)$.

▶ **13-2.**



In chemistry when one studies reactions involving three quantities A, B, C one can use an equilateral triangle with equal spaced lines drawn parallel to each side. This creates triangular coordinates which can be used to represent concentrations called mole fractions. A point P with coordinates $(y_A, y_B, y_C) = (0.5, 0.1, 0.4)$ is illustrated in the accompanying figure.
(a) Show that when using triangular coordinates $y_A + y_B + y_C = 1$. That is prove the sum of the distances associated with a point in triangular coordinate equals the length of one side of the triangle.
(b) One important property of triangular coordinates is associated with straight lines connecting a vertex and opposite edge as the red line illustrated. The significance of these special lines is that as the concentration of A increases the quantities B and C will always remain in the same initial proportion. To show this use similar triangles associated with any two different points p_1 and p_2 on the red line and show $\dfrac{y_{B1}}{y_{C1}} = \dfrac{y_{B2}}{y_{C2}}$

▶ **13-3.** How many triangular graphs are inside a hexagonal graph?

▶ **13-4.** Plot the following curves in polar coordinates.

$$(a) \quad r = \sin\theta \qquad (b) \quad r = \cos\theta$$

▶ **13-5.**
(a) Plot the points $(2, \frac{\pi}{6})$ and $(\sqrt{3}, \frac{\pi}{3})$ in polar coordinates.
(b) Find the distance between these points.

▶ **13-6.** Sketch the following conic sections

$$(a) \quad x^2 + y^2 = r^2 \qquad (b) \quad (x-h)^2 + (y-k)^2 = r^2$$

▶ **13-7.** Sketch the following conic sections

$$(a) \quad y = x^2 \qquad (b) \quad x = y^2$$

▶ **13-8.** Sketch the following conic sections

$$(a) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b \qquad (b) \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b$$

▶ **13-9.** Sketch the following conic sections

$$(a) \quad \frac{x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1, \quad a > b \qquad (b) \quad \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$$

▶ **13-10.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad y^2 + z^2 = \frac{x^2}{c^2} \qquad (b) \quad z^2 + x^2 = \frac{y^2}{c^2}$$

▶ **13-11.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} = \frac{x^2}{c^2} \qquad (b) \quad \frac{z^2}{a^2} + \frac{x^2}{b^2} = \frac{y^2}{c^2}$$

▶ **13-12.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} + \frac{x^2}{c^2} = 1 \qquad (b) \quad \frac{z^2}{a^2} + \frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$$

▶ **13-13.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad y^2 + z^2 = cx \qquad (b) \quad z^2 + x^2 = cy$$

▶ **13-14.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} = cx \qquad (b) \quad \frac{z^2}{a^2} + \frac{x^2}{b^2} = cy$$

▶ **13-15.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \qquad (b) \quad \frac{z^2}{a^2} + \frac{x^2}{b^2} = 1$$

▶ **13-16.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad \frac{y^2}{a^2} - \frac{z^2}{b^2} = cx \qquad (b) \quad \frac{z^2}{a^2} - \frac{x^2}{b^2} = cy$$

▶ **13-17.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} - \frac{x^2}{c^2} = 1 \qquad (b) \quad \frac{z^2}{a^2} + \frac{x^2}{b^2} - \frac{y^2}{c^2} = 1$$

▶ **13-18.** Sketch the following surfaces indicating the x, y, z axes and then describe the surface.

$$(a) \quad \frac{y^2}{a^2} - \frac{z^2}{b^2} - \frac{x^2}{c^2} = 1 \qquad (b) \quad \frac{z^2}{a^2} - \frac{x^2}{b^2} - \frac{y^2}{c^2} = 1$$

APPENDIX A
Units of Measurement

The following units, abbreviations and prefixes are from the Système International d'Unitès (designated SI in all Languages.)

Prefixes.

Abbreviations		
Prefix	Multiplication factor	Symbol
exa	10^{18}	W
peta	10^{15}	P
tera	10^{12}	T
giga	10^{9}	G
mega	10^{6}	M
kilo	10^{3}	K
hecto	10^{2}	h
deka	10	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p
femto	10^{-15}	f
atto	10^{-18}	a

Basic Units.

Basic units of measurement		
Unit	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	$^\circ$K
Luminous intensity	candela	cd

Supplementary units		
Unit	Name	Symbol
Plane angle	radian	rad
Solid angle	steradian	sr

DERIVED UNITS		
Name	Units	Symbol
Area	square meter	m^2
Volume	cubic meter	m^3
Frequency	hertz	Hz (s^{-1})
Density	kilogram per cubic meter	kg/m^3
Velocity	meter per second	m/s
Angular velocity	radian per second	rad/s
Acceleration	meter per second squared	m/s^2
Angular acceleration	radian per second squared	rad/s^2
Force	newton	N $(kg \cdot m/s^2)$
Pressure	newton per square meter	N/m^2
Kinematic viscosity	square meter per second	m^2/s
Dynamic viscosity	newton second per square meter	$N \cdot s/m^2$
Work, energy, quantity of heat	joule	J $(N \cdot m)$
Power	watt	W (J/s)
Electric charge	coulomb	C $(A \cdot s)$
Voltage, Potential difference	volt	V (W/A)
Electromotive force	volt	V (W/A)
Electric force field	volt per meter	V/m
Electric resistance	ohm	Ω (V/A)
Electric capacitance	farad	F $(A \cdot s/V)$
Magnetic flux	weber	Wb $(V \cdot s)$
Inductance	henry	H $(V \cdot s/A)$
Magnetic flux density	tesla	T (Wb/m^2)
Magnetic field strength	ampere per meter	A/m
Magnetomotive force	ampere	A

Physical Constants:

- $4 \arctan 1 = \pi = 3.14159\,26535\,89793\,23846\,2643\,\ldots$
- $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = 2.71828\,18284\,59045\,23536\,0287\,\ldots$
- Euler's constant $\gamma = 0.57721\,56649\,01532\,86060\,6512\,\ldots$
- $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$ Euler's constant
- Speed of light in vacuum $= 2.997925(10)^8\; m\; s^{-1}$
- Electron charge $= 1.60210(10)^{-19}\, C$
- Avogadro's constant $= 6.0221415(10)^{23}\, mol^{-1}$
- Plank's constant $= 6.6256(10)^{-34}\, J\, s$
- Universal gas constant $= 8.3143\, J\, K^{-1}\, mol^{-1} = 8314.3\, J\, Kg^{-1}\, K^{-1}$
- Boltzmann constant $= 1.38054(10)^{-23}\, J\, K^{-1}$
- Stefan–Boltzmann constant $= 5.6697(10)^{-8}\, W\, m^{-2}\, K^{-4}$
- Gravitational constant $= 6.67(10)^{-11}\, N\, m^2 kg^{-2}$

Index