Chapter 5

Applications of the Variational Calculus

In this chapter we present selected applications from the fields of science and engineering where the calculus of variations is employed to obtain an extremal associated with a derived functional. The resulting Euler-Lagrange equations are either ordinary differential equations or partial differential equations with boundary conditions. We begin with a presentation of some of the more classical applications where the calculus of variations is utilized.

The brachistrochrone problem

In 1696 John Bernoulli formulated the following problem. Imagine a bead sliding smoothly without friction on a thin wire. Given two points P_0 and P_1 on the wire, find the shape of the wire such that the bead moves under the action of gravity from point P_0 to point P_1 in the shortest time. This is called the brachistrochrone problem. The word brachistrochrone comes form the Greek *brachistos* for shortest and the Greek *chrones* for time. Assume the points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ are points in a vertical plane not on the same vertical line with $x_1 > x_0$ and $y_1 > y_0$. By constructing a coordinate system with the y-axis pointing downward the situation can be described by the illustration in figure 5-1.

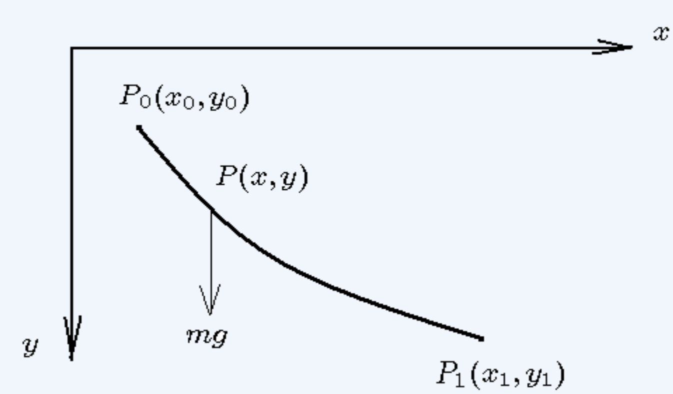


Figure 5-1. The brachistrochrone problem.

Let $y = y(x)$ denote the curve representing the shape of the thin wire which connects the points P_0 and P_1 and let V_0 denote the initial velocity of the bead of mass m at the point P_0 and let V denote the velocity of the bead at a general point $P(x, y)$ on the curve $y = y(x)$. Employ the work-energy theorem from physics which states that in the absence of frictional forces the work done by forces acting on a particle must equal the change in the kinetic energy of the particle. The work done is force times distance and kinetic energy is $T = \frac{1}{2}mV^2$ where m is the mass and V is the velocity.

The work done represents the change in potential[‡] energy in moving the bead from point y_0 to y and is given by $mg(y-y_0)$. Here m is the mass of the particle, and g is the acceleration of gravity. The change in kinetic energy of the particle is given by $\frac{1}{2}m(V^2-V_0^2)$, where V is the final velocity and V_0 is the initial velocity. The work-energy theorem requires

$$mg(y-y_0)=\frac{1}{2}m(V^2-V_0^2) \tag{5.1}$$

and solving for the velocity squared we find that

$$V^2=V_0^2+2gy-2gy_0.$$

Without loss of generality we can assume that the particle starts from rest so that $V_0=0$ and consequently the particle velocity $V=\frac{ds}{dt}$ is represented by the change in distance s with time t, and so one can write

$$\frac{ds}{dt}=V=\sqrt{2g(y-y_0)}. \tag{5.2}$$

The arc length along the extremal curve $y=y(x)$ is obtained by integrating $ds=\sqrt{1+y'^2}\,dx$ so that the total time T taken to move from P_0 to P_1 can then be expressed

$$T=\int_{x_0}^{x_1}\frac{ds}{V}=\frac{1}{\sqrt{2g}}\int_{x_0}^{x_1}\frac{\sqrt{1+y'^2}}{\sqrt{y-y_0}}\,dx. \tag{5.3}$$

We are to find the curve $y=y(x)$ such that the total time T is a minimum. Here the integrand is given by $f=\dfrac{1}{\sqrt{2g}}\sqrt{\dfrac{1+y'^2}{y-y_0}}=f(y,y')$ which is a function of y and y' with the independent variable absent. In this special case one can immediately obtain a first integral of the Euler-Lagrange equation. This first integral can be represented

$$y'\frac{\partial f}{\partial y'}-f=C_1 \quad \text{or} \quad \frac{y'^2}{\sqrt{(y-y_0)(1+y'^2)}}-\frac{\sqrt{1+y'^2}}{\sqrt{y-y_0}}=C_1. \tag{5.4}$$

where C_1 is a constant. This equation simplifies to

$$\frac{-1}{[(y-y_0)(1+y'^2)]^{1/2}}=C_1 \tag{5.5}$$

which can also be represented in the form

$$k^2=\left(\frac{-1}{C_1}\right)^2=(y-y_0)\left[1+\left(\frac{dy}{dx}\right)^2\right] \tag{5.6}$$

[‡] If $\vec{F}$ is a force acting on a particle, then one can define the potential energy function at a point P by $V(P)=-\int_\alpha^P \vec{F}\cdot d\vec{r}$ where α is some convenient reference point. The difference in potential between two points P_1 and P_2 is given by

$$V(P_2)-V(P_1)=-\int_\alpha^{P_2}\vec{F}\cdot d\vec{r}+\int_\alpha^{P_1}\vec{F}\cdot d\vec{r}=-\int_{P_1}^{P_2}\vec{F}\cdot d\vec{r}$$

and represents the negative of the work done in moving from P_1 to P_2. For a conservative force system $\oint_C \vec{F}\cdot d\vec{r}=0$ where C is a simple closed path. The choice of zero potential energy is arbitrary because we are only concerned with differences in potential energy.

where k^2 is some new constant. This form is easier to integrate because one can separate the variables in equation (5.6) to obtain

$$\frac{\sqrt{y-y_0}\,dy}{\sqrt{k^2-(y-y_0)}} = dx. \tag{5.7}$$

Make the substitution $y = y_0 + k^2 \sin^2\theta$ with $dy = 2k^2 \sin\theta\cos\theta\,d\theta$ and verify that equation (5.7) simplifies to

$$dx = 2k^2 \sin^2\theta\,d\theta. \tag{5.8}$$

This equation can now be integrated to obtain

$$x = x_0 + k^2\left(\theta - \frac{1}{2}\sin 2\theta\right).$$

This gives the solution of the Euler-Lagrange equation as the parametric equations

$$x = x_0 + \frac{1}{2}k^2\left(2\theta - \sin 2\theta\right), \qquad y = y_0 + \frac{1}{2}k^2\left(1-\cos 2\theta\right) \tag{5.9}$$

which represents a family of cycloids. Here the constants k and x_0 can be selected such that the curve passes through a given point (x_1, y_1). Note that the point y_0 is not arbitrary since by equation (5.2) we know that when $y = y_1$ we must have $V = V_1$ so that y_0 must satisfy $V_1 = \sqrt{2g(y_1-y_0)}$ or $y_0 = y_1 - V_1^2/2g$.

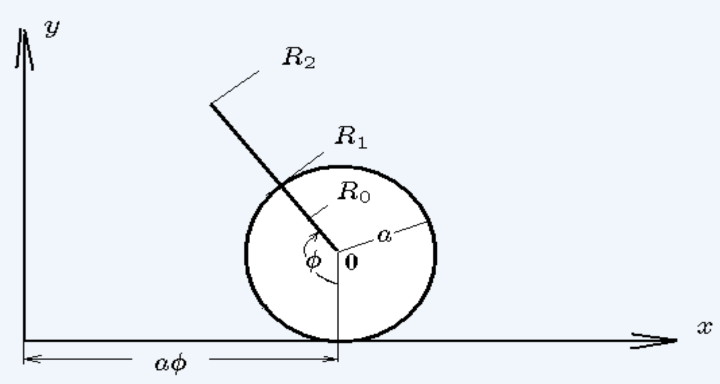


Figure 5-2. Circle rolling without slipping along x-axis.

There are many types of cycloids. Consider a circle of radius $R = a$ which rolls without slipping along the x-axis. Let $0R_0R_1R_2$ denote a line segment extended from the center of the circle, with $0 < R_0 < R_1 < R_2$ and $R_1 = a$, as illustrated in the figure 5-2.

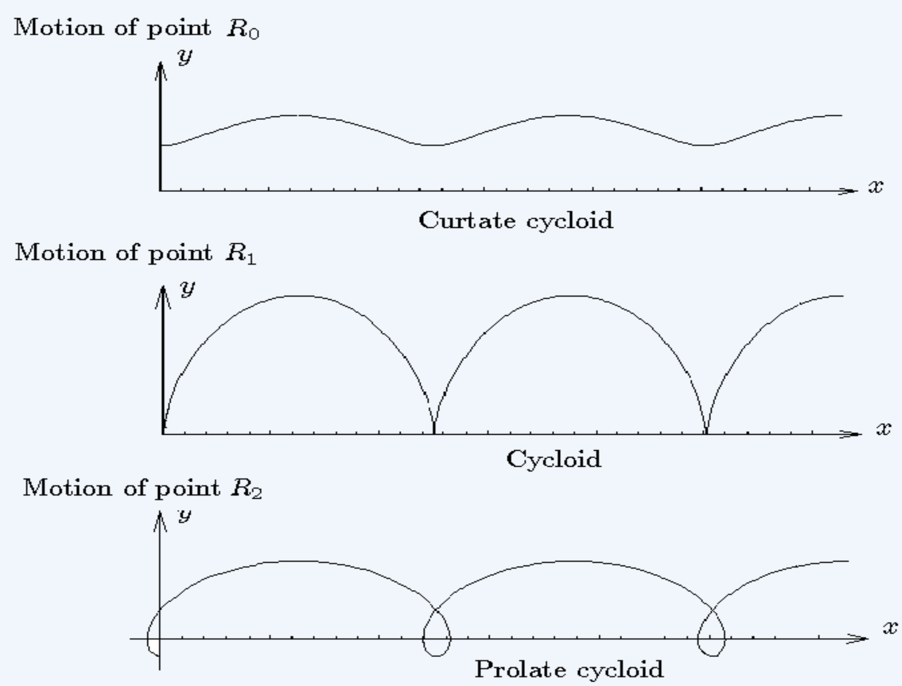


Figure 5-3. Various types of cycloids.

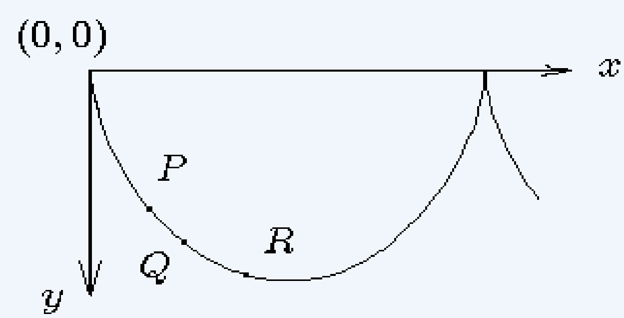


Figure 5-4. Inverted cycloid minimum time solution.

Let P denote one of the points on the line segment where the radius from the center of the circle has one of the values of R_0, R_1 or R_2. Denote the distance $OP = b$, then the motion of the point P, as the circle rolls without slipping, describes a cycloid. The conditions $b < a$, $b = a$ and $b > a$ produce three different types of cycloids. Each can be described by the parametric equations

$$x = a\phi - b\sin\phi, \qquad y = a - b\cos\phi \tag{5.10}$$

where ϕ is a parameter and a is the radius of the circle. The case $b > a$ gives a prolate cycloid. The case $b = a$ gives a cycloid and the case $b < a$ gives a curtate cycloid. These cases are illustrated in the figure 5-3. Note that the inverted cycloid, figure 5-4, is the shortest time problem solution. Additional properties of the brachistrochrone curve can be found in the exercises for chapter five.

The hanging chain, rope or cable

Consider a flexible chain, rope or cable of length L hanging between the two points (ℓ, h) and $(-\ell, h)$, with $h > 0$ as illustrated in the figure 5-5. One can use telephone poles and hanging telephone lines as an example of this problem. Assume the length L satisfies the inequality $L > 2\ell$.

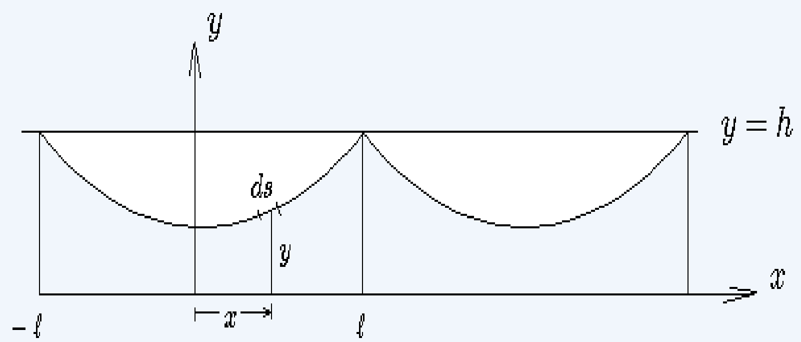


Figure 5-5. Hanging chain, rope or cable.

Consider a flexible cable whose center line gives a curve which defines the shape of the cable. This curve is assumed to lie in the xy-plane and in addition it is assumed that the cable is in a stable equilibrium configuration so that the potential energy associated with the cable is a minimum. Let ϱ denote the mass density per unit length of the cable and assume this lineal density is a constant. Consider an element of mass dm located at a general point (x, y) on the hanging cable. This element of mass has the potential energy $dV = (dm)gh = (\varrho ds)gy$ where $ds = \sqrt{1 + (y')^2}\, dx$ is a element of arc length along the cable. The potential energy of the cable is then obtained by integration over the length of the cable to obtain

$$V = \int_{-\ell}^{\ell} dV = \varrho g \int_{-\ell}^{\ell} y\sqrt{1 + (y')^2}\, dx, \qquad y' = \frac{dy}{dx}. \tag{5.11}$$

Here we must find the curve $y = y(x)$ such that the integral given by equation (5.11) is a minimum subject to the constraint conditions (i) the length of the cable between (ℓ, h) and $(-\ell, h)$ is a given length L so that $L = \int_{-\ell}^{\ell} \sqrt{1 + (y')^2}\, dx$ is a constant and (ii) the curve must satisfy the boundary conditions $y(-\ell) = h$ and $y(\ell) = h$. This is an isoperimetric problem in the calculus of variations. Introduce a Lagrange multiplier λ and form the integrand

$$f = f(x, y, y') = \varrho g y\sqrt{1 + (y')^2} + \lambda\sqrt{1 + (y')^2} \tag{5.12}$$

The Euler-Lagrange equation associated with this integrand is readily verified to have the representation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0, \qquad \Rightarrow \qquad \frac{d}{dx}\left[\frac{\varrho g y y' + \lambda y'}{\sqrt{1 + (y')^2}}\right] - \varrho g\sqrt{1 + (y')^2} = 0 \tag{5.13}$$

Note that the independent variable x is absent and consequently the equation (3.25) can be employed to immediately obtain the first integral $y' \frac{\partial f}{\partial y'} - f = \alpha =$ a constant. This first integral can be represented in the form

$$(\varrho g y + \lambda)\left[\frac{(y')^2}{\sqrt{1+(y')^2}} - \sqrt{1+(y')^2}\right] = \alpha \tag{5.14}$$

Solve the equation (5.14) for $(y')^2$ and verify that

$$(y')^2 = \frac{(\varrho g y + \lambda)^2}{\alpha^2} - 1 \tag{5.15}$$

and then make the change of variable $\varrho g y + \lambda = \alpha z$ and show that z can be obtained by integrating

$$\frac{dz}{\sqrt{z^2-1}} = \frac{\varrho g}{\alpha}\, dx \tag{5.16}$$

The equation (5.16) has the integral

$$\ln(z + \sqrt{z^2-1}) = \frac{\varrho g}{\alpha}\, x + b_1 \tag{5.17}$$

where b_1 is a constant of integration. To simplify the resulting algebra in solving for z, make the substitution $b_1 = -\frac{\varrho g}{\alpha}\beta$ where β is a new constant. Solving equation (5.17) for z produces the result

$$z = \frac{\varrho g y + \lambda}{\alpha} = \cosh\left[\frac{\varrho g(x-\beta)}{\alpha}\right] \tag{5.18}$$

and therefore the curve defining the shape of the cable is given by

$$y = y(x) = -\frac{\lambda}{\varrho g} + \frac{\alpha}{\varrho g}\cosh\left[\frac{\varrho g(x-\beta)}{\alpha}\right] \tag{5.19}$$

This gives the shape of the hanging rope, chain or cable as a hyperbolic cosine function. The Latin word for chain is *catena* and so the shape of a hanging chain, rope or cable is called a catenary.

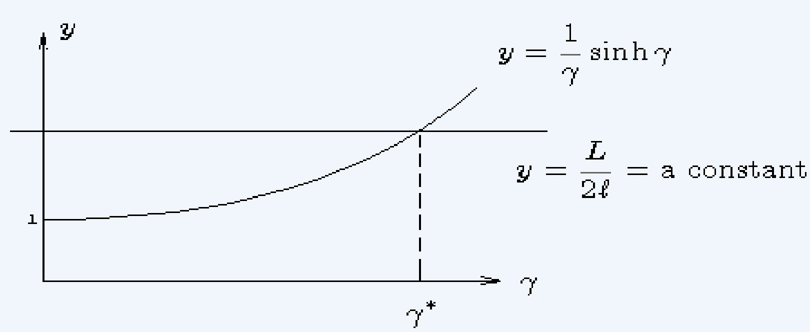


Figure 5-6. Sketch of $y = \frac{1}{\gamma}\sinh\gamma$ and $y =$a constant > 1.

Note the symmetry of the problem and select the constant $\beta = 0$ in equation (5.19) so that the boundary conditions $y(-\ell) = y(\ell) = h$ are satisfied by an appropriate selection for the constant λ once the constant α is known. The solution given by equation (5.19) must satisfy the length restriction that

$$L = \int_{-\ell}^{\ell} \sqrt{1 + (y')^2}\, dx = \int_{-\ell}^{\ell} \cosh\left(\frac{\varrho g x}{\alpha}\right) dx = \frac{2\alpha}{\varrho g} \sinh\left(\frac{\varrho g \ell}{\alpha}\right) \tag{5.20}$$

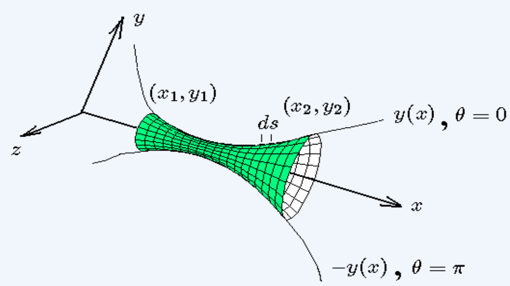
Here the fixed length L satisfies $L > 2\ell$ so that $L/2\ell > 1$ and so equation (5.20) will always have a solution for the constant α which can be determined from equation (5.20) using numerical methods as follows. Let $\gamma = \frac{\varrho g \ell}{\alpha}$ and write equation (5.20) in the form

$$\frac{L}{2\ell} = \frac{1}{\gamma} \sinh\gamma$$

A sketch of the curves $y = \frac{1}{\gamma}\sinh\gamma$ and $y = \frac{L}{2\ell}$ are illustrated in the figure 5-6 The problem is now reduced to finding the value γ^* where the two curves intersect. This is a root finding exercise in numerical methods.

Soap film problem

Consider a curve $y = y(x)$ passing through two given points (x_1, y_1) and (x_2, y_2) which is rotated about the x-axis as illustrated in the accompanying figure. One can formulate the variational problem to find the curve $y = y(x)$ such that the surface of revolution has a minimum surface area.



A ribbon element of surface area constructed on the surface is given by

$$dS = 2\pi y\, ds = 2\pi y \sqrt{1 + (y')^2}\, dx$$

where ds is an element of arc length along the original curve $y = y(x)$. One can then formulate the variational problem to find $y = y(x)$ such that

$$S = \int_{x_1}^{x_2} 2\pi y \sqrt{1 + (y')^2}\, dx$$

is a minimum. Here we have

$$f = f(x, y, y') = 2\pi y \sqrt{1 + (y')^2}$$

which is an integrand independent of x and consequently the Euler-Lagrange equation will have the first integral given by equation (3.25)

$$y' \frac{\partial f}{\partial y'} - f = \alpha \quad \text{or} \quad \frac{y (y')^2}{\sqrt{1 + (y')^2}} - y\sqrt{1 + (y')^2} = \alpha$$

where α is a constant. This equation can be simplified to the form

$$\frac{y}{\sqrt{1+(y')^2}} = C_1$$

where $C_1 = -\alpha$ is a new constant. This equation can be further simplified to the form

$$\frac{dx}{dy} = \frac{C_1}{\sqrt{y^2 - C_1^2}}$$

which can be integrated to produce the result

$$y = y(x) = C_1 \cosh\left(\frac{x - C_2}{C_1}\right)$$

where C_2 is a constant. The constants C_1 and C_2 are found from the requirement that the curve $y = y(x)$ passes through the given points (x_1, y_1) and (x_2, y_2). The surface generated by the curve $y = y(x)$ is a catenary of revolution which is called a catenoid.

Let us consider a special case of the above problem where we can make use of the symmetry properties of the hyperbolic cosine function. We set $C_2 = 0$ and require that the resulting curve $y = y(x) = C_1 \cosh\left(\frac{x}{C_1}\right)$ pass through the points $(-x_1, 1)$ and $(x_1, 1)$ where x_1 is a constant. This special case requires the constant C_1 to be selected to satisfy the equation

$$1 = C_1 \cosh\left(\frac{x_1}{C_1}\right). \tag{5.21}$$

To analyze this equation one can graph the functions $y = 1$ and $y = C_1 \cosh\left(\frac{x_1}{C_1}\right)$ vs C_1 for various fixed values of x_1 as illustrated in the figure 5-7.

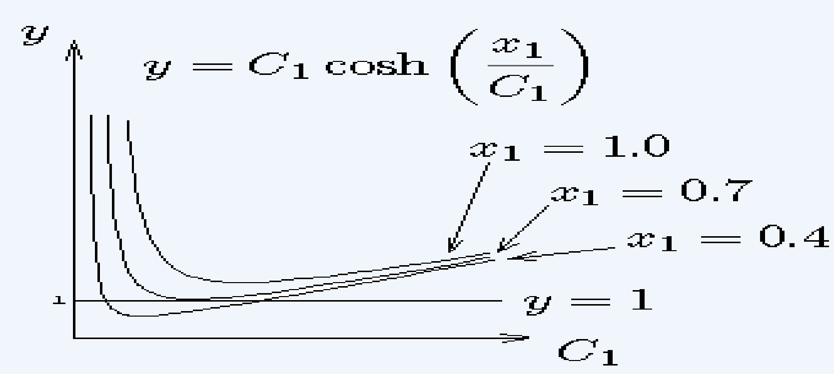


Figure 5-7. The function $C_1 \cosh\left(\frac{x_1}{C_1}\right)$ for $x_1 = 0.4, 0.7, 1.0$

The figure 5-7 illustrates the following cases. Case 1: There are values of x_1 for which equation (5.21) has no solutions for C_1. Case 2: There are values of x_1 for which one solution exists for C_1. Case 3: There are values of x_1 for which two solutions can exist for C_1. To understand what is happening consider a wire frame having two circles of radius unity where the circles lie in parallel planes separated by a distance $2x_1$. When the circles are dipped into a soap solution and removed a surface film is formed representing the minimal surface. If x_1 is small, then two possible values for C_1 can exist which gives rise to two different soap films. One of the soap films will correspond to the minimum surface area. The reference G.A. Bliss has given an examination of the two catenaries of revolution that produce the two soap films. The upper catenary is a flat curve for $y(x)$ and the lower catenary gives a deep curve for $y(x)$. The flat curve corresponds to an absolute minimum for the surface area while the lower curve does not produce a minimum value. As the end circles move further apart the quantity x_1 reaches a critical point where equation (5.21) has only one solution. Increasing x_1 beyond this critical point causes the soap film to break and no solution exists for equation (5.21). When this occurs, there is formed a circular film over each circle on the ends of the wire frame . This is known as the Goldschmidt discontinuous solution. For additional experiments with soap films the reference Courant and Robbins is suggested.

Hamilton's principle and Euler-Lagrange equations

Consider a system of N particles and denote by $\vec{r}_i = \vec{r}_i(t)$ the position vector of the ith particle of mass m_i. Newton's law of motion for the ith particle states that the time rate of change of linear momentum must equal the force applied to the ith particle. Assume that all the particles have a constant mass and write Newton's law of motion for the ith particle

$$m_i \ddot{\vec{r}}_i = \vec{F}_i \qquad \text{or} \qquad m_i \ddot{\vec{r}}_i - \vec{F}_i = \vec{0} \tag{5.22}$$

where $\vec{F}_i$ denotes the total force acting on the ith particle. Let $\delta\vec{r}_i$ denote the variation in the position of the ith particle and take the dot product of the ith equation (5.22) with the $\delta\vec{r}_i$ and then sum over all particles i to obtain

$$\sum_{i=1}^{N} \left(m_i \ddot{\vec{r}}_i - \vec{F}_i \right) \cdot \delta\vec{r}_i = \vec{0} \qquad \text{or} \qquad \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i - \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = \vec{0} \tag{5.23}$$

The second term on the left-hand side of equation (5.23) represents a summation of terms where each term represents a force times a distance or work done. The total work done δW is a summation of the forces $\vec{F}_i$ due to the displacements $\delta\vec{r}_i$. One can write

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i \tag{5.24}$$

Here the force $\vec{F}$ has units of $Newtons$ [N], $\delta\vec{r}$ has units of $meters$ [m] and the work W has units of $joules$ [J]. Here $1\ joule = 1\ newton \cdot meter$. To simplify the left-hand side of equation (5.23) we employ the identity

$$\sum_{i=1}^{N} \left[m_i \frac{d}{dt} \left(\dot{\vec{r}}_i \cdot \delta\vec{r}_i \right) - \delta \left(\frac{m_i}{2} \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \right) \right] = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i \tag{5.25}$$

The kinetic energy T of the system of particles is defined as a summation of one-half the mass times velocity squared for each particle. This gives the kinetic energy

$$T = \sum_{i=1}^{N} \frac{m_i}{2} \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \sum_{i=1}^{N} \frac{m_i}{2} v_i^2 \tag{5.26}$$

and so by using the identity (5.25) the left-hand side of equation (5.23) can be expressed

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\vec{r}}_i \cdot \delta\vec{r}_i) - \delta T \tag{5.27}$$

where δT is the variation in kinetic energy. Substitute the equations (5.27) and (5.24) into the equation (5.23) and express the result in the form

$$\delta T + \delta W = \sum_{i=1}^{N} m_i \frac{d}{dt} \left(\dot{\vec{r}}_i \cdot \delta\vec{r}_i \right) \tag{5.28}$$

Assume that the variations in path satisfy the conditions $\delta\vec{r}_i(t_1) = 0$ and $\delta\vec{r}_i(t_2) = 0$ and integrate the equation (5.28) from t_1 to t_2 to obtain

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \sum_{i=1}^{N} m_i (\dot{\vec{r}}_i \cdot \delta\vec{r}_i) \Big|_{t_1}^{t_2} = 0 \tag{5.29}$$

If there exists a potential function V such that $\delta W = -\delta V$, then the equation (5.29) can be written in the form

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0. \tag{5.30}$$

Define the quantity $L = T - V$ as the Lagrangian function associated with the motion of the system of particles, where T is the kinetic energy of the system of particles and V is the potential energy of the system of particles. Hamilton's principle can then be expressed

$$I = \int_{t_1}^{t_2} L\, dt \qquad \text{is an extremum, so that} \qquad \delta I = \delta \int_{t_1}^{t_2} L\, dt = 0.$$

The integral

$$I = \int_{t_1}^{t_2} L\, dt \tag{5.31}$$

is called a Hamilton integral and Hamilton's principle is that the path of motion of a particle or system of particles is such that the Hamilton integral is an extremum, usually a minimum. The equations of motion of the particles are then along the paths defined by the system of Euler-Lagrange equations. If the Lagrangian is a function of position coordinates $q_1, q_2, \ldots, q_N$ and velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N$, called generalized coordinates and their time derivatives, then the Lagrangian can be written $L = L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$ and the equations of motion of the system are represented by the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, N \tag{5.32}$$