Solutions Final Exam

No.1 Method 1: From finite Fourier Cosine Transform table we find

$$x \sim \frac{L}{2} + \frac{2}{L} \sum_{n=1}^{\infty} \left(\frac{L}{n\pi} \right)^2 [(-1)^n - 1] \cos \frac{n\pi x}{L}$$

so that with $L = \pi$ we have

$$x \sim \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos nx \tag{1}$$

Method 2: Use the definition of Fourier Cosine series

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

where for $f(x)$ even

$$A_0 = \frac{1}{L} \int_0^L x \, dx = \frac{1}{L} \left. \frac{x^2}{2} \right]_0^L = \frac{L}{2}$$

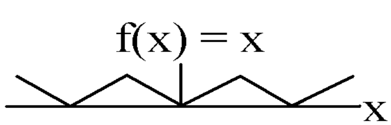
and

$$A_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} \, dx$$

Integrate by parts twice to obtain

$$A_n = \frac{2}{L} \left(\frac{L}{n\pi} \right)^2 [(-1)^n - 1]$$

For $L = \pi$ we obtain the same answer as equation (1).



No.2 Solve $\frac{d^2y}{dx^2} + \lambda y = 0$ over interval $0 < x < 1$ subject to the boundary conditions $y(0) = 0$ and $y'(1) = 0$ There are no solutions for $\lambda = 0$ and $\lambda = -\omega^2$. For $\lambda = \omega^2$ we obtain the solution

$$y = A\cos\omega x + B\sin\omega x$$

The first condition $y(0) = 0$ requires that $A = 0$. The second condition $y'(1) = 0$ requires that $B\omega\cos\omega = 0$ This gives the values

$$\omega = \omega_n = (2n-1)\pi/2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

(Many of you don't know where $cos\theta = 0$) Therefore,

$$\begin{aligned}
\text{Eigenvalues:} &\quad \omega_n^2 = (2n-1)^2\pi^2/4, \quad n = 1, 2, 3, \ldots \\
\text{Eigenfunctions:} &\quad y_n = y_n(x) = \sin\omega_n x = \sin\frac{(2n-1)\pi x}{2} \\
\text{Orthogonality:} &\quad (y_n, y_m) = \int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx = \begin{cases} 0, & m \neq n \\ ||y_n||^2, & m = n \end{cases}
\end{aligned}$$

where

$$||y_n||^2 = \int_0^1 \sin^2\frac{(2n-1)\pi x}{2}\,dx = \frac{1}{2}$$

No.3 Solve $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0,t) = T_0$ and $\frac{\partial u(L,t)}{\partial x} = 0$ and initial condition $u(x,0) = f(x)$. Assume a solution $u(x,t) = w(x,t) + h(x,t)$ and substitute into the PDE to obtain

$$\begin{aligned}
\frac{\partial w}{\partial t} + \frac{\partial h}{\partial t} =& K\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 h}{\partial x^2}\right) \\
w(0,t) + h(0,t) =& T_0 \\
\frac{\partial w(L,t)}{\partial x} + \frac{\partial h(L,t)}{\partial x} =& 0 \\
w(x,0) + h(x,0) =& f(x)
\end{aligned}$$

The equilibrium temperature is given by $\frac{\partial^2 h}{\partial x^2} = 0$ so that we can shift the non-homogeneous terms over to the equilibrium solution this gives the system of equations

$$\begin{aligned}
\frac{\partial w}{\partial t} =& K\frac{\partial^2 w}{\partial x^2} - \frac{\partial h}{\partial t} \\
w(0,t) =& 0 \\
\frac{\partial w(L,t)}{\partial x} =& 0 \\
w(x,0) =& f(x) - h(x,0)
\end{aligned}
\qquad\qquad
\begin{aligned}
\frac{\partial^2 h}{\partial x^2} =& 0 \\
h(0,t) =& T_0 \\
\frac{\partial h(L,t)}{\partial x} =& 0
\end{aligned}$$

The solution of the $h(x,t)$ equation is given by $h(x,t) = T_0$ Assume a solution to the $w(x,t)$ equation of the form $w(x,t) = F(x)G(t)$. By separation of variables one obtains the differential equations

$$F''(x) + \omega^2 F(x) = 0 \qquad \text{and} \qquad G'(t) + \omega^2 K G(t) = 0$$

where $F(x)$ is subject to the boundary conditions $F(0) = 0$ and $F'(L) = 0$.

(a) This gives the eigenvalues $\lambda_n = \omega_n^2$ where $\omega_n = \frac{(2n-1)\pi}{2L}$. The eigenfunctions are given by $F_n(x) = \sin\omega_n x$. The solution of the $G(t)$ equation is $G_n(t) = e^{-K\omega_n^2 t}$. By superposition of solutions of the form $u_n(x,t) = F_n(x)G_n(t)$ we obtain the general solution

$$w(x,t) = \sum_{n=1}^{\infty} B_n \sin\omega_n x e^{-K\omega_n^2 t}$$

(b) The $F_n(x)$ solutions are orthogonal functions satisfying the conditions

$$(F_n, F_m) = \int_0^L F_n(x)F_m(x)\,dx = \begin{cases} 0, & m \neq n \\ ||F_n||^2, & m = n \end{cases}$$

where

$$||F_n||^2 = \int_0^L \sin^2\omega_n x\,dx = \frac{L}{2}$$

(c) Combining the solutions for $h(x,t)$ and $w(x,t)$ gives the answer to part (c)

$$u(x,t) = T_0 + \sum_{n=1}^{\infty} B_n \sin\omega_n x e^{-K\omega_n^2 t}$$

where from the intial condition $w(x,0) = f(x) - T_0 = \sum_{n=1}^{\infty} B_n \sin\omega_n x$ This gives the Fourier coefficients $B_n = \frac{2}{L}\int_0^L (f(x) - T_0)\sin\omega_n x\,dx = \frac{\text{inner product}}{\text{Norm squared}}$

(d) In the special case $f(x) = T_1$ the above becomes

$$B_n = \frac{2(T_1 - T_0)}{L}\int_0^L \sin\omega_n x\,dx = \frac{4(T_1 - T_0)}{(2n-1)\pi}$$

with solution

$$u(x,t) = T_0 + \frac{4(T_1 - T_0)}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{(2n-1)\pi x}{L}}{(2n-1)} e^{-K\omega_n^2 t}$$

No. 4 Solve $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}$ where $u(x,0) = f(x)$ Let $\bar{u}(\omega,t) = \bar{u} = \mathcal{F}\{u(x,t); x \to \omega\}$. Now take the Fourier transform of the given PDE to obtain

$$\begin{aligned}\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} =& \mathcal{F}\left\{K\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}\left\{c\frac{\partial u}{\partial x}\right\} \\ \frac{d\bar{u}}{dt} =& -K\omega^2\bar{u} - i\omega c\bar{u}\end{aligned}$$

The transform of the initial condition gives $\mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$ or $\bar{u}(\omega,0) = F(\omega)$ The transformed equation has the solution

$$\bar{u} = \bar{u}(\omega,t) = F(\omega)e^{-K\omega^2 t}e^{-i\omega ct} = F(\omega)G(\omega)$$

which suggests a convolution integral for representation of the solution. We find by the shift theorem that

$$g(x) = \mathcal{F}^{-1}\left\{e^{-K\omega^2 t}e^{-i\omega ct}\right\} = \sqrt{\frac{\pi}{Kt}}e^{-(x+ct)^2/4Kt}$$

The convolution integral reduces to

$$u(x,t) = \frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty} f(\bar{x})e^{-(x-\bar{x}+ct)^2/4Kt}\,d\bar{x}$$

In the special case $f(x) = \delta(x)$ the above reduces to

$$u(x,t) = \frac{1}{\sqrt{4\pi Kt}}e^{-(x+ct)^2/4Kt}$$

A plot of $u(x,t)$ for various values of c shows that the temperature drops at a fixed time with increasing c values. This means more heat is being conducted away by the convection term.

No. 5 Solve $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ for $\frac{\partial u(0,t)}{\partial x} = 0$ and $u(x,0) = f(x)$.

(a) Using separation of variables we assume a solution $u(x,t) = F(x)G(t)$ and obtain the differential equations

$$G'(t) + \omega^2 KG(t) = 0 \qquad \text{and} \qquad F''(x) + \omega^2 F(x) = 0$$

These equations have the solutions

$$G(t) = e^{-\omega^2 Kt} \qquad \text{and} \qquad F(x) = A\cos\omega x + B\sin\omega x$$

This gives solutions of the form

$$u(x,t;\omega) = F(x)G(t) = [A(\omega)\cos\omega x + B(\omega)\sin\omega x]e^{-\omega^2 Kt}$$

By superposition

$$u(x,t) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]e^{-\omega^2 Kt}\,d\omega$$

The condition $\frac{\partial u(0,t)}{\partial x} = 0$ requires that $B(\omega) = 0$. The condition $u(x,0) = f(x)$ requires

$$u(x,0) = f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega$$

This is a Fourier integral representation of $f(x)$ with Fourier coefficient

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$$

This gives the solution $u(x,t) = \displaystyle\int_0^\infty A(\omega)\cos\omega x\, e^{-\omega^2 Kt}\,d\omega$

(b) Using Fourier Cosine Transforms we let $C[u(x,t)] = \bar{u} = \displaystyle\frac{2}{\pi}\int_0^\infty u(x,t)\cos\omega x\,dx$ Now take Cosine transform of given equation to obtain

$$\begin{aligned} C[\frac{\partial u}{\partial t}] =& KC[\frac{\partial^2 u}{\partial x^2}] \\ \frac{d\bar{u}}{dt} =& K\left[-\frac{2}{\pi}\frac{\partial u(0,t)}{\partial x} - \omega^2\bar{u}\right] \\ \frac{d\bar{u}}{dt} + K\omega^2\bar{u} =& 0 \end{aligned}$$

subject to the intial condition $C[u(x,0)] = \bar{u}(\omega,0) = C[f(x)] = F(\omega)$ This differential equation has the solution

$$\bar{u}(\omega,t) = F(\omega)e^{-K\omega^2 t}$$

The inverse Fourier Cosine transform gives the solution

$$u(x,t) = C^{-1}[\bar{u}(\omega,t)] = \int_0^\infty F(\omega)\cos\omega x\, e^{-K\omega^2 t}\,d\omega$$

where

$$F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$$

which is the same form of the solution as in part (a)

No. 6

$$S[1] = \frac{2}{\pi}\int_0^\infty \sin\omega x\,dx$$

doesn't exit because the sine function continues to oscillate forever and the integral does not converge. Sine transform does exist in the limiting sense though.

$$\begin{aligned} S[e^{-\epsilon x}] =& \frac{2}{\pi}\int_0^\infty e^{-\epsilon x}\sin\omega x\,dx \\ =& \frac{2}{\pi}\left[\frac{e^{-\epsilon x}}{\epsilon^2+\omega^2}\left(-\epsilon\sin\omega x - \omega\cos\omega x\right)\right]_0^\infty \\ =& \frac{2}{\pi}\frac{\omega}{\epsilon^2+\omega^2} \end{aligned}$$

and $$\lim_{\epsilon\to 0} S[e^{-\epsilon x}] = \frac{2}{\pi\omega} = S[1]$$

The inverse transform is given by

$$S^{-1}[\frac{2/\pi}{\omega}] = f(x) = \int_0^\infty \frac{2/\pi}{\omega}\sin\omega x\,d\omega = 1$$

from the given hint. i.e. The Fourier integral representation of $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ is given by

$$f(x) = \int_0^\infty (A(\omega)\cos\omega x + B(\omega)\sin\omega x)\,d\omega$$

where

$$\begin{aligned} A(\omega) =& \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos\omega x\,dx \\ B(\omega) =& \frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin\omega x\,dx \end{aligned}$$

The given $f(x)$ is only defined between -1 and 1 so that the above coefficients reduce to

$$\begin{aligned} A(\omega) =& \frac{1}{\pi}\int_{-1}^1 (1)\cos\omega x\,dx = \frac{2}{\pi}\frac{\sin\omega}{\omega} \\ B(\omega) =& \frac{1}{\pi}\int_{-1}^1 (1)\sin\omega x\,dx = 0 \end{aligned}$$

This gives the Fourier integral representation

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x\,d\omega$$

Note that when $x = 0$, we have $f(0) = 1$ so that

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega$$