Chapter 2

Functions of a Complex Variable

Basic concepts

Let $S$ denote a nonempty set of points in the complex $z$-plane. If there exists a rule $f$ which assigns to each value $z = x + iy$ belonging to $S$, one and only one complex number $\omega = u + iv$, then this correspondence is called a function or mapping of the point $z$ to the point $\omega$ and it is denoted using the notation $\omega = f(z) = u + iv$. The set $S$ is to be understood to represent some region of the $z$-plane where the term function denotes a single-valued function. The set $S$ is referred to as the domain of definition of the function.

The $z$-plane and $\omega$-plane

To geometrically illustrate a function of a complex variable it is convenient to consider two different planes with rectangular coordinates. These planes are called the $z$-plane and the $\omega$-plane. Functions of a complex variable can be illustrated graphically by indicating correspondences between sets of points in these two planes as illustrated in the figure 2-1. Let $z = x + iy$ represent a complex number in the $z$-plane and let $\omega = u + iv$ represent the image of $z$ in the complex $\omega$-plane. For $S$ a set of complex numbers in the $z$-plane we define a single valued function $f$ on $S$ as a rule which assigns to each value $z = x + iy$ in $S$ one and only one complex number $\omega = u + iv = u(x, y) + iv(x, y)$ of the $\omega$-plane. The functional relation is then denoted by

$$\omega = u + iv = f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where $z$ is considered as a complex variable in the domain of definition $S$ and $\omega = u + iv$ is the complex value of the function at $z$ and $f$ is the rule which assigns to each $z$ an image value for $\omega$. A function $f$ can be viewed as a mapping of the points from a set $S$ in the $z$–plane to a set of image points $S'$ in the $\omega$–plane. The transformations equations which describe the image in the $\omega$-plane are given by

$$u = u(x, y), \quad v = v(x, y) \quad (2.1)$$

Another way to denote this mapping symbolically is to write $f : S \rightarrow S'$. The set of all image points $S'$ is called the range of the function $f$ and the set $S$ is called the domain of definition for the function $f$. Whenever the domain of definition is a nonempty open connected set of points $z$, it will be denoted by the symbol $D$. We shall be primarily concerned with single valued functions, where there is a one-to-one correspondence between points in the $z$-plane
and \( \omega \)-plane. Note that if one assigns a relation between \( x \) and \( y \) in the \( z \)-plane, say \( y = G(x) \) for \( a \leq x \leq b \), then the image of this curve in the \( \omega \)-plane is given by the parametric equations

\[
\begin{align*}
    u &= u(x, G(x)), \\
    v &= v(x, G(x))
\end{align*}
\quad a \leq x \leq b
\tag{2.2}
\]

where \( x \) is a parameter. The image curve in the \( \omega \) plane is then obtained by eliminating the parameter \( x \) in the equations (2.2) to obtain an image curve having the general form \( v = F(u) \). The situation is illustrated in the figure 2-1. Note the image of straight lines and circles are usually easy to obtain. Many times it is convenient to place the \( z \)-plane on top of the \( \omega \)-plane in order to observe how the image point \( \omega \) has changed in relation to the original point \( z \).

**Other methods for representing functions**

Additional methods for creating a geometrical representation of the mapping \( \omega = f(z) \) are (i) the creation of a three-dimensional graphical representation of the mapping by plotting the modular surface \( H_1 = |f(z)| = \sqrt{u^2(x, y) + v^2(x, y)} \) on a set of axes \( (x, y, H_1) \). (ii) the plotting of the surface \( H_2 = \text{Re} \{f(z)\} \) on a three dimensional set of axes \( (x, y, H_2) \) (iii) the plotting of the surface \( H_3 = \text{Im} \{f(z)\} \) on a three dimensional set of axes \( (x, y, H_3) \). (iv) Contour plots
\[ u(x, y) = \text{Re}\{f(z)\} = \text{constant} \] and \[ v(x, y) = \text{Im}\{f(z)\} = \text{constant}. \]

(v) Some fortunate individuals have sophisticated graphics packages whereby they can assign a color to each point in the \( z \)-plane and then plot the image point \( \omega = f(z) \) with the same color. This produces some very sophisticated graphics. If you check out the internet you can find examples of these various kind of representations for various complex mappings \( \omega = f(z) \).

**Multiple-valued functions**

If corresponding to each value of \( z \) there is more than one value of \( \omega \), then the complex function \( \omega = f(z) \) is called a multiple-valued function of \( z \). For example, the square root function \( \omega = f(z) = \sqrt{z} \) is a multiple-valued function and two values of \( \omega \) exist for each value of \( z \). These two values are \( \omega_1 = +\sqrt{z} \) and \( \omega_2 = -\sqrt{z} \). We will investigate multiple-valued functions in more detail in the chapter four.

**One-to-One mappings**

A transformation or mapping \( \omega = f(z) \) is said to be one-to-one if distinct points \( z_1 \neq z_2 \) are mapped onto distinct points \( \omega_1 = f(z_1) \neq \omega_2 = f(z_2) \). A one-to-one mapping has the property that for each \( \omega \) in the range of the function there exists exactly one point \( z \) in the domain of the function. Under these conditions the inverse function \( z = f^{-1}(\omega) \) can be found.

Sometimes it is convenient to represent points \( z \) in the polar form \( z = re^{i\theta} \), and consequently the image point \( \omega = u + iv = f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \) produces transformation equations having the form \( u = u(r, \theta) \), \( v = v(r, \theta) \) in the \( \omega \)-plane. At other times it may prove to be convenient to represent points in polar form in the \( \omega \)-plane. If \( \omega = Re^{i\phi} \), then one can obtain transformation equations from the relation

\[ \omega = Re^{i\phi} = R \cos \phi + iR \sin \phi = f(z) = f(x + iy) = u(x, y) + iv(x, y) \]

Equating the real and imaginary parts of this relation produces the transformation equations

\[ R \cos \phi = u(x, y), \quad \text{and} \quad R \sin \phi = v(x, y) \]

From these equation one can then calculate

\[ R = \sqrt{u^2(x, y) + v^2(x, y)}, \quad \text{and} \quad \phi = \tan^{-1} \frac{v(x, y)}{u(x, y)} \]

for representation of the polar coordinates \((R, \phi)\).

**Example 2-1.** Some selected examples of functions or mappings are the following.

1. The function or mapping \( \omega = f(z) = z^2 \) can be expressed in rectangular coordinates by

\[ \omega = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy \]

The mapping is described by the transformation equations \( u = u(x, y) = x^2 - y^2 \) and \( v = v(x, y) = 2xy \). In polar coordinates the mapping \( \omega = f(z) = z^2 \) is represented

\[ \omega = Re^{i\phi} = z^2 = (re^{i\theta})^2 = r^2e^{i2\theta} \]

which produces the transformation equations \( R = r^2 \) and \( \phi = 2\theta \).
2. The function or mapping \( \omega = f(z) = \frac{1}{z} \) can be expressed in rectangular coordinates by

\[
\omega = u + iv = \frac{1}{z} = \frac{1}{x + iy} = \frac{1}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2}
\]

Equating real and imaginary parts of this relation produces the transformation equations

\[
u = u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = v(x, y) = \frac{y}{x^2 + y^2}.
\]

In polar coordinates the mapping \( \omega = f(z) = \frac{1}{z} \) is represented

\[
\omega = \text{Re}^i \phi = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}
\]

which produces the transformation equations \( R = \frac{1}{r} \) and \( \phi = -\theta \).

3. The function or mapping \( \omega = \alpha z \) where \( \alpha = \alpha_1 + i\alpha_2 \) is a complex constant, can be represented in rectangular form

\[
\omega = u + iv = \alpha z = (\alpha_1 + i\alpha_2)(x + iy) = (\alpha_1 x - \alpha_2 y) + i(\alpha_2 x + \alpha_1 y)
\]

which produces the transformation equations

\[
u = u(x, y) = \alpha_1 x - \alpha_2 y \quad \text{and} \quad v = v(x, y) = \alpha_2 x + \alpha_1 y.
\]

In polar coordinates the mapping \( \omega = \alpha z \) is represented

\[
\omega = \text{Re}^i \phi = \alpha z = |\alpha| e^{i \arg \alpha} r e^{i \theta} = |\alpha| r e^{i(\theta + \arg \alpha)}
\]

This produces the transformation equations \( R = |\alpha|r \) and \( \phi = \theta + \arg \alpha \).

4. The function or mapping \( \omega = f(z) = z + \frac{1}{z} \) can be viewed as a mapping which gives image points associated with \( z = re^{i\theta} = r \cos \theta + ir \sin \theta \) being described by the following transformation equations

\[
\omega = u + iv = f(z) = z + \frac{1}{z} = re^{i\theta} + \frac{1}{re^{i\theta}} = r(\cos \theta + i \sin \theta) + \frac{1}{r}(\cos \theta - i \sin \theta)
\]

This gives the transformation equations

\[
u = u(r, \theta) = \left( r + \frac{1}{r} \right) \cos \theta \quad \text{and} \quad v = v(r, \theta) = \left( r - \frac{1}{r} \right) \sin \theta
\]

**Example 2-2.** The function \( \omega = f(z) = z^2 \) can be represented in rectangular coordinates as

\[
\omega = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy.
\]

This produces the transformation equations

\[
u = u(x, y) = x^2 - y^2 \quad \text{and} \quad v = v(x, y) = 2xy.
\]
To geometrically see the effects that these transformation equations have on values of \( z \), it is customary to consider the images of various easy to recognize curves or figures such as straight lines and circles.

Figure 2-2. Mapping of the function \( \omega = f(z) = z^2 \) for straight lines.

Figure 2-3. Mapping for the function \( \omega = f(z) = z^2 \) for circular arcs and rays.

For example, the image of the straight line \( x = c_1 \) constant is the parametric curve

\[
\begin{align*}
u &= c_1^2 - y^2 \\
v &= 2c_1y
\end{align*}
\]

which has \( y \) as a parameter. Eliminating \( y \) from these equations we obtain the curve

\[
u = c_1^2 - \frac{v^2}{4c_1^2}
\]