Chapter 9

Miscellaneous Applications

In this chapter we consider selected methods where complex variable techniques can be employed to solve various types of problems. In particular, we consider the summation of series, complex potentials, two-dimensional fluid flow and a special mapping called the Schwarz-Christoffel transformation.

Summation of series

The function \( \pi \cot \pi z = \frac{\cos \pi z}{\sin \pi z} \) has singularities at the points \( z = n \) for \( n = 0, \pm 1, \pm 2, \ldots \) where the function \( \sin \pi z \) is zero. These singularities are simple poles because L'Hopital's rule gives

\[
\lim_{z \to n} \frac{z - n}{\sin \pi z} = \frac{1}{\pi \cos \pi n}
\]

for \( n = 0, \pm 1, \ldots \). If we assume that \( f(z) \) is a meromorphic function which has poles at the points \( z_1, z_2, z_3, \ldots, z_m \) not equal to any of the integer positions on the \( x \)-axis, then we can employ the residue theorem and integrate the function \( g(z) = \pi \cot \pi z f(z) \) around the square contour \( C_N \) bounded by the lines \( y = N + 1/2, y = -N - 1/2, x = N + 1/2, x = -N - 1/2 \), illustrated in the figure 9-1, to obtain the following result

\[
\oint_{C_N} \pi \cot \pi z f(z) \, dz = 2\pi i \left\{ \sum_{n=-N}^{N} \text{Res} \left[ g, n \right] + \sum_{k=1}^{m} \text{Res} \left[ g, z_k \right] \right\}, \quad \text{where } g = \pi \cot \pi z f(z) \quad (9.1)
\]

The residue of \( g = \pi \cot \pi z f(z) \) at the integer value \( z = n \), is given by

\[
\text{Res} \left[ g, n \right] = \lim_{z \to n} (z - n) \pi \cot \pi z f(z) = \lim_{z \to n} \frac{z - n}{\sin \pi z} \lim_{z \to n} \pi \cos \pi z f(z) = f(n) \quad (9.2)
\]

for \( n = 0, \pm 1, \pm 2, \ldots, \pm N \). Now if we can show that

\[
\lim_{N \to \infty} \oint_{C_N} \pi \cot \pi z f(z) \, dz = 0, \quad (9.3)
\]

then as \( N \to \infty \), the equation (9.1) simplifies to give the result

\[
\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \text{Res} \left[ g, z_k \right] \quad \text{where } g = g(z) = \pi \cot \pi z f(z) \quad (9.4)
\]

This is an application of the residue theorem for the summation of an infinite series. If in the limit as \( N \to \infty \) the function \( f(z) \) has an infinite number of poles inside the contour \( C_N \), then we must let \( m \to \infty \) in the equation (9.4).
In order to employ the summation formula given by equation (9.2) we must assume that the functions \( \cot \pi z \) and \( f(z) \) are bounded on the contour \( C_N \) in such a way that the contour integral given by equation (9.3) approaches zero as \( N \) increases without bound. Toward this purpose we consider the following three inequalities for the magnitude of \( \cot \pi z \) when \( z \) is on the contour \( C_N \).

(i) For \( z = N + \frac{1}{2} + iy \) with \(-1/2 \leq y \leq 1/2\) we have

\[
|\cot \pi z| = |\cot \pi(N + \frac{1}{2} + iy)| = |\cot(\frac{\pi}{2} + i\pi y)| = |\tanh \pi y| \leq \tanh \frac{\pi}{2}
\]

and for \( z = -N - \frac{1}{2} + iy \) and \(-1/2 \leq y \leq 1/2\) we have

\[
|\cot \pi z| = |\cot (-N - \frac{1}{2} + iy)| = |\cot(\frac{\pi}{2} - i\pi y)| = |\tanh \pi y| \leq \tanh \frac{\pi}{2}
\]

(ii) For \( y > 1/2 \), we have

\[
|\cot \pi z| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} \right| = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = \coth \frac{\pi}{2}
\]

(iii) For \( y < -1/2 \), we have

\[
|\cot \pi z| = |\cot(-N - \frac{1}{2} + iy)| = |\cot(\frac{\pi}{2} - i\pi y)| = |\tanh \pi y| \leq \tanh \frac{\pi}{2}
\]
An examination of the graphs of the hyperbolic tangent and hyperbolic cotangent function shows that at \( x = \pi/2 \) we have the inequality \( \coth \pi/2 > \tanh \pi/2 \). Therefore, one can say that the function \( \cot \pi z \) is bounded for values of \( z \) on the contour \( C_N \) and one can write \( |\cot \pi z| \leq \coth \pi/2 \). We also assume that the modulus of the function \( f(z) \), for \( z \) on the contour \( C_N \), satisfies the inequality \( |f(z)| \leq \frac{M}{|z|^k} \) where \( M \) and \( k > 1 \) are constants.

The above assumptions allow us to calculate the magnitude of the contour integral on the left-hand side of equation (9.1) in the limit as \( N \) increases without bound. One finds

\[
\lim_{N \to \infty} \left| \oint_{C_N} \pi \cot \pi z f(z) \, dz \right| \leq \lim_{N \to \infty} \oint_{C_N} \pi |\cot \pi z| |f(z)| \, dz \leq \lim_{N \to \infty} \frac{M}{N^k} (8N + 4) \coth \frac{\pi}{2} = 0
\]

where \( k > 1 \) and \( 8N + 4 \) is the length of the closed contour \( C_N \).

In the above development of a summation formula for infinite series, by use of the residue theorem and special contour of integration, we employed the function \( \pi \cot \pi z \). This function can be replaced by other types of functions having singularities on the real axis. Some example summation formulas are

\[
\sum_{n = -\infty}^{\infty} (-1)^n f(n) = -\sum_k \text{Res} [\pi \csc \pi z f(z), z_k] \quad (9.5)
\]

\[
\sum_{n = -\infty}^{\infty} \frac{2n + 1}{2} f\left(\frac{2n + 1}{2}\right) = \sum_k \text{Res} [\pi \tan \pi z f(z), z_k] \quad (9.6)
\]

\[
\sum_{n = -\infty}^{\infty} (-1)^n f\left(\frac{2n + 1}{2}\right) = \sum_k \text{Res} [\pi \sec \pi z f(z), z_k] \quad (9.7)
\]

where the summation \( \sum_k \) denotes a summation over all the poles of \( f(z) \). These summation formula are derived in a manner very similar to that of equation (9.4).

**Example 9-1.** (Summation of series)

For \( S = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots \), use the results from the equation (9.4) to find \( S \).

**Solution:** Let \( f(z) = \frac{1}{z^2 + 1} \) and \( g(z) = \pi \cot \pi z f(z) = \frac{\pi \cot \pi z}{z^2 + 1} \). The function \( f(z) \) has simple poles at \( z_1 = i \) and \( z_2 = -i \) and so the residues of \( g(z) \) at these poles are given by

\[
\text{Res} [g(z), i] = \lim_{z \to i} (z - i) \frac{\pi \cot \pi z}{z^2 + 1} = \frac{\pi \cot \pi z}{z + i} \bigg|_{z = i} = -\frac{\pi}{2} \coth \pi
\]

\[
\text{Res} [g(z), -i] = \lim_{z \to -i} (z + i) \frac{\pi \cot \pi z}{z^2 + 1} = \frac{\pi \cot \pi z}{z - i} \bigg|_{z = -i} = -\frac{\pi}{2} \coth \pi
\]
Using equation (9.4) we obtain
\[\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} = -2 \sum_{k=1}^{\infty} \text{Res}[g, z_k] = \pi \coth \pi\]

Note that \(f(-n) = f(n)\) and so \(f(n)\) is an even function of \(n\). Consequently, one can write
\[\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=\infty}^{0} f(n) + f(0) + \sum_{n=1}^{\infty} f(n) = \pi \coth \pi\] (9.8)

In the first summation in equation (9.8) replace \(n\) by \(-n\) and write
\[\sum_{n=1}^{\infty} f(n) + \sum_{n=1}^{\infty} f(n) = 2 \sum_{n=1}^{\infty} f(n) + 1 = \pi \coth \pi\]

which simplifies to
\[\sum_{n=1}^{\infty} f(n) = \frac{1}{2} + \frac{1}{2} (\pi \coth \pi - 1)\]

The argument principle
Recall that if \(f(z)\) has a zero of order \(n_1\) at the point \(z = z_1\), then \(f(z)\) can be expressed in the form
\[f(z) = (z - z_1)^{n_1} G(z)\] (9.9)

where \(G(z)\) is analytic in some neighborhood of the point \(z_1\). The function \(f(z)\) has the derivative
\[f'(z) = n_1 (z - z_1)^{n_1-1} G(z) + (z - z_1)^{n_1} G'(z)\] (9.10)

Note that the ratio \(f'(z)/f(z)\) can be expressed
\[\frac{f'(z)}{f(z)} = \frac{n_1}{z - z_1} + \frac{G'(z)}{G(z)}\] (9.11)

If we construct a small circle of radius \(r_1\) about the point \(z_1\) where \(\frac{G'(z)}{G(z)}\) is analytic everywhere inside the small circle, then we can define the contour integral
\[I_1 = \frac{1}{2\pi i} \oint_{|z-z_1|=r_1} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{|z-z_1|=r_1} \left[ \frac{n_1}{z - z_1} + \frac{G'(z)}{G(z)} \right] dz = n_1\] (9.12)

In equation (9.12) the first integral gives the order \(n_1\) and the second integral is zero because of the Cauchy theorem.

Similarly, if \(f(z)\) has a pole at the point \(z = \zeta_1\), which is a pole of order \(m_1\), then \(f(z)\) can be represented in the form
\[f(z) = \frac{c_{m_1}}{(z - \zeta_1)^{m_1}} + \cdots + \frac{c_1}{z - \zeta_1} + b_0 + b_1(z - \zeta_1) + \cdots = \frac{H(z)}{(z - \zeta_1)^{m_1}}\] (9.13)
This function has the derivative
\[ f'(z) = \frac{H'(z)}{(z - \zeta_1)^{m_1}} - \frac{m_1 H(z)}{(z - \zeta_1)^{m_1+1}} \] (9.14)

If we construct a small circle of radius \( r_2 \) about the point \( z = \zeta_1 \) where \( \frac{H'(z)}{H(z)} \) is analytic everywhere inside the circle, then we can define the integral
\[ P_1 = \frac{1}{2 \pi i} \oint_{|z - \zeta_1| = r_2} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2 \pi i} \oint_{|z - \zeta_1| = r_2} \left[ \frac{H'(z)}{H(z)} - \frac{m_1}{z - \zeta_1} \right] \, dz = -m_1 \] (9.15)

Here the first integral is zero by Cauchy’s theorem and the second integral produces the order \( m_1 \) of the pole with a sign change.

Let \( |z - z_1| = r_1 \) and \( |z - \zeta_1| = r_2 \) be disjoint circles and construct a simple closed path which encircles both of the smaller circles centered at \( z_1 \) and \( \zeta_1 \) so that \( f(z) \) is analytic everywhere inside \( C \) except for the pole at \( \zeta_1 \) or order \( m_1 \). One can then write
\[ I = \frac{1}{2 \pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = I_1 + P_1 = n_1 - m_1 \] (9.16)
which represents the difference between the order of the zero and the order of the pole.

If there are \( N \)-zeros inside the simple closed curve \( C \) at the points \( z_1, z_2, \ldots, z_N \) with multiplicities \( n_1, n_2, \ldots, n_N \) and there are \( M \)-poles inside \( C \) at the points \( \zeta_1, \zeta_2, \ldots, \zeta_M \) of orders \( m_1, m_2, \ldots, m_M \) respectively, then one can surround these zeros and poles with small disjoint circles \( C_{z_j} \) for \( j = 1, 2, \ldots, N \) and \( C_{\zeta_j} \) for \( j = 1, 2, \ldots, M \), and then employ the extended Cauchy theorem to show
\[ \frac{1}{2 \pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{N} \frac{1}{2 \pi i} \oint_{C_{z_j}} \frac{f'(z)}{f(z)} \, dz + \sum_{j=1}^{M} \frac{1}{2 \pi i} \oint_{C_{\zeta_j}} \frac{f'(z)}{f(z)} \, dz \] (9.17)

where \( N \) is the number of zeros of \( f(z) \) (counting multiplicities) inside \( C \) and \( P \) is the summation of the orders associated with the poles of \( f(z) \) within the simple closed contour \( C \).

If \( \omega = f(z) \) with \( d\omega = f'(z) \, dz \), then equation (9.17) can be expressed
\[ \frac{1}{2 \pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \frac{1}{2 \pi i} \oint_{C'} \frac{d\omega}{\omega} \] (9.18)
where \( C' \) is the image of the curve \( C \) under the mapping \( \omega = f(z) \). If \( z = z(t) \), for \( t_0 \leq t \leq t_1 \) is the parametric representation of the curve \( C \), then \( \omega = \omega(t) = f(z(t)) \), for \( t_0 \leq t \leq t_1 \) gives the