Chapter 7

Evaluation of Integrals

The following is an introduction into selected applications where the theory of complex integration can be employed. We begin by studying the evaluation of certain real definite integrals and improper integrals that can be evaluated by using a limiting process associated with the integration of a suitable complex function around a special simple closed path or contour of integration. These special integrals are associated with complex functions \( f(z) \) which take on real values along a portion or on all of the path of integration. In some applications we will require that the contour of integration be allowed to change in a limiting process and along such contours the integrand function \( f(z) \) will at times be required to be bounded. The residue theorem will play an important part in the evaluation of these special integrals and so we restate this important theorem.

**Cauchy Residue Theorem**

If \( f(z) \) is analytic everywhere inside and on a simple closed curve \( C \) except for a finite number of isolated poles \( z_1, z_2, \ldots, z_n \) inside \( C \) and not on the boundary \( C \), then the integration of \( f(z) \) taken around the boundary curve \( C \) in the positive sense is equal to \( 2\pi i \) times the sum of the residues of \( f(z) \) interior to the curve \( C \). This result can be expressed

\[
\int_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k]
\]

In the following sections we define improper integrals and Cauchy principal values associated with integrals. We introduce and investigate special paths of integration in evaluating contour integrals. In the methods developed be sure to note how limiting processes and the residue theorem are employed in the complex domain to evaluate certain real integrals.

**Improper integrals**

When evaluating real integrals of the form \( \int_a^b f(x) \, dx \), \( x \) real, it is usually assumed that \( f(x) \) is well defined at all points \( x \) satisfying \( a \leq x \leq b \). If there exists a point \( \xi \) satisfying \( a \leq \xi \leq b \) where \( f(x) \) is undefined, then one can introduce small positive quantities \( \epsilon \) and \( \eta \) and write the integral as

\[
\int_a^b f(x) \, dx = \lim_{\epsilon \to 0} \int_a^{\xi-\epsilon} f(x) \, dx + \lim_{\eta \to 0} \int_{\xi+\eta}^b f(x) \, dx
\]

provided that the limits exist. If the limits exist, then the integral is called convergent, otherwise it is called divergent. Another way to define the integral is to introduce just one small positive quantity \( \epsilon \) and write
\[ \int_a^b f(x) \, dx = \lim_{\epsilon \to 0} \int_a^{\xi-\epsilon} f(x) \, dx + \int_{\xi+\epsilon}^b f(x) \, dx \]  

(7.2)

provided the limits exist. The limits in equations (7.1) and (7.2) do not always produce the same results. Whenever the integral given by equation (7.2) exists, then the value of the integral is called the Cauchy principal value. Note that the Cauchy principal value is symmetric with the limits decreasing at the same rate, while in comparison the limits in equation (7.1) can decrease at different rates.

The same idea applies to integration involving infinite limits. The integration of rational functions \( f(x) \) over the interval \([0, \infty)\) are called improper integrals and are defined by a limiting process

\[ \int_0^\infty f(x) \, dx = \lim_{R \to \infty} \int_0^R f(x) \, dx \]

if this limit exists. Similarly, the integration of real rational functions \( f(x) \), defined for all real values of \( x \), over the interval \((-\infty, \infty)\) are calculated from the limiting process

\[ \int_{-\infty}^\infty f(x) \, dx = \lim_{R_1 \to \infty} \int_{-R_1}^0 f(x) \, dx + \lim_{R_2 \to \infty} \int_0^{R_2} f(x) \, dx \]  

(7.3)

provided that both limits exist. The limiting process given by equation (7.3) is not the same as the limiting process

\[ \int_{-\infty}^\infty f(x) \, dx = \lim_{R \to \infty} \int_{-R}^R f(x) \, dx \]  

(7.4)

because in one case the limiting process is symmetric and nonsymmetric in the other case. If the limit in equation (7.3) exists, then the same value can be obtained from the limiting process given by equation (7.4). However, if the limit given by equation (7.4) exists, then the limit given by equation (7.3) may or may not exist. This can be illustrated with a simple example. Consider the limiting process given by equation (7.4) in the special case \( f(x) = x \)

\[ \lim_{R \to \infty} \int_{-R}^{R} x \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{2} \, dx = \lim_{R \to \infty} \left[ \frac{x^2}{2} \right]_{-R}^{R} = \lim_{R \to \infty} \left[ \frac{R^2}{2} - \frac{(-R)^2}{2} \right] = 0 \]  

(7.5)

However, the limiting process defined by equation (7.3) for the function \( f(x) = x \) produces

\[ \lim_{R_1 \to \infty} \int_{-R_1}^{0} x \, dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} x \, dx = \lim_{R_1 \to \infty} \int_{-R_1}^{0} \frac{x^2}{2} \, dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} \frac{x^2}{2} \, dx = \lim_{R_1 \to \infty} \left[ \frac{R_1^2}{2} - \frac{(-R_1)^2}{2} \right] + \lim_{R_2 \to \infty} \left[ \frac{R_2^2}{2} - \frac{R_2^2}{2} \right] \]  

(7.6)

which becomes an indeterminate form in the limit as \( R_1 \) and \( R_2 \) increase without bound. The limiting process defined by equation (7.4) is used to define the Cauchy principal value of an improper integral.

The Cauchy principal value of the improper integral \( \int_{-\infty}^{\infty} f(x) \, dx \), of a real rational function \( f(x) \) is defined by the limiting process

\[ \text{Cauchy principal value} \quad \int_{-\infty}^{\infty} f(x) \, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx \]  

(7.7)
provided that the limit exists. If the limit exists, then the integral is said to be a convergent integral in the Cauchy sense, otherwise it is called divergent.

Consider the contour integral of a complex rational function \( f(z) \) taken around the special contour illustrated in the figure 7-1. Assume that the rational function \( f(z) \) has poles at the points \( z_1, z_2, \ldots, z_n \) lying in the upper half plane. The contour of integration consists of two parts. The first part is the upper semicircle \( C_R \) centered at the origin and having a radius \( R \) large enough to enclose all the poles of \( f(z) \) in the upper half plane. The second part of the contour is the straight line segment from \((-R, 0)\) to \((R, 0)\). The integration of \( f(z) \) around this special contour \( C \) is performed in the positive sense. The semicircular part of the path of integration is given the notation \( C_R \) to emphasize that this portion of the path depends upon the radius \( R \).

**Figure 7-1.** Contour path \( C \) consisting of semicircle \( C_R \) and straight line path from \(-R\) to \( R \) on the \( x \)-axis. It is assumed that \( f(z) \) has poles at points \( z_1, z_2, \ldots, z_n \) in the upper half-plane which lie inside the contour \( C \).

Consider the contour integral of \( f(z) \) around the path \( C \) of figure 7-1 in the limit as the radius \( R \) increases without bound. This type of integral can be evaluated using the residue theorem by writing

\[
\lim_{R \to \infty} \oint_C f(z) \, dz = \lim_{R \to \infty} \left[ \int_{-R}^{R} f(x) \, dx + \int_{C_R} f(Re^{i\theta}) \, iRe^{i\theta} \, d\theta \right] = 2\pi i \sum_{j=1}^{n} \text{Res} \left[ f, z_j \right] \tag{7.8}
\]

where \( z = x \) is the value of \( z \) on the line segment and \( z = Re^{i\theta} \) is the value of \( z \) on the semicircular path \( C_R \). If we can show that the line integral along \( C_R \) approaches zero as \( R \) increases without bound, then one can write

\[
\lim_{R \to \infty} \int_{C_R} f(z) \, dz = \lim_{R \to \infty} \int_{0}^{\pi} f(Re^{i\theta}) \, Re^{i\theta} \, d\theta = 0 \tag{7.9}
\]
and consequently the integral in equation (7.8) reduces to the Cauchy principal value of an improper integral

$$\lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx = 2\pi i \sum_{j=1}^{n} \text{Res} \{ f, z_j \} \quad (7.10)$$

If the integrand \( f(z) \), for \( z = \text{Re}^{i\theta} \) on the semicircle \( C_R \), satisfies \( |f(z)| \leq \frac{M}{R^k} \) where \( M \) and \( k \) are constants with \( k > 1 \), then integrals of the type given by equations (7.9) and (7.10) can be shown to be valid. If \( f(z) \) satisfies the above boundedness property, then one can employ the magnitude of a line integral property, equation (3.9), to obtain the result

$$\left| \int_{C_R} f(z) \, dz \right| \leq \frac{M}{R^k} \cdot \pi R = \frac{\pi M}{R^{k-1}}, \quad k > 1 \quad (7.11)$$

where \( \pi R \) is the length of the semicircle arc \( C_R \). It then follows that \( \lim_{R \to \infty} \int_{C_R} f(z) \, dz = 0 \).

Whenever \( f(z) = \frac{P(z)}{Q(z)} \) is a rational function with \( P(z) \) a polynomial of degree \( m \) with real coefficients \( a_0, a_1, \ldots, a_m \) and \( Q(z) \) is a polynomial of degree \( n \geq m + 2 \) with real coefficients \( b_0, b_1, \ldots, b_n \), then one can write

$$P(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_{m-1} z + b_m} = \frac{z^m p(z)}{z^n q(z)}$$

where

$$p(z) = a_0 + a_1 z^{-1} + \cdots + a_{m-1} z^{-m+1} + a_m z^{-m} \quad \text{and} \quad q(z) = b_0 + b_1 z^{-1} + \cdots + b_{n-1} z^{-n+1} + b_n z^{-n}$$

In the special case \( z = \text{Re}^{i\theta} \) lies on the path \( C_R \) in figure 7-1 one finds that

$$|z f(z)| = \left| \frac{z P(z)}{Q(z)} \right| \leq \frac{|R^{m+1}|}{|R^n|} \frac{|p(z)|}{|q(z)|}$$

and in the limit as \( R \) increases without bound we can write

$$\lim_{R \to \infty} \frac{|R^{m+1}|}{|R^n|} \frac{|p(z)|}{|q(z)|} \leq \lim_{R \to \infty} \frac{|R^{m+1} a_0}{R^n b_0} = 0$$

because \( n \geq m + 2 \). Therefore, one can select a radius \( R \) so large that \( \left| \frac{z P(z)}{Q(z)} \right| \leq \epsilon \) which implies the inequality \( \left| \frac{P(z)}{Q(z)} \right| < \frac{\epsilon}{R} \) whenever \( z = \text{Re}^{i\theta} \) is on the semicircle \( C_R \). It then follows from the magnitude of an integral property, equation (3.9), that

$$\left| \int_{C_R} \frac{P(z)}{Q(z)} \, dz \right| \leq \int_{C_R} \frac{\epsilon}{R} |dz| = \frac{\epsilon}{R} \pi R = \pi \epsilon \quad (7.12)$$

In the limit as \( R \) increases without bound the integral on the left-hand side of equation (7.12) is some constant which is less than \( \pi \epsilon \) which can be made arbitrarily small by making \( \epsilon \) small. This implies that the constant on the left-hand side must be zero. Hence, for \( P(z) \) and \( Q(z) \) polynomials of degree \( m \) and \( n \) respectively, with \( n \geq m + 2 \), one can write

$$\lim_{R \to \infty} \int_{C_R} \frac{P(z)}{Q(z)} \, dx = 0 \quad (7.13)$$
Example 7-1. (Evaluation of improper integral)

Evaluate the Cauchy principal value of \( \int_{-\infty}^{\infty} \frac{dx}{\alpha^2 + x^2} \), where \( \alpha > 0 \) is a real constant.

**Solution:** Use the path of integration illustrated in the figure 7-1 and replace the integrand by the complex function \( f(z) = \frac{1}{\alpha^2 + z^2} \). The residue theorem gives

\[
\oint_C f(z) \, dz = 2\pi i \text{Res} \left[ f(z), i\alpha \right]
\]

since only the simple pole at \( z = i\alpha \) lies within the contour \( C \) of figure 7-1. The residue of \( f(z) \) at \( z = i\alpha \) is found to be \( 1/2i\alpha \). In the limit as \( R \to \infty \) we write

\[
\lim_{R \to \infty} \oint_C f(z) \, dz = \lim_{R \to \infty} \left[ \int_{C_R} f(z) \, dz + \int_{-R}^{R} \frac{dx}{\alpha^2 + x^2} \right] = 2\pi i \frac{1}{2i\alpha} = \frac{\pi}{\alpha}
\]

If we can show that \( \lim_{R \to \infty} |\int_{C_R} f(z) \, dz| = 0 \), then we are left with the Cauchy principal value \( \int_{-\infty}^{\infty} \frac{dx}{\alpha^2 + x^2} = \frac{\pi}{\alpha} \). We can use the previous discussion to show \( \lim_{R \to \infty} \int_{C_R} f(z) \, dz = 0 \).

Alternatively, for \( z \) on \( C_R \) we can use the inequality \( |z_1 + z_2| \geq |z_1| - |z_2| \) to show for \( z \) on \( C_R \) that \( |f(z)| = \left| \frac{1}{z^2 + \alpha^2} \right| \leq \left| \frac{1}{R^2 - \alpha^2} \right| \). One can then show \( \lim_{R \to \infty} \int_{C_R} f(z) \, dz \leq \lim_{R \to \infty} \frac{\pi R}{\alpha R^2 - \alpha^2} = 0 \).

Here \( f(z) \) satisfies the boundedness property required for the integral along \( C_R \) to approach zero as \( R \) increases without bound. One can use the rational function \( P(z)/Q(z) \) approach or some other inequality to show a boundedness property on a curve.

Example 7-2. (Evaluation of improper integral)

Evaluate the improper integral \( \int_{0}^{\infty} \frac{dx}{x^6 + 1} \).

**Solution:** Replace the integrand by the complex function \( f = f(z) = \frac{1}{z^6 + 1} \) and use the residue theorem to evaluate the integral of \( f(z) \) around the path \( C \) illustrated in the figure 7-1 in the limit as \( R \) increases without bound. One can then write

\[
\lim_{R \to \infty} \oint_C f(z) \, dz = \lim_{R \to \infty} \left[ \int_{-R}^{R} f(x) \, dx + \int_{C_R} f(R e^{i\theta}) e^{i\theta} \, d\theta \right] = 2\pi i \sum_{k=1}^{n} \text{Res} \left[ f, z_k \right]
\]

where \( n \) represents the number of singularities of \( f(z) \) in the upper half plane. The line integral along the semicircular path \( C_R \) approaches zero as \( R \) increases without bound. Note that if \( |z_1 + z_2| \geq |z_1| - |z_2| \), then \( \frac{1}{|z_1 + z_2|} \leq \frac{1}{|z_1| - |z_2|} \) consequently one can show

\[
\lim_{R \to \infty} \left| \int_{C_R} f(z) \, dz \right| \leq \lim_{R \to \infty} \int_{C_R} \frac{1}{|z^6 + 1|} \, dz \leq \lim_{R \to \infty} \frac{\pi R}{R^6 - 1} = 0
\]

The singularities of \( f(z) \) occur where \( z^6 + 1 = 0 \). We can write

\[
z^6 = -1 = e^{i(\pi + 2k\pi)}, \quad \text{or} \quad z = e^{i(\pi/6 + 2k\pi/6)} \quad \text{where for} \ k = 0, 1, 2, 3, 4, 5 \ \text{we obtain}
\]