

Chapter 4

Additional Variational Concepts

In the previous chapter we considered calculus of variation problems which had fixed boundary conditions. That is, in one dimension the end point conditions were specified. In two and three dimensions, a boundary condition or surface condition was specified. In this chapter we shall consider other types of boundary conditions together with subsidiary conditions or constraints imposed upon the class of admissible solutions which produce an extremal value for a given functional. We shall also examine necessary conditions for a maximum or minimum value of a functional to exist. Let us begin by examining the one-dimensional case where either one or both boundary conditions are not prescribed.

Natural boundary conditions

Consider the problem of finding a function $y = y(x)$ such that the functional

$$I = I(y) = \int_{x_1}^{x_2} f(x, y, y') dx \quad (4.1)$$

has a stationary value where either one or both of the end point conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ are not prescribed. One proceeds exactly as we have done previously. Assume that $y = y(x)$ produces an extremal value and construct the class of comparison functions $Y(x) = y(x) + \epsilon\eta(x)$. Substitute $Y(x)$ into the functional given by equation (4.1) to obtain

$$I = I(\epsilon) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta') dx.$$

By hypothesis, the functional I has a stationary value at $\epsilon = 0$ so that

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = I'(0) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' \right) dx = 0. \quad (4.2)$$

Now integrate the second term in equation (4.2) using integration by parts to obtain

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = I'(0) = \left. \frac{\partial f}{\partial y'} \eta \right|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \eta dx = 0. \quad (4.3)$$

Assume that the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad (4.4)$$

is satisfied over the interval $x_1 \leq x \leq x_2$, then in order for equation (4.3) to be satisfied it is necessary that the first term of equation (4.3) also equal zero. This requires that

$$\left. \frac{\partial f}{\partial y'} \eta \right|_{x_1}^{x_2} = \left. \frac{\partial f}{\partial y'} \right|_{x=x_2} \eta(x_2) - \left. \frac{\partial f}{\partial y'} \right|_{x=x_1} \eta(x_1) = 0. \quad (4.5)$$

In the case of fixed end points, the Euler-Lagrange equation together with the end conditions $\eta(x_1) = 0$ and $\eta(x_2) = 0$ guarantees that the equation (4.5) is satisfied. In the case both end points are variable, then the terms $\eta(x_1)$ and $\eta(x_2)$ are arbitrary, and so in order for equation (4.5) to be satisfied one must require in addition to the Euler-Lagrange equation, the end point conditions

$$\left. \frac{\partial f}{\partial y'} \right|_{x=x_1} = 0, \quad \text{and} \quad \left. \frac{\partial f}{\partial y'} \right|_{x=x_2} = 0. \quad (4.6)$$

These conditions are called the natural boundary conditions or transversality conditions associated with the extremum problem. The case of mixed boundary conditions occurs when one end of the curve is fixed and the other end can vary. The above considerations give rise to the following boundary cases.

Case 1: (Fixed end points) If $y(x_1) = y_1$ and $y(x_2) = y_2$ are prescribed, then there is no variation of the end points so that $\eta(x_1) = 0$ and $\eta(x_2) = 0$.

Case 2: (Mixed end condition) If $y(x_1) = y_1$ is given and $y(x_2)$ is unknown, then one must impose the natural boundary condition $\left. \frac{\partial f}{\partial y'} \right|_{x=x_2} = 0$ and $\eta(x_1) = 0$.

Case 3: (Mixed end condition) If $y(x_2) = y_2$ is given and $y(x_1)$ is unknown, then one must impose the natural boundary condition $\left. \frac{\partial f}{\partial y'} \right|_{x=x_1} = 0$ and $\eta(x_2) = 0$.

Case 4: (Variable end points) If $y = y(x)$ is not prescribed at the end points of the given interval $x_1 \leq x \leq x_2$, then the partial derivative $\frac{\partial f}{\partial y'}$ must satisfy the natural boundary conditions given by equations (4.6).

Note that the requirement that $\eta = 0$ at an end point is equivalent to specifying the value of the dependent variable y at an end point. Boundary conditions where $\eta = 0$ or $\delta y = 0$, where the values of y are specified at a boundary, are called essential boundary conditions sometimes referred to as Dirichlet boundary conditions or geometric boundary conditions. Natural boundary conditions are sometimes referred to as Neumann boundary conditions or dynamic boundary conditions. Mixed boundary value problems, sometimes referred to as Robin boundary value problems, occur when both essential and natural boundary are specified on portions of the boundary. A general rule is that the vanishing of the variation η on a boundary is an essential boundary condition and the vanishing of the term that multiplies η is called a natural boundary condition. This general rule, and the above terminology, with slight modifications, can be applied to one, two and three dimensional problems.

Example 4-1. Natural boundary condition

Find the curve $y = y(x)$ producing the shortest distance between the points x_0 and x_1 subject to natural boundary conditions.

Solution: Let $ds^2 = dx^2 + dy^2$ denote an element of distance squared in Cartesian coordinates. One can then form the functional

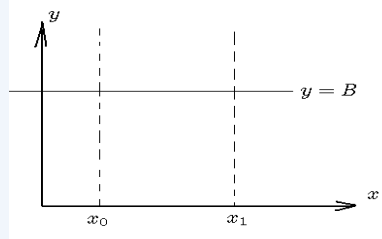
$$I = \int_{x_0}^{x_1} ds = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

The integrand for this functional is $f = \sqrt{1 + (y')^2}$ with partial derivatives

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

The Euler-Lagrange equation associated with this functional is $y'' = 0$ which is subject to the natural boundary conditions

$$\begin{aligned} \frac{\partial f}{\partial y'} \Big|_{x=x_0} &= \frac{y'(x_0)}{\sqrt{1 + (y'(x_0))^2}} = 0 \\ \frac{\partial f}{\partial y'} \Big|_{x=x_1} &= \frac{y'(x_1)}{\sqrt{1 + (y'(x_1))^2}} = 0 \end{aligned}$$



The Euler-Lagrange equation has the solution $y = y(x) = Ax + B$ where A, B are constants. The natural boundary conditions require that $A = 0$ and so the solution is $y = B = \text{constant}$. ■

Natural boundary conditions for other functionals

Consider the functional

$$I = \int_a^b f(x, y, y', y'') dx \quad (4.7)$$

where we hold the variation in x constant and consider only the variation in y . This functional has first variation

$$\delta I = \int_a^b \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' + \frac{\partial f}{\partial y''} \delta y'' \right) dx = 0. \quad (4.8)$$

Use integration by parts on the second term of equation (4.8) using

$$U = \frac{\partial f}{\partial y'}, \quad dU = \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx, \quad dV = \delta y' dx, \quad V = \delta y$$

and then use integration by parts on the third term in equation (4.8) using

$$U = \frac{\partial f}{\partial y''}, \quad dU = \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) dx, \quad dV = \delta y'' dx, \quad V = \delta y'$$

and verify that equation (4.8) can be expressed in the form

$$\delta I = \int_a^b \left[\frac{\partial f}{\partial y} \delta y - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \delta y - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \delta y' \right] dx + \frac{\partial f}{\partial y'} \delta y \Big|_a^b + \frac{\partial f}{\partial y''} \delta y' \Big|_a^b = 0 \quad (4.9)$$

Now use integration by parts again on the third term under the integral in equation (4.9) to show

$$\delta I = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) \right] \delta y dx + \left[\frac{\partial f}{\partial y'} - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \right] \delta y \Big|_a^b + \frac{\partial f}{\partial y''} \delta y' \Big|_a^b = 0. \quad (4.10)$$

The Euler-Lagrange equation associated with the functional given by equation (4.7) is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0.$$

This is a fourth order ordinary differential equation so that the general solution will contain four arbitrary constants. The boundary conditions come from the requirement that the remaining terms on the right-hand side of equation (4.10) must equal zero. This requires

$$\left[\frac{\partial f}{\partial y'} - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \right] \delta y \Big|_a^b = 0 \quad \text{and} \quad \frac{\partial f}{\partial y''} \delta y' \Big|_a^b = 0. \quad (4.11)$$

This produces the essential boundary conditions δy and $\delta y'$ equal to zero at $x = a$ and $x = b$ producing the conditions that $y(a), y(b)$ and $y'(a), y'(b)$ are specified at the end points. Alternatively, one can use the natural boundary conditions that $\frac{\partial f}{\partial y'} - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right)$ and $\frac{\partial f}{\partial y''}$ are zero at the end points $x = a$ and $x = b$, or one can employ some combination of mixed boundary conditions determined from the equation (4.11). For example, one can select $\delta y = 0$ at $x = a$ and $x = b$ together with $\frac{\partial f}{\partial y''} = 0$ at $x = a$ and $x = b$ as a mixed boundary condition involving both essential and natural boundary conditions. The other combination is to assume $\delta y' = 0$ at $x = a$ and $x = b$ together with $\frac{\partial f}{\partial y'} - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) = 0$ at the end points. Note that the conditions $\delta y = 0$ and $\delta y' = 0$ implies that $y(a), y(b), y'(a), y'(b)$ are all specified so that there is no variation of these quantities at the end points.

Example 4-2. Natural boundary condition

Find the function $y = y(x)$ such that the functional

$$I = \int_a^b [a(x)(y'')^2 - b(x)(y')^2 + c(x)y^2] dx$$

is an extremum.

Solution: Here $f = a(x)(y'')^2 - b(x)(y')^2 + c(x)y^2$ with partial derivatives

$$\frac{\partial f}{\partial y} = 2c(x)y, \quad \frac{\partial f}{\partial y'} = -2b(x)y', \quad \frac{\partial f}{\partial y''} = 2a(x)y''$$

The Euler-Lagrange equation associated with the above functional is the fourth order ordinary differential equation

$$c(x)y + \frac{d}{dx} (b(x)y') + \frac{d^2}{dx^2} (a(x)y'') = 0, \quad a \leq x \leq b$$

Various types of boundary conditions can be applied to obtain a solution to this differential equation.

Case 1 If $\delta y = 0$ and $\delta y' = 0$ at the end points, then $y(a), y'(a), y(b), y'(b)$ must be prescribed.

Case 2 If $\delta y = 0$ and $\delta y' \neq 0$ at the end points, then $y(a)$ and $y(b)$ must be prescribed and the natural boundary conditions

$$\frac{\partial f}{\partial y''} \Big|_{x=a} = 0, \quad \text{and} \quad \frac{\partial f}{\partial y''} \Big|_{x=b} = 0$$

must be satisfied. This implies $y''(a)$ and $y''(b)$ have prescribed values.

Case 3 If $\delta y \neq 0$ and $\delta y' = 0$ at the end points, then $y'(a)$ and $y'(b)$ are prescribed and the natural boundary conditions

$$\left[\frac{\partial f}{\partial y'} - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \right]_{x=a} = 0, \quad \text{and} \quad \left[\frac{\partial f}{\partial y'} - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \right]_{x=b} = 0,$$

must be satisfied.

Case 4 If $\delta y \neq 0$ and $\delta y' \neq 0$ at the end points, then the natural boundary conditions

$$\frac{\partial f}{\partial y''} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) = 0$$

at the end points $x = a$ and $x = b$ must be satisfied. ■

More natural boundary conditions

Consider the functional

$$I = \iint_R f(x, y, w, w_x, w_y) dx dy \quad (4.12)$$

where $w = w(x, y)$ is a function of x and y and R is a region enclosed by a simple closed curve $C = \partial R$. We use the variational notation and require that at an extremum the first variation is zero so that

$$\delta I = \iint_R \left[\frac{\partial f}{\partial w} \delta w + \frac{\partial f}{\partial w_x} \delta w_x + \frac{\partial f}{\partial w_y} \delta w_y \right] dx dy = 0. \quad (4.13)$$

Now use the Green's theorem in the plane

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \oint_C M dx + N dy \quad (4.14)$$

with $N = \frac{\partial f}{\partial w_x} \delta w$ and $M = -\frac{\partial f}{\partial w_y} \delta w$ to integrate the last two terms under the integral on the right-hand side of equation (4.13). Note that for the above choices for M and N we have

$$\frac{\partial N}{\partial x} = \frac{\partial f}{\partial w_x} \delta w_x + \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial w_x} \right) \delta w \quad \text{and} \quad \frac{\partial M}{\partial y} = -\frac{\partial f}{\partial w_y} \delta w_y - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial w_y} \right) \delta w$$

and so the use of the Green's theorem produces the result

$$\begin{aligned} \iint_R \left(\frac{\partial f}{\partial w_x} \delta w_x + \frac{\partial f}{\partial w_y} \delta w_y \right) dx dy &= \oint_C \left(-\frac{\partial f}{\partial w_y} \delta w dx + \frac{\partial f}{\partial w_x} \delta w dy \right) \\ &\quad - \iint_R \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial w_x} \right) \delta w + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial w_y} \right) \delta w \right] dx dy \end{aligned} \quad (4.15)$$

Substitute the result from equation (4.15) into the equation (4.13) and simplify to obtain

$$\begin{aligned} \delta I &= \iint_R \left[\frac{\partial f}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial w_y} \right) \right] \delta w dx dy \\ &\quad + \oint_C \delta w \left(-\frac{\partial f}{\partial w_y} dx + \frac{\partial f}{\partial w_x} dy \right) = 0. \end{aligned} \quad (4.16)$$

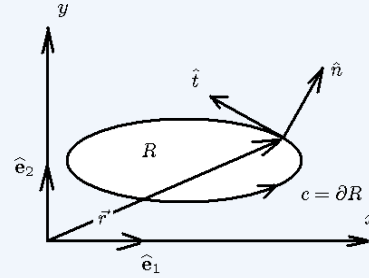
If the equation (4.16) is to be zero, then $w = w(x, y)$ must satisfy the Euler-Lagrange equation

$$\frac{\partial f}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial w_y} \right) = 0$$

and the boundary conditions come from an analysis of the remaining line integral term. If $\delta w = 0$, then the boundary conditions required are for the function $w = w(x, y)$ to be specified everywhere along the boundary curve $C = \partial R$. If $\delta w \neq 0$, then we require that

$$-\frac{\partial f}{\partial w_y} dx + \frac{\partial f}{\partial w_x} dy \Big|_{(x,y) \in \partial R} = 0. \quad (4.17)$$

This condition can be written in terms of the unit normal vector to the boundary curve $C = \partial R$. Note that if $\vec{r} = x(s)\hat{e}_1 + y(s)\hat{e}_2$ is the position vector defining the boundary curve C , then $\frac{d\vec{r}}{ds} = \hat{t} = \frac{dx}{ds}\hat{e}_1 + \frac{dy}{ds}\hat{e}_2$ is the unit tangent vector to a point on the boundary curve. The cross product



$$\hat{n} = \hat{t} \times \hat{e}_3 = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds}\hat{e}_1 - \frac{dx}{ds}\hat{e}_2$$

gives the unit normal vector to the boundary curve. The boundary condition given by equation (4.17) can then be written in the form

$$\hat{n} \cdot \left(\frac{\partial f}{\partial w_x} \hat{e}_1 + \frac{\partial f}{\partial w_y} \hat{e}_2 \right) \Big|_{(x,y) \in \partial R} = 0 \quad (4.18)$$

where $\hat{n} = \frac{dy}{ds}\hat{e}_1 - \frac{dx}{ds}\hat{e}_2$ is a unit normal vector to the boundary curve $C = \partial R$. The condition (4.18), or (4.17), is called the natural boundary condition associated with the functional given by equation (4.12).

Tests for maxima and minima

So far we have been solving the Euler-Lagrange equation and stating that either a maximum or minimum value exists. We have not actually proved these results. The following are two tests for extremum problems associated with the functional given by equation (4.1). These tests are known as Legendre's test and Jacobi's test.

The Legendre and Jacobi analysis

Consider the functional

$$I = I(y) = \int_{x_1}^{x_2} f(x, y, y') dx \quad (4.19)$$

and assume that $y = y(x)$ produces an extremum. If we replace y by the set of comparison functions $Y = y + \epsilon\eta$, where we assume weak variations with $\eta(x_1) = 0$ and $\eta(x_2) = 0$. One can then treat I as a function of ϵ and write $I = I(\epsilon)$ which can be expanded in a Taylor series about $\epsilon = 0$. This produces the result

$$I = I(\epsilon) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta') dx = I(0) + I'(0)\epsilon + I''(0)\frac{\epsilon^2}{2!} + I'''(0)\frac{\epsilon^3}{3!} + \cdots \quad (4.20)$$

The change in the value of the functional I is given by

$$\Delta I = \int_{x_1}^{x_2} [f(x, y + \epsilon\eta, y' + \epsilon\eta') - f(x, y, y')] dx \quad (4.21)$$

Expanded the integrand of equation (4.21) in a Taylor series about $\epsilon = 0$ to produced

$$\begin{aligned} f(x, y + \epsilon\eta, y' + \epsilon\eta') - f(x, y, y') = \\ \frac{\epsilon}{1!} \frac{\partial f(x, y + \epsilon\eta, y' + \epsilon\eta')}{\partial \epsilon} \Big|_{\epsilon=0} + \frac{\epsilon^2}{2!} \frac{\partial^2 f(x, y + \epsilon\eta, y' + \epsilon\eta')}{\partial \epsilon^2} \Big|_{\epsilon=0} + \dots \\ + \frac{\epsilon^{n-1}}{(n-1)!} \frac{\partial^{(n-1)} f(x, y + \epsilon\eta, y' + \epsilon\eta')}{\partial \epsilon^{n-1}} \Big|_{\epsilon=0} + R_n \end{aligned} \quad (4.22)$$

An integration of both sides of equation (4.22) from x_1 to x_2 produces

$$\begin{aligned} \Delta I = \frac{\epsilon}{1!} \int_{x_1}^{x_2} \frac{\partial f(x, y + \epsilon\eta, y' + \epsilon\eta')}{\partial \epsilon} \Big|_{\epsilon=0} dx + \frac{\epsilon^2}{2!} \int_{x_1}^{x_2} \frac{\partial^2 f(x, y + \epsilon\eta, y' + \epsilon\eta')}{\partial \epsilon^2} \Big|_{\epsilon=0} dx + \dots \\ + \frac{\epsilon^{n-1}}{(n-1)!} \int_{x_1}^{x_2} \frac{\partial^{(n-1)} f(x, y + \epsilon\eta, y' + \epsilon\eta')}{\partial \epsilon^{(n-1)}} \Big|_{\epsilon=0} dx + \int_{x_1}^{x_2} R_n dx \end{aligned} \quad (4.23)$$

The coefficients of quantities $\epsilon/1!, \epsilon^2/2!, \dots, \epsilon^{(n-1)}/(n-1)!$ are referred to as the first, second, \dots , $(n-1)$ st variations of the functional I and written as $\delta I_1, \delta I_2, \dots, \delta I_{(n-1)}$. One can calculate the derivatives in equation (4.23) and verify that these variations are given by

$$\delta I_1 = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' \right] dx \quad (4.24)$$

$$\delta I_2 = \int_{x_1}^{x_2} \left[\frac{\partial^2 f}{\partial y^2} \eta^2 + 2 \frac{\partial^2 f}{\partial y \partial y'} \eta \eta' + \frac{\partial^2 f}{\partial y'^2} (\eta')^2 \right] dx \quad (4.25)$$

$$\delta I_3 = \int_{x_1}^{x_2} \left[\frac{\partial^3 f}{\partial y^3} \eta^3 + 3 \frac{\partial^3 f}{\partial y^2 \partial y'} \eta^2 \eta' + 3 \frac{\partial^3 f}{\partial y \partial y'^2} \eta \eta'^2 + \frac{\partial^3 f}{\partial y'^3} \eta'^3 \right] dx \quad (4.26)$$

\vdots

Then the change in the value of the functional I can be written

$$\Delta I = \frac{\epsilon}{1!} \delta I_1 + \frac{\epsilon^2}{2!} \delta I_2 + \frac{\epsilon^3}{3!} \delta I_3 + \dots + \frac{\epsilon^{(n-1)}}{(n-1)!} \delta I_{n-1} + \int_{x_1}^{x_2} R_n dx \quad (4.27)$$

Some textbooks write these variations using the notations

$$\Delta I = \delta I + \frac{1}{2!} \delta^2 I + \frac{1}{3!} \delta^3 I + \dots \quad (4.28)$$

The relation between these different notations is

$$\delta^m I = \epsilon^m \delta I_m = I^{(m)}(0) \epsilon^m \quad \text{for } m = 1, 2, 3, \dots \quad (4.29)$$

Examine the first variation and assume that $\eta = 0$ at the end points of the integration interval. Use integration by parts on the second term in equation (4.24) to obtain

$$\delta I_1 = \left[\frac{\partial f}{\partial y'} \eta \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \eta dx. \quad (4.30)$$

The end point conditions insure that the first term in equation (4.30) is zero. In order for the second term in equation (4.30) to be zero the function $y = y(x)$ must satisfy the Euler-Lagrange equation subject to boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. If these conditions are satisfied one can write that $\left. \frac{\delta I}{\delta \epsilon} \right|_{\epsilon=0} = 0$ is a necessary condition that the functional assume a stationary value.

We assume that the extremal curves y which are solutions of the Euler-Lagrange equations and the derivatives of y are continuous functions. However, this is not always the case. Discontinuous solutions to the Euler-Lagrange equation are also possible. In these cases the function y must be continuous but the first or higher derivatives may be discontinuous functions of the independent variable x . Discontinuous solutions are represented by calculating continuous curves over different sub-intervals of the solution domain and then requiring continuity of these solutions at the end points of the sub-intervals. In this way a continuous curve is constructed by joining the solution pieces at the end points of the sub-intervals so that the resulting curve is continuous. In this type of construction one is confronted with corners at the junction points where the left and right-hand derivatives are not always the same. At all corner points the quantities $\frac{\partial f}{\partial y'}$ and $f - \frac{\partial f}{\partial y'} y'$ must be continuous. These conditions are associated with the Weierstrass-Erdmann corner conditions to be discussed later in this chapter. For present we assume continuity for the functions y and its derivative $\frac{dy}{dx}$, where y is a solution of the Euler-Lagrange equation.

If the first variation is zero, $\delta I_1 = 0$, then the sign of the functional change ΔI is determined by the sign of δI_2 . If $\delta I_2 > 0$ is positive and of constant sign at all points of the extremal arc $y(x)$, then a relative minimum exists. If $\delta I_2 < 0$, is negative and of constant sign at all points of the extremal arc $y(x)$, then a relative maximum exists. Note that the maximum or minimum value assigned to the functional must be independent of the value of η and ϵ . If δI_2 changes sign over the extremal arc, then the stationary value is neither a maximum or minimum. Examine equation (4.27) and note that if both δI_1 and δI_2 are zero, then the sign of ΔI depends upon the sign of $\epsilon^3 \delta I_3$ which changes sign for $\epsilon > 0$ and $\epsilon < 0$, so that there can be no maximum or minimum value unless δI_3 is also zero. Consequently, for these conditions the sign of ΔI is determined by an examination of the sign of the δI_4 variation, if it is different from zero.

We wish to examine the second variation

$$\delta I_2 = \int_{x_1}^{x_2} \left[\frac{\partial^2 f}{\partial y^2} \eta^2 + 2 \frac{\partial^2 f}{\partial y \partial y'} \eta \eta' + \frac{\partial^2 f}{\partial y'^2} (\eta')^2 \right] dx \quad (4.31)$$

under the conditions of a weak variation where $\eta(x_1) = 0$, $\eta(x_2) = 0$ and the Euler-Lagrange equation $f_y - \frac{d}{dx}(f_{y'}) = 0$ is satisfied by a function $y(x)$ which satisfies end point conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. If $y(x)$ is a solution of the Euler-Lagrange equation and the functional I has a minimum value, then δI_2 must be positive for all values of η over the interval $x_1 \leq x \leq x_2$. If $y(x)$ is a solution of the Euler-Lagrange equation and the functional I has a maximum value, then δI_2 must be negative for all values of η over the interval $x_1 \leq x \leq x_2$.

The second variation can be written in several different forms for analysis. One form is to integrate the middle term in equation (4.31) using integration by parts to obtain

$$\delta I_2 = \left[\eta^2 \frac{\partial^2 f}{\partial y \partial y'} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left\{ \eta^2 \left[\frac{\partial^2 f}{\partial y^2} - \frac{d}{dx} \left(\frac{\partial^2 f}{\partial y \partial y'} \right) \right] + (\eta')^2 \left[\frac{\partial^2 f}{\partial y'^2} \right] \right\} dx \quad (4.32)$$

Assume η is zero at the end points, then one can say that a necessary and sufficient condition for I to be a minimum is for $\delta I_2 > 0$ which requires that

$$\frac{\partial^2 f}{\partial y^2} - \frac{d}{dx} \left(\frac{\partial^2 f}{\partial y \partial y'} \right) > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y'^2} > 0 \quad (4.33)$$

for all smooth weak variations η .

One can also write the second variation in the form

$$\delta I_2 = \int_{x_1}^{x_2} \{ [\eta^2 f_{yy} + \eta \eta' f_{yy'}] + \eta' [\eta f_{yy'} + \eta' f_{y'y'}] \} dx \quad (4.34)$$

where subscripts denote partial differentiation. Integrate the second half of the integral (4.34) using integration by parts to obtain after simplification

$$\delta I_2 = [\eta^2 f_{yy'} + \eta \eta' f_{y'y'}]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\eta^2 f_{yy} - \eta^2 \frac{d}{dx} (f_{yy'}) - \eta \frac{d}{dx} (\eta' f_{y'y'}) \right] dx \quad (4.35)$$

By assumption the first term is zero at the end points. We now express the equation (4.35) using a differential operator $L(\)$ and write

$$\delta I_2 = - \int_{x_1}^{x_2} \eta L(\eta) dx \quad (4.36)$$

where $L(\eta)$ is the differential operator

$$L(\eta) = \frac{d}{dx} \left[f_{y'y'} \frac{d\eta}{dx} \right] - \left[f_{yy} - \frac{d}{dx} (f_{yy'}) \right] \eta \quad (4.37)$$

The differential operator (4.37) has the basic form

$$L(u) = \frac{d}{dx} \left[p(x) \frac{du}{dx} \right] - q(x)u \quad (4.38)$$

where

$$p(x) = f_{y'y'} \quad \text{and} \quad q(x) = f_{yy} - \frac{d}{dx} (f_{yy'}) \quad (4.39)$$

and the differential equation

$$L(u) = \frac{d}{dx} \left[p(x) \frac{du}{dx} \right] - q(x)u = 0, \quad x_1 \leq x \leq x_2 \quad (4.40)$$

is known as Jacobi's differential equation. The operator $L(\)$ is a self-adjoint Sturm-Liouville operator which satisfies the Lagrange identity

$$uL(\eta) - \eta L(u) = \frac{d}{dx} [p(x)(u\eta' - \eta u')] \quad (4.41)$$

We make use of the Lagrange identity and write the second variation (4.36) in the form

$$\delta I_2 = - \int_{x_1}^{x_2} \frac{\eta}{u} u L(\eta) dx = - \int_{x_1}^{x_2} \frac{\eta}{u} \left(\eta L(u) + \frac{d}{dx} [p(x)(u\eta' - \eta u')] \right) dx \quad (4.42)$$

This form can be simplified by using integration by parts on the last integral with

$$\begin{aligned} U &= \eta/u & dV &= \frac{d}{dx} [p(x)(u\eta' - \eta u')] dx \\ dU &= \left(\frac{u\eta' - \eta u'}{u^2} \right) dx & V &= p(x)(u\eta' - \eta u') \end{aligned} \quad (4.43)$$

The second variation can now be expressed in the form

$$\delta I_2 = - \int_{x_1}^{x_2} \frac{\eta^2}{u} L(u) dx - \frac{\eta}{u} p(x)(u\eta' - \eta u') \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} p(x) \left(\frac{u\eta' - \eta u'}{u} \right)^2 dx$$

We substitute the value $p(x) = f_{y'y'}$ from equation (4.39) and rearrange terms to write the second variation as

$$\delta I_2 = - \int_{x_1}^{x_2} \frac{\eta^2}{u} L(u) dx + \frac{\eta}{u} f_{y'y'} (\eta u' - u\eta') \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} f_{y'y'} \left(\eta' - \eta \frac{u'}{u} \right)^2 dx \quad (4.44)$$

Let us analyze the second variation given by equation (4.44). For the time being we shall assume that the following conditions are satisfied.

Condition (i)

Assume u is a nonzero solution to the Jacobi differential equation

$L(u) = 0$ for $x_1 \leq x \leq x_2$, with initial conditions $u(x_1) = 0$ and $u'(x_1) \neq 0$.

Condition (ii)

Assume the end point conditions $\eta(x_1) = 0$ and $\eta(x_2) = 0$ so that we are assured that the middle term in equation (4.44) is zero.

Condition (iii)

Assume $\left(\eta' - \eta \frac{u'}{u} \right)^2 \neq 0$ so that the sign of the third term in equation (4.44) is determined by the sign of $f_{y'y'}$.

then one can say that

The condition $f_{y'y'} > 0$ for $x_1 \leq x \leq x_2$ is a necessary condition for the functional $I(y)$ to be a minimum.

The condition $f_{y'y'} < 0$ for $x_1 \leq x \leq x_2$ is a necessary condition for the functional $I(y)$ to be a maximum.

These are known as Legendre's necessary conditions for an extremal to exist. We shall return to analyze the above conditions after first having developed some necessary background material associated with the Jacobi differential equation.

Background material for the Jacobi differential equation

We assume that $y = y(x, c_1, c_2)$ is a two-parameter solution of the Euler-Lagrange equation

$$\frac{\partial f(x, y, y')}{\partial y} - \frac{d}{dx} \left(\frac{\partial f(x, y, y')}{\partial y'} \right) = 0, \quad x_1 \leq x \leq x_2 \quad (4.45)$$