Chapter 6

Additional Applications

Applications of the variational calculus can be found in such diverse areas as chemistry, biology, physics, engineering and mechanics. Many of these applications require special knowledge of a subject area in order to formulate a variational problem. In this chapter we consider a few such applications which do not require excessive background material for their development. We also consider numerical methods and approximation techniques for obtaining extreme values associated with functions and functionals.

The vibrating string

A string is placed under tension between given fixed points. The zero displacement state of the string is called the equilibrium position of the string. The string is given an initial displacement from its equilibrium position and then released from rest. We desire to develop the equation describing the motion of the string. We introduce symbols, use some basic physics and use Hamilton's principle to construct the equation of motion.

Let x denote distance with the points $x = 0$ and $x = L$ the fixed end points of the string. Further, let $u = u(x,t)$ [m] denote the string displacement of a general point x, [m] for $0 \le x \le L$, at a time t [sec], then at a general point x at time t the quantities $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ denote respectively the slope and velocity of a general point on the string.

We neglect damping forces and assume that the slope of the string $\frac{\partial u(x,t)}{\partial x}$ at any point x is small such that $\mid \frac{\partial u}{\partial x} \mid << 1$. To employ Hamilton's principle we must calculate the kinetic energy and potential energy for the string. To accomplish this let ϱ [kg/m] denote the lineal mass density of the string so that $\varrho\, dx$ denotes the mass of a string element of length dx. The kinetic energy of the string can then be expressed as a summation of one-half the mass times velocity squared for all elements along the string. This gives the kinetic energy

$$T = \frac{1}{2}\int_0^L \varrho \left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx \qquad \left[\frac{kg}{m}\right]\left[\frac{m}{sec}\right]^2 [m] = [joule] \tag{6.1}$$

The potential energy of the string is written as $V = V_1 + V_2$ where V_1 is the work done in distorting the string from its equilibrium position together with V_2 denoting the work done by the stretching that occurs at a boundary point. Assume that each element of the string experiences a constant stretching force of τ [nt]. The work done in distorting a string element of length dx in its equilibrium position to a length $ds = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx$ at some general time t is given by a force times a distance or

$$dV_1 = \tau(ds - dx) = \tau \left(\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} - 1\right) dx \tag{6.2}$$

We can expand the square root term using a binomial expansion to obtain

$$\left[1+\left(\frac{\partial u}{\partial x}\right)^2\right]^{1/2} = 1+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial u}{\partial x}\right)^4 + \frac{1}{16}\left(\frac{\partial u}{\partial x}\right)^6 - \cdots$$

and if $\mid \frac{\partial u}{\partial x} \mid << 1$ we can neglect higher order terms in this expansion. The equation (6.2) can then be expressed in the simplified form

$$dV_1 = \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{6.3}$$

so that by summing over all values of x we obtain the first part of the potential

$$V_1 = \int_0^L \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2 dx \qquad [nt]\,[m] = [joule] \tag{6.4}$$

There is also work done in stretching the ends of the string. The ends of the string can be thought of as springs with spring constant κ_1 at $x=0$ and spring constant κ_2 at $x=L$. The springs are treated as linear springs so that the spring force is proportional to the displacement. If the displacement of a boundary point is denoted by ξ, then the element of work done by these end point forces is given by force $(\kappa\xi)$ times displacement $(d\xi)$. The work done at the boundaries can therefore be expressed

$$\int_0^{u(0,t)} \kappa_1\xi\, d\xi = \frac{1}{2}\kappa_1 u^2(0,t) \qquad \text{and} \qquad \int_0^{u(L,t)} \kappa_2\xi\, d\xi = \frac{1}{2}u^2(L,t) \tag{6.5}$$

These integrals produce the potential function

$$V_2 = \frac{1}{2}\kappa_1 u^2(0,t) + \frac{1}{2}\kappa_2 u^2(L,t) \tag{6.6}$$

associated with the displacement of the boundaries. The Lagrangian is then found to be

$$L = T - V_1 - V_2 = \int_0^L \left[\frac{1}{2}\varrho\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}\right)^2\right] dx - \frac{1}{2}\kappa_1 u^2(0,t) - \frac{1}{2}\kappa_2 u^2(L,t) \tag{6.7}$$

Hamilton's principle requires that the integral

$$\begin{aligned} I &= \int_{t_1}^{t_2} L\, dt \\ I &= \int_{t_1}^{t_2}\int_0^L \left[\frac{1}{2}\varrho\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}\right)^2\right] dxdt - \frac{1}{2}\kappa_1\int_{t_1}^{t_2} u^2(0,t)\, dt - \frac{1}{2}\kappa_2\int_{t_1}^{t_2} u^2(L,t)\, dt \end{aligned} \tag{6.8}$$

be a minimum for arbitrary times t_1 and $t_2 > t_1$.

Assume that $u(x,t)$ is the function that minimizes the Hamiltonian integral given by equation (6.8) and then consider a comparison function $U = U(x,t) = u(x,t) + \epsilon\eta(x,t)$ where $\eta(x,t_1) = 0$ and $\eta(x,t_2) = 0$ for all x satisfying $0 \le x \le L$. That is, we assume there is no variation at the beginning time t_1 and ending time t_2. The equation (6.8) then becomes the functional

$$\begin{aligned} I = I(\epsilon) = &\int_{t_1}^{t_2}\int_0^L \left[\frac{1}{2}\varrho\left(\frac{\partial u}{\partial t}+\epsilon\frac{\partial \eta}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}+\epsilon\frac{\partial \eta}{\partial x}\right)^2\right] dxdt \\ &- \frac{1}{2}\kappa_1\int_{t_1}^{t_2} \left(u(0,t)+\epsilon\eta(0,t)\right)^2\, dt - \frac{1}{2}\kappa_2\int_{t_1}^{t_2} \left(u(L,t)+\epsilon\eta(L,t)\right)^2\, dt \end{aligned} \tag{6.9}$$

which has a minimum at $\epsilon = 0$. A necessary condition for a minimum is that $\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$. We differentiate the equation (6.9) with respect to ϵ and set the result equal to zero. The resulting equation can be written in the form

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2}\int_0^L [\varrho u_t \eta_t - \tau u_x \eta_x]\, dxdt - \kappa_1 \int_{t_1}^{t_2} u(0,t)\eta(0,t)\, dt - \kappa_2 \int_{t_1}^{t_2} u(L,t)\eta(L,t)\, dt = 0 \tag{6.10}$$

where the subscripts denote partial derivatives. Observe that for ϱ and τ constant one can calculate the derivatives

$$\begin{aligned}\frac{\partial}{\partial x}[-\tau u_x(x,t)\eta(x,t)] =& -\tau u_x(x,t)\eta_x(x,t) - \tau u_{xx}(x,t)\eta(x,t)\\ \frac{\partial}{\partial t}[\varrho u_t(x,t)\eta(x,t)] =& \varrho u_t(x,t)\eta_t(x,t) + \varrho u_{tt}(x,t)\eta(x,t)\end{aligned} \tag{6.11}$$

which can be substituted into the equation (6.10) to obtain

$$\begin{aligned}\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} =& \int_{t_1}^{t_2}\int_0^L [-\varrho u_{tt} + \tau u_{xx}]\,\eta\, dxdt - \kappa_1 \int_{t_1}^{t_2} u(0,t)\eta(0,t)\, dt - \kappa_2 \int_{t_1}^{t_2} u(L,t)\eta(L,t)\, dt\\ &+ \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}[-\tau u_x(x,t)\eta(x,t)]\, dxdt + \int_{t_1}^{t_2} \frac{\partial}{\partial t}[\varrho u_t(x,t)\eta(x,t)]\, dxdt = 0\end{aligned} \tag{6.12}$$

One can integrate the last integral in equation (6.12) with respect to t and show the result is zero. This is because of our original assumption the η is zero at t_1 and t_2. The second to last integral in equation (6.12) can be integrated with respect to x to obtain

$$\int_{t_1}^{t_2} [-\tau u_x(x,t)\eta(x,t)]_0^L\, dt = \int_{t_1}^{t_2} \tau\,[u_x(0,t)\eta(0,t) - u_x(L,t)\eta(L,t)]\, dt \tag{6.13}$$

The equation (6.12) can then be written in the form

$$\begin{aligned}\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} =& \int_{t_1}^{t_2}\int_0^L [-\varrho u_{tt} + \tau u_{xx}]\,\eta\, dxdt\\ &- \int_{t_1}^{t_2} [\kappa_1 u(0,t) - \tau u_x(0,t)]\,\eta(0,t)\, dt\\ &- \int_{t_1}^{t_2} [\kappa_2 u(L,t) + \tau u_x(L,t)]\,\eta(L,t)\, dt = 0\end{aligned} \tag{6.14}$$

The equation (6.14) can now be assigned boundary conditions and analyzed.

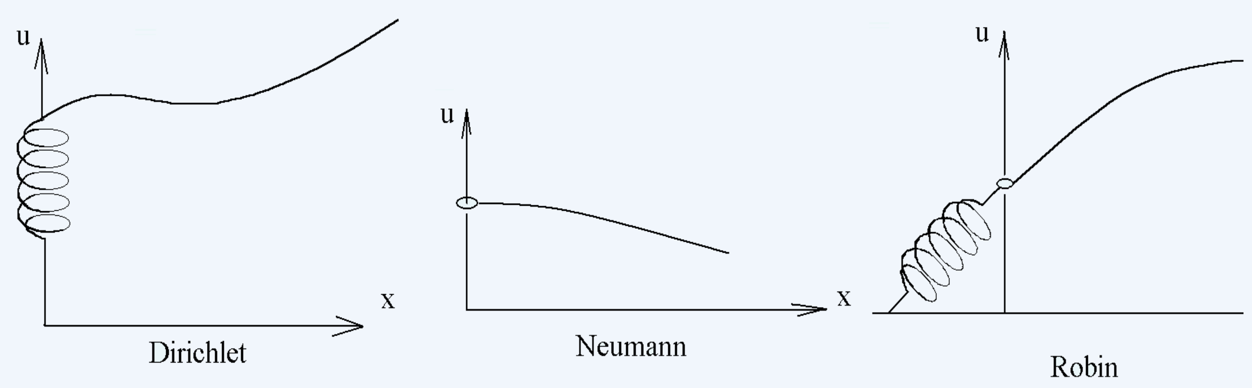


Figure 6-1 Boundary elements

(i) **Dirichlet boundary value problem**

Assume the ϱ and τ are constants. If $\eta(0,t)=0$ and $\eta(L,t)=0$ with $\eta(x,t)$ otherwise arbitrary we obtain the Euler-Lagrange equation

$$u_{tt}(x,t)=\alpha^2 u_{xx}(x,t), \qquad 0\le x\le L, \quad t>0 \text{ where } \alpha^2=\tau/\varrho \tag{6.15}$$

which is known as the wave equation. The wave equation is subject to the initial condition that $u(x,0)=f(x)$ describes the initial shape of the string and the boundary conditions $u(0,t)$ and $u(L,t)$ are specified. The resulting boundary value problem is referred to as a Dirichlet boundary value problem.

(ii) **Robin boundary value problem**

If $\eta(0,t)$ and $\eta(L,t)$ are arbitrary, then in addition to the Euler-Lagrange equation (6.15) we require the natural boundary conditions

$$\begin{aligned}\tau u_x(0,t)-\kappa_1 u(0,t)=&0\\ \tau u_x(L,t)+\kappa_2 u(L,t)=&0\end{aligned} \tag{6.16}$$

The normal vectors at the end of string are $\hat{n}=-\widehat{\mathbf{e}}_1$ at $x=0$ and $\hat{n}=+\widehat{\mathbf{e}}_1$ at $x=L$ so that in terms of normal derivatives the above boundary conditions can be written in the form

$$\begin{aligned}\frac{\partial u(0,t)}{\partial n}+h_1 u(0,t)=&0, \qquad h_1=\kappa_1/\tau\\ \frac{\partial u(L,t)}{\partial n}+h_2 u(L,t)=&0, \qquad h_2=\kappa_2/\tau\end{aligned} \tag{6.17}$$

The wave equation (6.15) with given initial condition and boundary conditions of the form given by equations (6.17) is referred to as a Robin boundary value problem.

(iii) **Neumann boundary value problem**

In the special case where $\kappa_1=0$ and $\kappa_2=0$ the boundary conditions given by equation (6.17) reduce to the form

$$\frac{\partial u(0,t)}{\partial x}=0 \qquad \text{and} \qquad \frac{\partial u(L,t)}{\partial x}=0 \tag{6.18}$$

This corresponds to free end conditions where the slopes of the string maintain a zero slope at the end points. The wave equation (6.15) with given initial condition and boundary conditions of the form given by equations (6.18) are referred to as a Neumann boundary value problem.

Other variational problems similar to the vibrating string

There are many variational problems similar to the vibrating string problem previously considered. Two examples are the deformation of a beam and the longitudinal deformation of a rod. The variational problem associated with the deformation $u=u(x,t)$ of a beam subject to an external force f per unit length is to minimize the functional

$$I=\int_{t_1}^{t_2}\int_0^L\left[\frac{1}{2}\rho\,(u_t)^2-\frac{1}{2}EI\,(u_{xx})^2+fu\right]dxdt$$

where E is Young's modulus of elasticity and I is the moment of inertia of the beam cross section. The variational problem associated with the deformation of $u = u(x,t)$ of a rod subject to an external force per unit length is to minimize the functional

$$I = \int_{t_1}^{t_2} \int_0^L \left[\frac{1}{2}\rho\,(u_t)^2 - \frac{1}{2}EA\,(u_x)^2 + fu \right] dxdt$$

where A is the cross sectional area of the rod with f the external force per unit length.

The vibrating membrane

The equations of motion for a vibrating membrane is developed in a manner which is very similar to our previous development for the equations of motion of the vibrating string. Let $u = u(x,y,t)$ denote the displacement of a thin elastic membrane from its equilibrium position $u = 0$. The thin membrane extends over a region R which is bounded by a simple closed curve $C = \partial R$. The boundary curve C is assumed to be fixed. Let $\varrho = \varrho(x,y)$ $[kg/m^2]$ denote the mass per unit area of the membrane. The kinetic energy per unit area of the membrane is given by $\frac{1}{2}\varrho\left(\frac{\partial u}{\partial t}\right)^2$ $[kg/m^2][m/sec]^2 = [Nt\,m]/m^2]$. This is summed over the area R to obtain the total kinetic energy

$$T = \frac{1}{2}\iint_R \varrho\left(\frac{\partial u}{\partial t}\right)^2 dxdy \qquad [Nt\cdot m] = [joule] \tag{6.19}$$

The potential energy is composed of two parts. The first part represents the work needed to move the membrane from its equilibrium position $u=0$ to some shape defined by $u = u(x,y,t)$ and the second part represents the work done in stretching the boundary from its fixed position. For the first part of the potential energy assume the membrane experiences a constant surface tension per unit of length which is given by τ $[Nt/m]$. The work done in moving an element of area $dA = dxdy$ in the equilibrium position to an element of area dS on the surface is given by

$$dV_1 = \tau\,(dS - dA) \quad [Nt/m][m^2] = [Nt\cdot m] \tag{6.20}$$

Recall that if $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + u\,\widehat{\mathbf{e}}_3$ is a position vector to a point on the surface, then the differential of $\vec{r}$ is given by $d\vec{r} = \frac{\partial \vec{r}}{\partial x}\,dx + \frac{\partial \vec{r}}{\partial y}\,dy$ which represents a small change on the surface. The components $\frac{\partial \vec{r}}{\partial x}\,dx$ and $\frac{\partial \vec{r}}{\partial y}\,dy$ of this vector are the sides of an element of surface area dS. This element of area is given by

$$dS = \left|\frac{\partial \vec{r}}{\partial x}\,dx \times \frac{\partial \vec{r}}{\partial y}\,dy\right| = \sqrt{\left(\frac{\partial \vec{r}}{\partial x}\cdot\frac{\partial \vec{r}}{\partial x}\right)\left(\frac{\partial \vec{r}}{\partial y}\cdot\frac{\partial \vec{r}}{\partial y}\right) - \left(\frac{\partial \vec{r}}{\partial x}\cdot\frac{\partial \vec{r}}{\partial y}\right)^2}\,dxdy = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\,dxdy \tag{6.21}$$

The element of work done in moving dA to dS can therefore be expressed in the form

$$dV_1 = \tau\,(\sqrt{1 + u_x^2 + u_y^2} - 1)\,dxdy \tag{6.22}$$

We make the assumption that the slopes $\mid \frac{\partial u}{\partial x} \mid$ and $\mid \frac{\partial u}{\partial y} \mid$ are small so that we can make the approximation

$$\sqrt{1 + u_x^2 + u_y^2} = 1 + \frac{1}{2}(u_x^2 + u_y^2) - \frac{1}{4}(u_x^2 + u_y^2)^2 + \cdots \tag{6.23}$$

If we neglect terms $(u_x^2 + u_y^2)^m$ for $m \geq 2$ in equation (6.23), then the potential energy of moving an element of area can be represented in the form

$$dV_1 = \frac{1}{2}\tau\,(u_x^2 + u_y^2)\,dxdy \tag{6.24}$$

Summing over the region R gives the potential energy function

$$V_1 = \frac{1}{2}\iint_R \tau\,(u_x^2 + u_y^2)\,dxdy \qquad [Nt\cdot m] \tag{6.25}$$

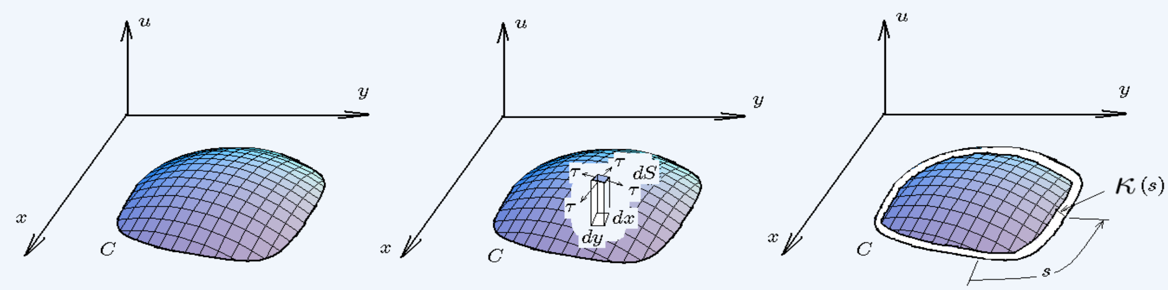


Figure 6-2 Boundary and surface elements for vibrating membrane.

The stretching of the membrane along the boundary is treated in a manner similar to that of the stretched string. Assume that each point of the boundary has associated with it a spring constant $\kappa(s)$ which varies with arc length s measured from some reference point on the boundary. The spring force is assumed to be linear and consequently the total work done in stretching the boundary can be written as a sum of the forces around the bounding curve C. Following the example of the vibrating string we construct the potential function

$$V_2 = \frac{1}{2}\int_C \kappa(s)u^2(s,t)\,ds$$

where s is an element of arc length and $u(s,t)$ denotes the displacement of a boundary point from its equilibrium position.

The Lagrangian can then be written $L = T - V_1 - V_2$ and the Hamiltonian can be expressed as

$$I = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\iint_R \left(\frac{1}{2}\varrho u_t^2 - \frac{1}{2}\tau\,[u_x^2 + u_y^2]\right)dxdydt - \frac{1}{2}\int_{t_1}^{t_2}\int_c \kappa(s)u^2(s,t)\,dt \tag{6.26}$$

Hamilton's principle requires that the Hamiltonian be minimized. To find the function $u = u(x,y,t)$ which minimizes the integral given by equation (6.26) we assume that $u(x,y,t)$ minimizes I and then consider the family of comparison functions given by the variation $U = U(x,y,t) = u(x,y,t) + \epsilon\eta(x,y,t)$. Here we assume that the conditions $\eta(x,y,t_1) = 0$ and $\eta(x,y,t_2) = 0$ are satisfied at the end points of the arbitrary time interval. Substituting the comparison function into the integral (6.26) gives

$$I = I(\epsilon) = \int_{t_1}^{t_2}\iint_R \left(\frac{1}{2}\varrho U_t^2 - \frac{1}{2}\tau\,[U_x^2 + U_y^2]\right)dxdydt - \frac{1}{2}\int_{t_1}^{t_2}\int_c \kappa(s)U^2(s,t)\,dt \tag{6.27}$$

which is to have a minimum at $\epsilon = 0$. A necessary condition for a minimum is that $\frac{dI}{d\epsilon}\Big|_{\epsilon=0} = 0$. Differentiate the equation (6.27) with respect to ϵ to obtain the equation

$$\frac{dI}{d\epsilon}\Big|_{\epsilon=0} = \int_{t_1}^{t_2} \iint_R \{\varrho u_t \eta_t - \tau\,[u_x \eta_x + u_y \eta_y]\}\, dxdydt - \int_{t_1}^{t_2} \kappa(s) u(s,t)\eta\, ds = 0 \tag{6.28}$$

One can employ the derivatives

$$\frac{\partial}{\partial t}(u_t \eta) = u_t \eta_t + u_{tt}\eta\ , \qquad \frac{\partial}{\partial x}(u_x \eta) = u_x \eta_x + u_{xx}\eta\ , \qquad \frac{\partial}{\partial y}(u_y \eta) = u_y \eta_y + u_{yy}\eta$$

to write equation (6.28) in the form

$$\begin{aligned} \frac{dI}{d\epsilon}\Big|_{\epsilon=0} =& \int_{t_1}^{t_2} \iint_R [-\varrho u_{tt} + \tau\,(u_{xx} + u_{yy})]\;\eta\, dxdydt - \int_{t_1}^{t_2} \kappa(s) u(s,t)\,\eta\, ds \\ & - \int_{t_1}^{t_2} \iint_R \tau\,\left[\frac{\partial}{\partial x}(u_x \eta) + \frac{\partial}{\partial y}(u_y \eta)\right]\, dxdydt + \int_{t_1}^{t_2} \frac{\partial}{\partial t}(u_t \eta)\, dxdydt = 0 \end{aligned} \tag{6.29}$$

The last integral in equation (6.29) can be integrated with respect to time t with the result being zero because of the assumption that $\eta(x, y, t_1) = 0$ and $\eta(x, y, t_2) = 0$ at the end points of the arbitrary time interval. The two-dimensional form of the Green's theorem can then be employed to replace the second to last integral in equation (6.29) in terms of a line integral. For τ constant, one can verify that

$$\iint_R \left[\frac{\partial}{\partial x}(u_x \eta) + \frac{\partial}{\partial y}(u_y \eta)\right] dxdy = \int_C (u_x \eta\, dy - u_y \eta\, dx) = \int_C \frac{\partial u}{\partial n} \eta ds \tag{6.30}$$

where

$$\frac{\partial u}{\partial n} = \text{grad}\, u \cdot \hat{n} = \left(\frac{\partial u}{\partial x}\,\widehat{\mathbf{e}}_1 + \frac{\partial u}{\partial y}\,\widehat{\mathbf{e}}_2\right) \cdot \left(\frac{dy}{ds}\,\widehat{\mathbf{e}}_1 - \frac{dx}{ds}\,\widehat{\mathbf{e}}_2\right) \tag{6.31}$$

That is, if $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2$ defines the boundary curve C for the membrane, then the unit normal to the boundary is given by $\hat{n} = \frac{dy}{ds}\,\widehat{\mathbf{e}}_1 - \frac{dx}{ds}\,\widehat{\mathbf{e}}_2$.

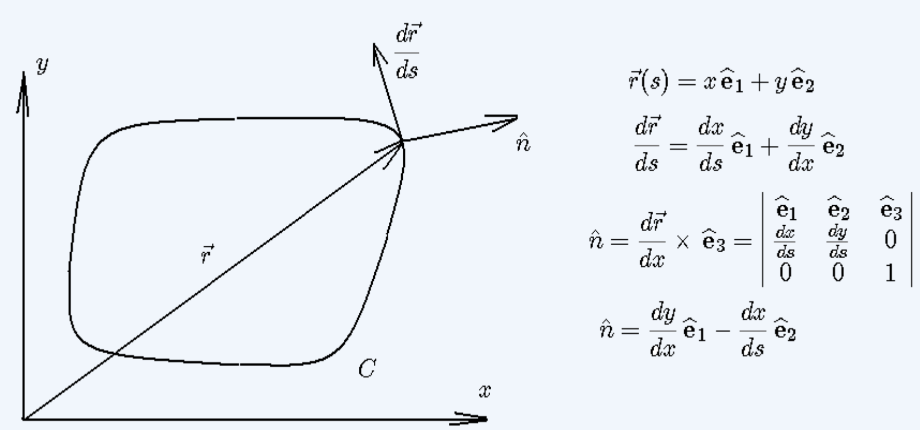


Figure 6-3 Boundary curve C with unit tangent and normal vectors to boundary.

The equation (6.29) can then be simplified to the form

$$\frac{dI}{d\epsilon}\bigg|_{\epsilon=0} = \int_{t_1}^{t_2} \iint_R \left[-\varrho u_{tt} + \tau\left(u_{xx} + u_{yy}\right)\right] \eta\, dxdydt - \int_{t_1}^{t_2} \int_C \left(\kappa u + \tau \frac{\partial u}{\partial n}\right) \eta\, dsdt = 0 \qquad (6.32)$$

which can now be analyzed.

(i) **Fixed boundary conditions**

If $\eta(x, y, t) = 0$ for all $x, y \in C$ and all times t, then the last term in equation (6.32) vanishes. We can then employ a form of the basic lemma to obtain the Euler-Lagrange equation

$$\varrho u_{tt} = \tau\left(u_{xx} + u_{yy}\right) \qquad \text{or} \qquad u_{tt} = \alpha^2\left(u_{xx} + u_{yy}\right), \qquad \alpha^2 = \tau/\varrho \qquad (6.33)$$

This equation is called the two-dimensional wave equation. It is to be solved subject to the initial condition $u(x, y, 0) = f(x, y)$ which defines the initial shape of the membrane. Here $\eta = 0$ along the boundary and therefore the boundary condition is assumed to be specified, say $u(x, y, t) = G(x, y, t)$ for $x, y \in C$.

(ii) **Elastic boundary condition**

If η is arbitrary, then the natural boundary conditions from the boundary term occurring in equation (6.32) requires that

$$\kappa(s)u(s, t) + \tau\frac{\partial u}{\partial n} = 0 \qquad \text{for} \quad s \in C = \partial R \qquad (6.34)$$

This type of boundary condition is called an elastic boundary condition.

(iii) **Free and fixed boundary condition**

If $\kappa(s) = 0$, then a free boundary condition is said to exist. This requires that $\frac{\partial u}{\partial n} = 0$ for $x, y \in C = \partial R$. The other extreme is for $\kappa(s)$ to increase without bound. Dividing equation (6.34) by κ and then letting κ increase without bound produces the fixed boundary condition that $u(x, y, t) = 0$ for $x, y \in C = \partial R$.

The reference

> R. Courant, "Variational Methods for the Solution of Problems of Equilibrium and Vibrations", Bulletin of the American Mathematical Society, Vol. 49, January 1943, Pp 1-23.

gives a generalized presentation for developing variational methods for the solution of the following type of problems.

(i) Problems of stable equilibrium of a plate or membrane subject to an external pressure

(ii) Variational problems associated with vibrations of plates and membranes.

The reader should have some basic background knowledge from the theory of elasticity before consulting the above reference.

Generalized brachistrochrone problem

A generalization of the brachistrochrone problem is to develop the equations of motion of a particle which is acted upon by a conservative force system. Find the path of motion such that the particle moves from point P_1 to a point P_2 in the shortest time. Another form of this problem is to find the path or curve from one surface to another surface such that the particle moves in the shortest time. All solution paths or curves which represent solutions to the above type of problems are called brachistrochrones. Let $PE = m\phi(x, y, z)$ denote the potential energy of the force system and let $KE = \frac{1}{2}mv^2$ denote the kinetic energy of the particle where m is the mass of the particle, v denotes the velocity of the particle and ϕ is the potential energy per unit mass. The force per unit mass acting on the particle is

$$\vec{F} = -\text{grad}\,\phi = -\frac{\partial \phi}{\partial x}\,\widehat{\mathbf{e}}_1 - \frac{\partial \phi}{\partial y}\,\widehat{\mathbf{e}}_2 - \frac{\partial \phi}{\partial z}\,\widehat{\mathbf{e}}_3$$

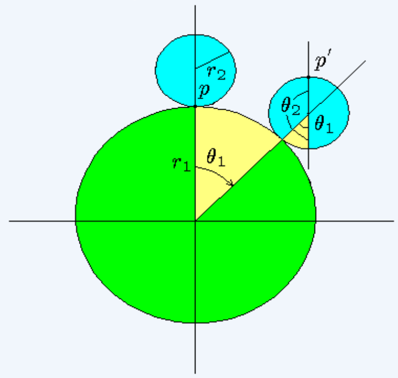
and the components represent the rate of decrease of the potential energy because of the minus sign. The total energy E of the particle is the sum of the kinetic energy and potential energy and can be written

$$E = KE + PE = \frac{1}{2}mv^2 + m\phi = \text{Constant}$$

Holonomic and nonholonomic systems

Dynamical systems are classified as scleronomic if there are constraint equations independent of time t of the form $f_i(q^1, q^2, \ldots, q^n) = 0$ where q^i, $i = 1, 2, \ldots, n$ are generalize coordinates. The system is classified as rheonomic if the time variable t is needed for the representation of the constraint conditions so that one would have constraint equations of the form $f_i(q^1, q^2, \ldots, q^n, t) = 0$. If there exists constraint equations of the form $f_j(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n, t) = 0$ which are all integrable or constraint equations exist of the form $f_j(q^1, q^2, \ldots, q_n, t) = 0$ for $j = 1, 2, \ldots, m$, then a definite relation can be found between the generalized coordinates and so the system is termed a holonomic system. If at least one constraint equation is not integrable or a definite relationship is not specified by any of the constraint equations, then the system is said to be nonholonomic.

An example of a holonomic constraint would be that of a particle constrained to move on a given surface $G(x, y, z) = 0$. Under such conditions one of the coordinate unknowns, either x, y or z can be eliminated from the problem. That is, it is theoretically possible to solve the equation $G(x, y, z) = 0$, for say the variable z as a function of x and y. Constraints which can not be integrated to find a definite relation between the variables or where inequalities are involve, then these type of constraints are called nonholonomic constraints. As an example of such a constraint consider the motion of gas particles moving inside a cylinder container of radius r_0. The radial position r of a gas molecule is subject to the constraint condition $r \leq r_0$. Another example of a nonholonomic constraint is that of a particle released on the top of a sphere. Here the particle will roll on the surface of a sphere, due to the gravitational force acting on the particle, and then it eventually falls off the sphere. If r_0 is the radius of the sphere, then the constraint condition can be expressed in the form $r^2 - r_0^2 \geq 0$, where r is the radial position of the particle from the center of the coordinate system defining the sphere.



Example 6-1. Rolling cylinders

Consider a cylinder of radius r_2 and mass m which rolls without slipping on the surface of another fixed cylinder of radius $r_1 > r_2$. At some point the outer cylinder falls off the inner cylinder. Find the equations of motion of the outer cylinder. Find the angle θ_1^* at which the cylinders separate.

Solution: Introduce as generalized coordinates the angle θ_1 measured clockwise and the angle θ_2 measured counterclockwise from the vertical lines through the center of the cylinders together with the generalized coordinate r representing the distance of the center of the smaller cylinder from the origin of the larger cylinder. The angles θ_1 and θ_2 are illustrated in the above sketch of the problem which illustrates the point p moving to the point p' as θ_1 changes. The constraint condition that the smaller cylinder rolls without slipping can be represented by the constraint equation that

$$s = \text{arc length} = r_1\theta_1 = r_2(\theta_2 - \theta_1) \tag{6.35}$$

which states that the arc lengths swept out by the point p must be the same on both cylinders. The further constraint

$$r = r_1 + r_2 = \text{a constant} \tag{6.36}$$

states that the smaller cylinder remains on the larger cylinder as it rolls. These constraint conditions can be represented

$$\begin{aligned} G_1 &= (r_1 + r_2)\theta_1 - r_2\theta_2 = 0 \\ G_2 &= \qquad r - (r_1 + r_2) = 0 \end{aligned} \tag{6.37}$$

The kinetic energy of the smaller cylinder can be broken up into a translational component involving the change in arc length $s = r\theta_1$ and change in radial distance r together with a rotational component $\frac{1}{2}I\omega^2$ involving the moment of inertial $I = \frac{1}{2}m{r_2}^2$ of the smaller cylinder and angular velocity $\omega = \dot{\theta}_2$. The kinetic energy T can then be written as

$$T = \underbrace{\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}_1^2\right)}_{\text{translational}} + \underbrace{\frac{1}{2}\left[\frac{1}{2}mr_2^2\right]\dot{\theta}_2^2}_{\text{rotational}} \tag{6.38}$$

The potential energy can be represented

$$V = mgh + \text{constant} = mgr\cos\theta_1 + \text{constant} \tag{6.39}$$