Chapter 3

Introduction to the Calculus of Variations

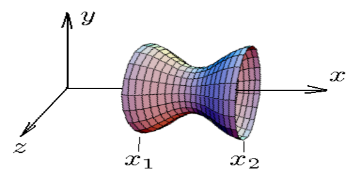
We continue our investigation of finding maximum and minimum values associated with various quantities. Instead of finding points where functions have a relative maximum or minimum value over some domain $x_1 \le x \le x_2$, we examine situations where certain curves have the property of producing a maximum or minimum value. For example, the problem of determining the curve or shape which minimizes drag force and maximizes lift force on an airplane wing moving at a given speed requires that one find a special function $y = y(x)$ defining the shape of the curve which "best" achieves the desired objective.

We begin by studying a function and possibly some of its derivatives that determine the value of an integral. We vary the function and determine its effect on the value of the integral. We try to find the function which makes the integral have a maximum or minimum value. This is a basic calculus of variations problem. The methods developed in the study of the variational calculus introduce new concepts and principles. These new variational principles can then be employed to view certain problems in physics and mechanics from a new and different viewpoint. We begin with the simplest calculus of variations problems which have fixed end points.

Functionals

A functional is a mapping which assigns a real number to each function or curve associated with some class of functions. Some examples of functionals are the following.

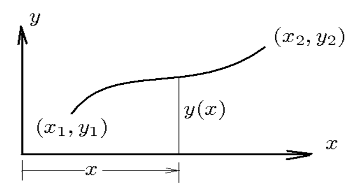
1. In Cartesian coordinates consider all plane curves $y = y(x)$ which pass through two given points (x_1, y_1) and (x_2, y_2). When one of these curves is rotated about the x-axis a surface of revolution is produced. The surface area S which is generated is given by



$$S = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{3.1}$$

and the scalar value obtained depends upon the function $y(x)$ selected. Finding the particular function $y = y(x)$ which produces the minimum surface area is an example of a calculus of variation problem.

2. In Cartesian coordinates consider all plane curves $y = y(x)$ which pass through two given points (x_1, y_1) and (x_2, y_2). The length ℓ along one particular curve between the given points is obtained by integrating the element of arc length $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ between the limits x_1 and x_2 to obtain



$$\ell = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{3.2}$$

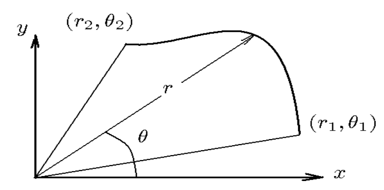
The scalar value representing the length depends upon the curve $y(x)$ selected. The problem of finding the curve $y = y(x)$, which produces the minimum length, is a calculus of variations problem.

3. In Cartesian coordinates consider all plane curves $x = x(y)$ which pass through two given points (x_1, y_1) and (x_2, y_2). The length ℓ along one particular curve between the given end points is determined from the integral

$$\ell = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{3.3}$$

and this length depends upon the curve $x = x(y)$ selected. Finding the curve which produces the minimum length is a calculus of variations problem. This is the same problem as the previous example but formulated with x as the dependent variable and y as the independent variable.

4. In polar coordinates consider all plane curves $r = r(\theta)$ which pass through the points (r_1, θ_1) and (r_2, θ_2). The length ℓ along one particular curve between the given points is obtained by integrating the element of arc length $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$



$$\ell = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{3.4}$$

and the value of this integral depends upon the curve selected. Here the problem of calculating length along a curve is posed in polar coordinates. Note that the same problem can be formulated in many different ways by making a change of variables.

5. In Cartesian coordinates consider all parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$ which satisfy the end point conditions $x(a) = x_1,\ y(a) = y_1$ and $x(b) = x_2,\ y(b) = y_2$ where (x_1, y_1) and (x_2, y_2) are given fixed points. The length ℓ of one of these curves between the given points is determined by integrating the element of arc length to obtain the relation

$$\ell = \ell(x, y) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{3.5}$$

and the calculated length depends upon the curve selected through the end points. This is the same type of problem as the previous two problems. However, it is formulated in the form of finding parametric equations $x = x(t)$ and $y = y(t)$ which define the curve producing a minimum value for the arc length.

6. In three dimensional space consider all curves which pass through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2). This family of curves can be represented by the position vector

$$\vec{r} = \vec{r}(t) = t\,\widehat{\mathbf{e}}_1 + y(t)\,\widehat{\mathbf{e}}_2 + z(t)\,\widehat{\mathbf{e}}_3 \qquad x_1 \le t \le x_2$$

where $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$, $\widehat{\mathbf{e}}_3$ are unit base vectors in the directions of the x, y and z axes and the functions $y = y(t)$ and $z = z(t)$ are assumed to have continuous derivatives and satisfy the conditions $y(x_1) = y_1$, $y(x_2) = y_2$ and $z(x_1) = z_1$, $z(x_2) = z_2$. The arc length ℓ of one of these curves is given by the integral

$$\ell = \ell(y, z) = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

If we desire to find the minimum value for ℓ, then one must find the parametric functions $y = y(t)$ and $z = z(t)$ which produce this minimum value. This is the three-dimensional version of the previous problem.

7. Consider a family of curves $y = y(x)$ which are continuous and twice differentiable and pass through the points (x_1, y_1) and (x_2, y_2). Let each curve $y = y(x)$ in the family satisfy the conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. Consider an integral of the form

$$I = I(y) = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx \tag{3.6}$$

where the integrand $f = f(x, y(x), y'(x))$ is a given continuous function of x, y, y'. This is an example of a general functional where the value of the integral I depends upon the smooth function $y = y(x)$ through the given points. The problem of finding a smooth function $y = y(x)$ from the family of curves which makes the functional have a maximum or minimum value is a typical calculus of variations problem. In this introductory development we restrict our study to finding smooth curves which produce an extreme value. Later we shall admit discontinuous curves into our class of functions that can be substituted into the given integrals.

Basic lemma used in the calculus of variations

Consider the integral

$$\delta I = \int_a^b \eta(x)\beta(x)\, dx \tag{3.7}$$

where $\eta(x)$ is an arbitrary function which is defined and continuous over the interval $[a, b]$ and satisfies the end conditions $\eta(a) = 0$ and $\eta(b) = 0$. If $\delta I = 0$ for all arbitrary functions $\eta(x)$, satisfying the given conditions, then what can be said about the function $\beta(x)$? The answer to this question is given by the following lemma.

Basic Lemma

If $\beta = \beta(x)$ is continuous over the interval $a \le x \le b$ and if the integral

$$\delta I = \delta I(\eta) = \int_a^b \eta(x)\beta(x)\, dx = 0$$

for every continuous function $\eta(x)$ which satisfies $\eta(a) = \eta(b) = 0$, then necessarily $\beta(x) = 0$ for all values of $x \in [a, b]$.

The proof of the above lemma is a proof by contradiction. Assume that $\beta(x)$ is nonzero, say positive, for some point in the interval $[a,b]$. By hypothesis, the function $\beta(x)$ is continuous and so it must be positive for all values of x in some subinterval $[x_1,x_2]$ contained in the interval $[a,b]$. If this is true, then the integral $I = I(\eta)$ cannot be identically zero for every function $\eta(x)$. Consider the special function

$$\eta(x) = \begin{cases} 0, & a \le x \le x_1 \\ (x-x_1)^2(x-x_2)^2, & x_1 \le x \le x_2 \\ 0, & x_2 \le x \le b \end{cases} \tag{3.8}$$

which is positive over the subinterval where $\beta(x)$ is positive. Substituting this special function into the integral (3.7) and using the mean value theorem for integrals, there results

$$\delta I = \delta I(\eta) = \int_a^b \eta(x)\beta(x)\,dx = \int_{x_1}^{x_2} \eta(x)\beta(x)\,dx = (x_2 - x_1)\eta(x^*)\beta(x^*) > 0$$

for some value x^* satisfying $x_1 < x^* < x_2$. This is a contradiction to our original assumption that $I = I(\eta) = 0$ for all functions $\eta(x)$.

If we assume that the given integral (3.7) is zero for all possible functions $\eta = \eta(x)$ then we need only select $\eta = \beta(x)$ in order to establish the lemma. In this text we are only concerned with those $\eta = \eta(x)$ which have derivatives which are continuous functions.

The above lemma can be generalized to double, triple and multiple integrals of product functions $\eta\beta$. For example, consider an integral of the form

$$\delta I = \delta I(\eta) = \iint_R \eta(x,y)\beta(x,y)\,dxdy, \tag{3.9}$$

where the integration is over a region R of the x,y-plane. We use the notation ∂R to denote the curve representing the boundary of the region R. It is assumed that the region R is bounded by a simple closed curve ∂R. If the integral given by equation (3.9) is zero for all continuous functions η and in addition the function η is zero when evaluated at a boundary point (x,y) on the boundary curve ∂R of the region R, then it follows that $\beta(x,y) = 0$ for $(x,y) \in R$. The proof is very similar to the proof given for the single integral.

Notation

Let $f = f(x_1, x_2, \ldots, x_n)$ denote a real function of n-real variables defined and continuous over a region R. The notation, "f belongs to the class $\mathcal{C}^{(n)}$ in a region R", is employed to denote the condition that f and all its derivatives up to and including the nth order, exist and are continuous in the region R. This notation is sometimes shortened to the form, $f \in \mathcal{C}^{(n)}$ in a region R. For example, if $f = f(x,y,y')$, for $x_1 \le x \le x_2$, is such that $f \in \mathcal{C}^{(2)}$, then

$$f,\ \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial y'},\ \frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial y^2},\ \frac{\partial^2 f}{\partial x \partial y},\ \frac{\partial^2 f}{\partial x \partial y'},\ \frac{\partial^2 f}{\partial y \partial y'} \text{ and } \frac{\partial^2 f}{\partial y'^2}$$

all exist and are continuous over the interval $x_1 \le x \le x_2$.

General approach

We present an overview of the basic ideas that will be employed to analyze functionals such as

$$I = I(y) = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx \tag{3.10}$$

We consider changes in $I(y)$ as we vary the functions $y = y(x)$ that are used in the evaluation of the functional. Our general approach is as follows.

(i) If we can find a curve $y = y(x)$ for $x \in [x_1, x_2]$ such that for all other curves $Y = Y(x)$, belonging to some class, we have $I(y) \geq I(Y)$, then the curve y produces a maximum value for the functional $I(y)$.

(ii) If we can find a function $y = y(x)$ for $x \in [x_1, x_2]$ such that for all other curves $Y = Y(x)$, the inequality $I(y) \leq I(Y)$ is satisfied, then the curve y produces a minimum value for the functional $I(y)$.

(iii) Curves $Y = Y(x)$, which differ slightly from the curve y which produces an extreme value, can be denoted by $Y = Y(x) = y(x) + \epsilon\eta(x)$ where ϵ is a small quantity and the function η can be any arbitrary curve through the end points of the integral. Note that by varying the parameter ϵ we are varying the function which occurs in the functional. Upon substituting $Y = Y(x)$ into the functional (3.10) one obtains

$$I = I(\epsilon) = \int_{x_1}^{x_2} f(x, Y, Y')\, dx = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta')\, dx$$

which can then be viewed as a function of ϵ which has an extreme value at $\epsilon = 0$. If $I(\epsilon)$ is a continuous function of ϵ, then $\frac{dI}{d\epsilon}\Big|_{\epsilon=0} = 0$ corresponds to a stationary value for the functional.

We will use this general approach to find a way of determining all curves y which produce an extreme value for the functional I. We develop necessary conditions to be satisfied by each member of a family of curves which make a given functional have a extreme value.

We use the terminology "find an extremum for the functional" to mean– find the necessary conditions to be satisfied by the functions which produce a stationary value associated with the functional I above. To determine if a given curve produces an extreme value of a maximum or minimum value associated with a given functional requires further testing. Recall that the first derivative $\frac{dI}{d\epsilon} = 0$ is only a necessary condition for an extreme value to exist. The condition $\frac{dI}{d\epsilon} = 0$ gives us stationary values. An examination of the second derivative is required to analyze the stationary values to determine if the stationary values correspond to a maximum, minimum or saddle point.

In this chapter we study various types of functionals and develop necessary conditions to be satisfied such that these functionals take on a stationary value. We begin by considering only variations of functions which have fixed values at the end points of an interval or functions which have specified values on the boundary of a region in two or three-dimensions. These special functions are easier to handle. We examine other types of boundary conditions in the next chapter.

In the following discussions we examine functionals represented by various types of integrals over some region R. We label these different functionals by the type of integrand they have and we use the notation $[f1]$, $[f2]$, $[f3]$, $\ldots$ to denote these integrands for future reference.

[f1]: Integrand $\mathbf{f(x, y, y')}$

Consider the functional

$$I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx \tag{3.11}$$

where $f = f(x, y, y')$ is a given integrand and $f \in \mathcal{C}^{(2)}$ over the interval (x_1, x_2). Let us examine how the integral given by equation (3.11) changes as the function $y(x)$ changes. The variation of a function $y(x)$ and its derivative $y'(x)$ within an integral (3.11) is called a variational problem and is the most fundamental problem in the development of the variational calculus.

We require that the functions available for substitution into the functional (3.11) are such that (i) the integral exists and (ii) $y = y(x) \in \mathcal{C}^{(2)}$ over the region $R = \{x \,|\, x_1 \le x \le x_2\}$ and (iii) the functions $y = y(x)$ considered are required to satisfy the end point conditions

$$y_1 = y(x_1) \qquad \text{and} \qquad y_2 = y(x_2). \tag{3.12}$$

We now illustrate a procedure that can be used to construct a differential equation whose solution family makes the integral I, given by equation (3.11), have a stationary value. We begin by assuming that we know the function $y(x)$ which defines a curve C that makes the given functional have a stationary value and then consider a comparison function

$$Y = Y(x) = y(x) + \epsilon\eta(x) \tag{3.13}$$

which defines a curve C^*, where ϵ is a small parameter and $\eta(x)$ is an arbitrary function which is defined and continuous over the interval $[x_1, x_2]$ and satisfies the end conditions $\eta(x_1) = 0$ and $\eta(x_2) = 0$.

Here the end conditions on $\eta(x)$ have been selected such that the comparison function satisfies the same end conditions as the curve which produces a stationary value. The comparison function therefore satisfies the end point conditions $Y(x_1) = y_1$ and $Y(x_2) = y_2$ for all values of the parameter ϵ. The situation is illustrated in the figure 3-1.

A weak variation is said to exist if the function η is independent of ϵ and in the limit as ϵ tends to zero the comparison curve C^* approaches the optimal curve C and simultaneously the slopes along C^* approach the slopes of the curve C for all values of x satisfying $x_1 \le x \le x_2$. That is, for a weak variation we assume that

$$|Y(x) - y(x)| \qquad \text{and} \qquad |Y'(x) - y'(x)|,$$

are both small and approach zero as epsilon tends toward zero. If these conditions are not satisfied, then a strong variation is said to exist. Note that comparison functions of the form $Y(x) = y(x) + \epsilon\eta(x, \epsilon)$, where η is a function of both x and ϵ, produces the functional

$$I(\epsilon) = \int_{x_1}^{x_2} f(x, y(x) + \epsilon\eta(x, \epsilon), y'(x) + \epsilon\eta'(x, \epsilon))\, dx \tag{3.14}$$

which may or may not approach the functional of equation (3.11) as ϵ tends toward zero. Weierstrass gave the following example of a strong variation, $\eta = \eta(x,\epsilon) = \sin\left[\frac{(x-x_1)\pi}{\epsilon^n}\right]$, where n is a positive integer. In this case we have the limits $\lim_{\epsilon\to 0} |Y(x)-y(x)| = \lim_{\epsilon\to 0} \epsilon\eta(x,\epsilon) = 0$ but $\lim_{\epsilon\to 0} |Y'(x)-y'(x)| \neq 0$. We begin by considering only weak variations because weak variations can lead to maximum and minimum values of the functional given by the equation (3.11). In contrast, maximum and minimum values for the functional (3.11) may or may not occur if strong variations are considered.

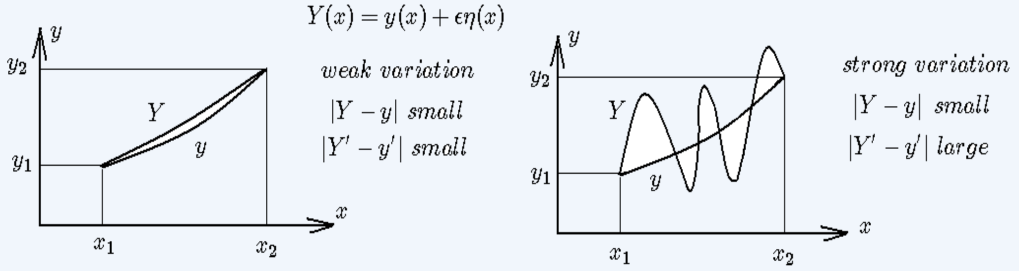


Figure 3-1. Variations from the optimal path.

Substituting the comparison function given by equation (3.13) into the integral given by equation (3.11) gives

$$I = I(\epsilon) = \int_{x_1}^{x_2} f(x,Y,Y')\,dx = \int_{x_1}^{x_2} f(x, y(x)+\epsilon\eta(x), y'(x)+\epsilon\eta'(x))\,dx \tag{3.15}$$

which by assumption has a stationary value when $\epsilon = 0$. Treating $I = I(\epsilon)$ as a continuous function of ϵ which has a stationary value at $\epsilon = 0$ requires that the condition $\frac{dI(\epsilon)}{d\epsilon} = I'(\epsilon)$ equal zero at $\epsilon = 0$. This is a necessary condition which must be satisfied in order for a stationary value to exist at $\epsilon = 0$. We calculate the derivative $I'(\epsilon)$ and find

$$\frac{dI}{d\epsilon} = I'(\epsilon) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \epsilon}\right) dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y}\eta + \frac{\partial f}{\partial Y'}\eta'\right) dx. \tag{3.16}$$

The necessary condition for a stationary value is obtained by setting $\epsilon = 0$ in this derivative. This is equivalent to letting $Y = y$ and $Y' = y'$ in equation (3.16) so that at the stationary value we have

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = I'(0) = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta\,dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'\,dx = 0. \tag{3.17}$$

In equation (3.17) we integrate the second term by parts to obtain

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = I'(0) = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta\,dx + \left.\frac{\partial f}{\partial y'}\eta(x)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx = 0$$

which can also be written in the form

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = I'(0) = \left.\frac{\partial f}{\partial y'}\eta(x)\right|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx = 0. \tag{3.18}$$

Observe that the boundary conditions $\eta(x_1) = \eta(x_2) = 0$ insures that the first term of equation (3.18) is zero. Consequently, the necessary condition for a stationary value can be written

$$I'(0) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx = 0. \tag{3.19}$$

For arbitrary functions $\eta(x)$ the basic lemma previously considered requires that the condition

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \quad x_1 \le x \le x_2 \tag{3.20}$$

must be true. This equation is called the Euler-Lagrange equation associated with the integral given by equation (3.11). The Euler-Lagrange equation (3.20) is a necessary condition to be satisfied by the optimal trajectory $y = y(x)$. It is not a sufficient condition for an extreme value to exist. That is, the function $y = y(x)$ which satisfies the Euler-Lagrange equation does not always produce a maximum or minimum value of the integral. The condition $dI/d\epsilon = 0$ is a necessary condition to insure that $\epsilon = 0$ is a stationary point associated with the functional given by equation (3.11). The function $y = y(x)$ which satisfies the Euler-Lagrange equation may be such that I has an extreme value of being either a maximum value or minimum value. It is also possible that the solution y could produce a horizontal inflection point with no extreme value. We won't know which condition is satisfied until further testing is done. In many science and engineering investigations the physics of the problem and the way the problem is formulated, sometimes suggests that a maximum or minimum value for the functional exists. Under such circumstances the solution $y = y(x)$ of the Euler-Lagrange equation can be said to produce an extreme value for the functional.

Note that the variables x, y, y' occurring in the integrand of equation (3.11) have been treated as independent variables. Consequently, the derivative term in the Euler-Lagrange equation (3.20) is evaluated

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial^2 f}{\partial y' \partial x} + \frac{\partial^2 f}{\partial y' \partial y} y' + \frac{\partial^2 f}{\partial y'^{\,2}} y'' \tag{3.21}$$

and so the expanded form of the Euler-Lagrange equation (3.20) can be written

$$\left(\frac{\partial^2 f}{\partial y'^{\,2}}\right)\frac{d^2 y}{dx^2} + \left(\frac{\partial^2 f}{\partial y' \partial y}\right)\frac{dy}{dx} + \left(\frac{\partial^2 f}{\partial y' \partial x} - \frac{\partial f}{\partial y}\right) = 0. \tag{3.22}$$

Observe that if the second partial derivative $\dfrac{\partial^2 f}{\partial y'^{\,2}}$ is different from zero, then the equation (3.22) is a second order ordinary differential equation which is to be solved. This second order differential equation may be linear or nonlinear depending upon the functional considered. In many instances when a difficult nonlinear differential equation arises, one must resort to numerical methods or approximation techniques to solve the equation. Recall that second order ordinary differential equations require two independent constants in the general solution and so the general solution of the Euler-Lagrange equation is called a two-parameter family of solution curves. Under favorable conditions the Euler equation can be solved and the general solution will contain two arbitrary constants. These constants occurring in the general solution must be selected to satisfy the end conditions given by equation (3.12).

It is left as an exercise to show that the Euler-Lagrange equation (3.22) can also be written in the equivalent form

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial y'}\frac{dy}{dx}\right) - \frac{\partial f}{\partial x} = 0. \tag{3.23}$$

Example 3-1. Special cases for the Euler-Lagrange equation

We consider special conditions under which the functional $I = \int_{x_1}^{x_2} f(x,y,y')\,dx$ assumes a stationary value. We assume that the solution $y = y(x)$ satisfies the end point conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ for the following special cases.

Case 1: Dependent variable absent

If $f = f(x,y')$, then $\frac{\partial f}{\partial y} = 0$ so that the Euler-Lagrange equation (3.20) reduces to the form $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$. An integration of this equation gives an equation of the form $\frac{\partial f}{\partial y'} = C_1 = constant$. This equation can now be solved to represent y' as a function of x and C_1 to obtain an equation of the form $y' = \frac{dy}{dx} = G(x, C_1)$. This equation can now be solved by an integration to obtain $y = \int_{x_1}^{x} G(x, C_1)\,dx + C_2$.

Case 2: Independent variable absent

For $f = f(x,y,y')$ one can verify the Beltrami identity

$$\begin{aligned}\frac{d}{dx}\left[y'\frac{\partial f}{\partial y'} - f\right] =& y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + y''\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}y' - \frac{\partial f}{\partial y'}y'' \\ =& -y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] - \frac{\partial f}{\partial x}.\end{aligned} \tag{3.24}$$

In the special case $f = f(y,y')$ the term inside the brackets of Beltrami's identity is zero because the Euler-Lagrange equation is assumed to be satisfied. The last term $\frac{\partial f}{\partial x} = 0$ because f is assumed to be independent of the variable x. The Beltrami identity implies that in the special case where $f = f(y,y')$, then a first integral of the Euler-Lagrange equation can be written as

$$y'\frac{\partial f}{\partial y'} - f = \alpha = constant. \tag{3.25}$$

This first integral is a first order ordinary differential equation containing only y and y' terms.

Case 3: Exact or total derivative exists

Assume there exists a function $\phi = \phi(x,y)$ such that

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}y' = f(x,y,y'). \tag{3.26}$$

In this special case the functional can be written in a form which is easily integrated. That is,

$$I = \int_{x_1}^{x_2} f(x,y,y')\,dx = \int_{x_1}^{x_2} d\phi = \phi(x_2,y_2) - \phi(x_1,y_1) \tag{3.27}$$

In this special case the integral is independent of the path connecting the points (x_1, y_1) and (x_2, y_2). This implies that all admissible functions $y(x)$ yield stationary values. Hence, the Euler-Lagrange equation degenerates into an identity. If $f = \phi_x + \phi_y y'$, then

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \phi_{xy} + \phi_{yy} y' - \frac{d}{dx}\left(\phi_y\right) = \phi_{xy} + \phi_{yy} y' - \phi_{yx} - \phi_{yy} y' = 0$$

is an identity. Conversely, let us assume that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial y' \partial x} - \frac{\partial^2 f}{\partial y' \partial y} y' - \frac{\partial^2 f}{\partial y'^{\,2}} y'' = 0 \tag{3.28}$$

is satisfied for all values of x, y, y', y''. The term y'' does not occur in the first three terms of equation (3.28) and since y'' can be selected arbitrarily, then the coefficient of y'' in equation (3.28) must equal zero in order for this equation to hold for arbitrary functions y. Hence, for arbitrary functions y we require that the following equations are satisfied

$$\frac{\partial^2 f}{\partial y'^{\,2}} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial y' \partial x} - \frac{\partial^2 f}{\partial y' \partial y} y' = 0. \tag{3.29}$$

The first of these equations is integrated to obtain $\frac{\partial f}{\partial y'} = N(x, y)$ where $N(x, y)$ is an arbitrary function of x and y. Another integration gives

$$f = N(x, y) y' + M(x, y) \tag{3.30}$$

where $M(x, y)$ is another arbitrary function of x and y. Now substitute the results from equation (3.30) into the second condition from equation (3.29) to obtain

$$\frac{\partial N}{\partial y} y' + \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} - \frac{\partial N}{\partial y} y' = 0$$

which implies

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{3.31}$$

The equation (3.31) is a necessary condition that the integral

$$I = \int_{x_1}^{x_2} \left[N(x, y) y' + M(x, y)\right] dx = \int_{x_1}^{x_2} N(x, y)\, dy + M(x, y)\, dx = \int_{x_1}^{x_2} d\phi \tag{3.32}$$

be independent of the path connecting the points (x_1, y_1) and (x_2, y_2). In this case the functional has a constant value for every admissible curve and so the variational problem is of no interest.

Consider the addition of a term to the integrand of a given integral. In order that the added term not affect the resulting Euler-Lagrange equation, we must impose the condition that the added term be the exact or total derivative with respect to x of some function $\phi(x, y)$. This is a necessary and sufficient condition.

Case 4: No solution exists

There can arise situations where the resulting Euler-Lagrange equation does not have a solution. Usually, these situations are not of interest or a trivial solution will exist. ■

The following is a well known example illustrating how a simple problem can be formulated in different ways. These various formulations can also be achieved by using a change of variable to transform the problem into a new coordinate system.