

Chapter 2

Maxima and Minima

In this chapter we investigate methods for determining points where the maximum and minimum values of a given function occur. We begin by studying extreme values associated with real functions $y = y(x)$ of a single independent real variable x . These functions are assumed to be well defined and continuous over a given interval $R = \{x \mid a \leq x \leq b\}$. Concepts introduced in the study of extreme values associated with functions of a single variable are then generalized to investigate extreme values associated with real functions $f = f(x, y)$ of two independent real variables, x and y . The functions $f = f(x, y)$ investigated are assumed to be well defined and continuous over a given region R of the x, y -plane. In addition to the functions of one and two real variables being well defined, we assume these functions have derivatives through the second order that are also well defined and continuous. The concepts introduced in the study of maximum and minimum values for real functions of one and two real variables are then generalize to investigate maximum and minimum values associated with real functions $f = f(x_1, x_2, \dots, x_n)$ of several real variables. These functions are assumed to be well defined and have derivatives everywhere in a given region R of n -dimensional space. In the investigation of local maximum and minimum values of functions we develop several new concepts such as directional derivative, Hessian matrices and Lagrange multipliers to aid in the investigation of extreme values of a function. We conclude this chapter with an introduction to mathematical programming.

Functions of a single real variable

Let $y = f(x)$ denote a function that is well defined and continuous over an interval $a \leq x \leq b$. A δ -neighborhood of a point x_0 within this interval is defined as the set of points x satisfying $|x - x_0| < \delta$, where δ is a small positive number. Functions of a real single variable $y = f(x)$ are said to have a relative or local maximum or minimum value at a point x_0 if for all values of x in a δ -neighborhood of x_0 certain inequalities are satisfied. In particular, at points x_0 where a local maximum value occurs, one can say

$$f(x_0) \geq f(x) \text{ for all values of } x \text{ satisfying } |x - x_0| < \delta.$$

At points x_0 where a local minimum occurs, one can say

$$f(x_0) \leq f(x) \text{ for all values of } x \text{ satisfying } |x - x_0| < \delta,$$

where δ is a small positive number.

Associated with a function $y = f(x)$, that is well defined over a closed interval $[a, b]$, there are certain points called critical points. Critical points are defined as either

- (i) Those points x where the derivative of the function is zero, or
- (ii) Those points x where the derivative $f'(x)$ fails to exist.

The point or points where the derivative is zero satisfies the equation $f'(x) = 0$. These points are called stationary points.

A curve $y = f(x)$ is called concave upward over an interval if the graph of f lies above all of the tangent lines to the curve on the interval. A curve $y = f(x)$ is called concave downward over an interval if the graph of f lies below all of its tangent lines on the interval. A point on a curve $y = f(x)$ where the concavity changes is called an inflection point.

A smooth curve is found to be concave up in regions where $f''(x) > 0$ and concave downward in regions where $f''(x) < 0$. Those points x that satisfy $f''(x) = 0$ are the points of inflection. These are the x values where the concavity of the curve changes. Note that relative maximum and minimum values of a function $y = f(x)$, within the interval (a, b) of definition, can occur at critical points and at stationary points. The end points $x = a$ and $x = b$ must be tested separately to determine if a local maximum or minimum value exists. The figure 2-1 illustrates several functions defined over an interval $a \leq x \leq b$ that have critical points. These critical points illustrate local maximum values or local minimum values that occur at points where either the slope is zero or not defined. Observe in figure 2-1 that at points where a sharp corner occurs the left and right-handed derivatives are not the same at these points and so the derivative fails to exist at these points. Also note that at the end points of a given interval a function can have relative or local maximum or minimum values. When testing for relative or local maximum and minimum values over an interval $a \leq x \leq b$, the end points at $x = a$ and $x = b$ should be tested separately.

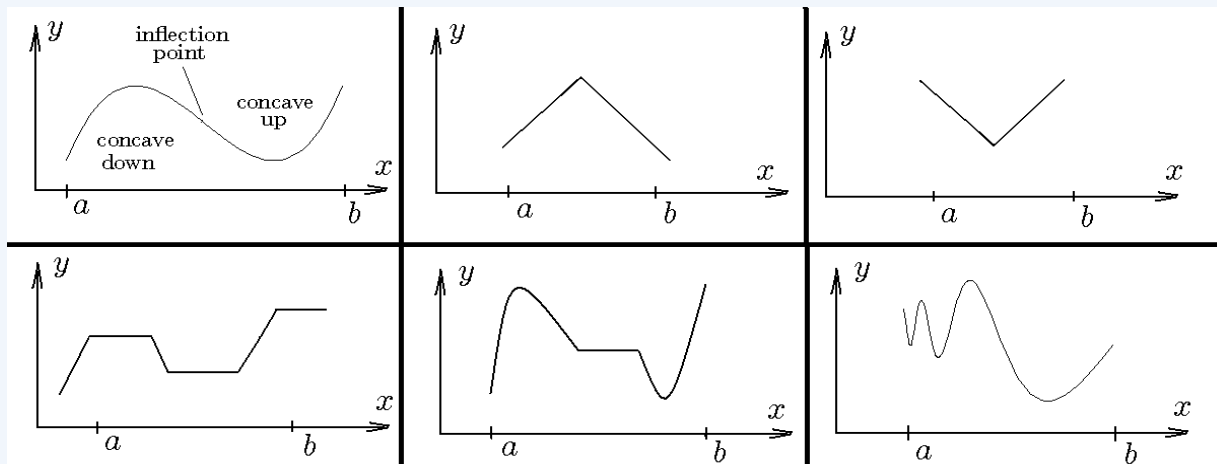


Figure 2-1. Sketches illustrating various ways a local maximum or minimum value can occur.

The terminology extrema, extremum or relative extreme values is a way of referring to both the relative maximum and minimum values associated with a real function. Relative maxima or minima values are referred to as local extrema of the function $y = f(x)$ over the interval $[a, b]$ where the function is defined. Points where $f'(x) = 0$ are referred to as stationary points since these are points where the slope is zero.

A function $y = f(x)$ is said to have an absolute maximum at a point x_0 in an interval (a, b) if $f(x_0) \geq f(x)$ for all $x \in (a, b)$. A function $y = f(x)$ is said to have an absolute minimum at a point x_0 in an interval (a, b) if $f(x_0) \leq f(x)$ for all $x \in (a, b)$. The absolute maximum or

minimum value of a function $y = f(x)$ can occur either at a critical point of $f(x)$ within the interval or at one of the end points of a closed interval. One should make it a habit to always test separately the end points of a closed interval for absolute maximum or minimum values of the function.

On an open interval ($a < x < b$), where the end points are not included, some functions do not possess a maximum or minimum value. For example, the function $y = y(x) = x$ on the interval ($0 < x < 1$). On closed intervals ($a \leq x \leq b$) the extremum values may occur at the end points or boundaries. The Weierstrass theorem states that a continuous function on a closed interval attains its maximum or minimum values on a boundary or at an interior point.

Tests for maximum and minimum values

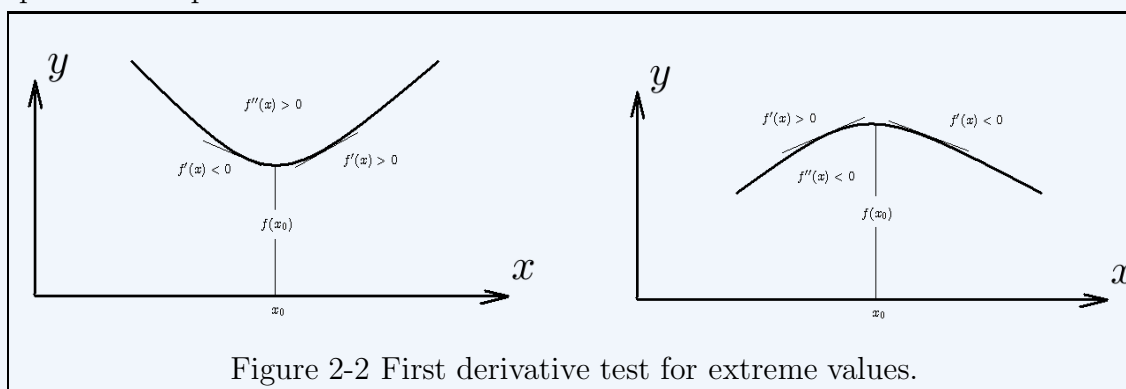
In every calculus course one learns to test the critical points associated with a continuous functions $y = f(x)$ defined over an interval $a \leq x \leq b$. The critical points can then be tested to see if a relative maximum or minimum value occurs. The first derivative test and second derivative test are familiar tests for examining the behavior of smooth functions at a critical point.

First derivative test

Recall that the first derivative test states that if $x = x_0$ is a critical point associated with a continuous differentiable function $y = f(x)$, then $f(x_0)$ is called a relative minimum value of the function $f(x)$ if the following conditions hold true

$$(i) \quad f'(x) < 0 \text{ for } x < x_0 \quad \text{and} \quad (ii) \quad f'(x) > 0 \text{ for } x > x_0 \quad (2.1)$$

That is, the derivative changes sign from negative to zero to positive as x moves across the critical point in the positive direction.



If $x = x_0$ is a critical point associated with the a continuous differentiable function $y = f(x)$, then $f(x_0)$ is called a relative maximum value of the function $f(x)$ if the following conditions hold

$$(i) \quad f'(x) > 0 \text{ for } x < x_0 \quad \text{and} \quad (ii) \quad f'(x) < 0 \text{ for } x > x_0 \quad (2.2)$$

That is, the derivative changes sign from positive to zero to negative as x moves across the critical point in the positive direction. Conditions for a first derivative test for an extreme value are illustrated in the figure 2-2.

Second derivative test

The second derivative test for relative maximum and minimum values is based upon the concavity of the curve in the vicinity of a critical point. Assume the second derivative $f''(x)$ is continuous in the vicinity of a critical point x_0 . If the second derivative at the critical point is positive, then the given curve will be concave upward. If the second derivative at the critical point is negative, then the given curve will be concave downward. Consequently, one has the following second derivative test for local extreme values

- (i) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ represents a local minimum value.
- (ii) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ represents a local maximum value.
- (iii) If $f'(x_0) = 0$ and $f''(x_0) = 0$ or $f''(x_0)$ does not exist,
then the second derivative test fails.

If the second derivative test fails, then the first derivative test can be applied.

Further investigation of critical points

Assume a given function $y = f(x)$ is defined over a closed interval $[a, b]$ and is such that it has a Taylor series expansion about an interior point x_0 so that one can write

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + \cdots + f^{(n-1)}(x_0)\frac{h^{n-1}}{(n-1)!} + f^{(n)}(\xi)\frac{h^n}{n!}$$

where the last term represents the Lagrange remainder term with $x_0 < \xi < x_0 + h$.

Now if x_0 is a critical point that satisfies $f'(x_0) = 0$ and if in addition one has

$$f''(x_0) = f'''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

and $f^{(n)}(x_0) \neq 0$, then one can obtain from the Taylor series with Lagrange remainder the relation

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f^{(n)}(\xi)\frac{h^n}{n!} \quad (2.3)$$

where it is assumed that the n th derivative $f^{(n)}(x)$ is continuous in some neighborhood of the point x_0 and ξ lies in this neighborhood. If the n th derivative $f^{(n)}(x)$ is continuous, then we can assume $f^{(n)}(\xi)$ has the same sign as $f^{(n)}(x_0)$. The equation (2.3) can then be interpreted as follows. The change in the value of $f(x_0)$ in the neighborhood of the point x_0 depends upon the value of n , h and the sign of the n -th derivative.

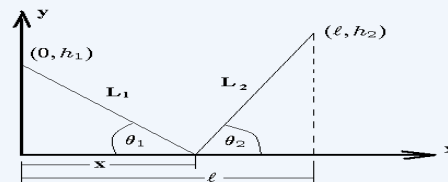
Case 1: If n is even, then h^n is always positive and so the sign of the change $\Delta f(x_0)$ depends upon the sign of the n th derivative $f^{(n)}(x_0)$. If $f^{(n)}(x_0) > 0$, then $\Delta f(x_0) > 0$ for all values of h and so $x = x_0$ corresponds to a relative minimum value. If $f^{(n)}(x_0) < 0$, then $\Delta f(x_0) < 0$ for all values of h and so $x = x_0$ corresponds to a relative maximum value.

Case 2: If n is odd, then h^n takes on different signs depending upon whether $h > 0$ or $h < 0$. Hence $\Delta f(x_0)$ changes signs in the neighborhood of the critical point x_0 . That is, for $x = x_0 + h$ there will be values of h for which $f(x) > f(x_0)$ and values for which $f(x) < f(x_0)$ and hence the point $x = x_0$ corresponds to a point of inflection.

One can use arguments similar to those given above to analyze critical points associated with functions of several variables.

Example 2-1. Law of reflection

We examine the situation where light travels in a straight line and is reflected from a smooth polished surface, say a mirror. Assume ℓ , h_1 and h_2 are positive quantities and we are given the two points $(0, h_1)$ and (ℓ, h_2) together with a general point x on the x -axis satisfying $0 < x < \ell$. Let L_1 denote the distance from the point $(0, h_1)$ to the point x and let L_2 denote the distance from the point x to the point (ℓ, h_2) . Find the position of the point x such that the sum of the distances L_1 and L_2 is a minimum.



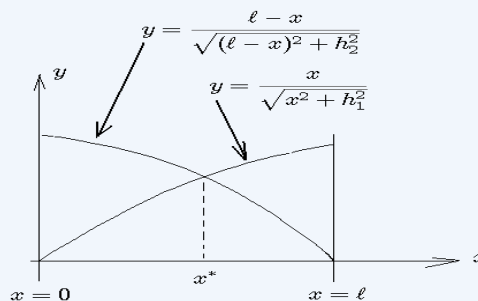
Solution: The sum of the distances L_1 and L_2 can be written as a function of x . We have $L_1 = \sqrt{x^2 + h_1^2}$ and $L_2 = \sqrt{(\ell - x)^2 + h_2^2}$ so that the sum can be written

$$S = L_1 + L_2 = \sqrt{x^2 + h_1^2} + \sqrt{(\ell - x)^2 + h_2^2}, \quad \text{with} \quad \frac{dS}{dx} = \frac{x}{\sqrt{x^2 + h_1^2}} + \frac{x - \ell}{\sqrt{(\ell - x)^2 + h_2^2}}.$$

At an extremum we require $\frac{dS}{dx} = 0$ which requires that

$$\frac{x}{\sqrt{x^2 + h_1^2}} = \cos \theta_1 = \frac{\ell - x}{\sqrt{(\ell - x)^2 + h_2^2}} = \cos \theta_2 \quad (2.4)$$

The equation (2.4) can be interpreted as finding the point x^* where two graphs intersect. We find that the equation (2.4) has a unique solution x^* where the curves $y = x/\sqrt{x^2 + h_1^2}$ and $y = (\ell - x)/\sqrt{(\ell - x)^2 + h_2^2}$ intersect as illustrated in the accompanying figure.



The equation (2.4) implies that $\theta_1 = \theta_2$ at this critical value for x . When the x -axis is a mirror this requires that the angle of incidence equal the angle of reflection. The second derivative test can be used to show that S has a minimum value under these conditions. One can verify that the second derivative reduces to

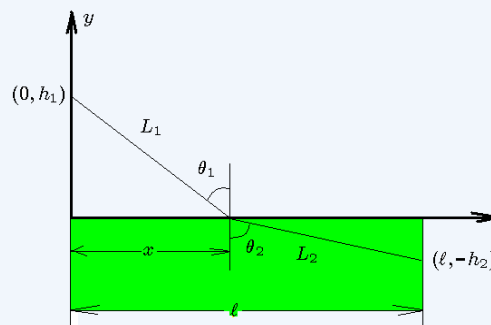
$$\frac{d^2S}{dx^2} = \frac{h_1^2}{\sqrt{x^2 + h_1^2}} + \frac{h_2^2}{\sqrt{(\ell - x)^2 + h_2^2}}.$$

This is a quantity that is always positive and hence, by the second derivative test, the distance S has a minimum value at the critical point x^* where $\theta_1 = \theta_2$. ■

Example 2-2. Snell's law of refraction

We examine the situation where light changes direction as it travels from one medium to another. Assume ℓ , h_1 and h_2 are positive quantities and we are given the two points $(0, h_1)$ and $(\ell, -h_2)$ together with a general point x on the x -axis.

The region $y > 0$ represents a medium where the speed of light is c_1 and the region $y < 0$ represents a medium where the speed of light is c_2 . Find the point $(x, 0)$ such that light travels from $(0, h_1)$ to $(x, 0)$ and then to $(\ell, -h_2)$ in the shortest time.



Solution Let L_1 and L_2 denote the distances that the light travels in the media above and below the x -axis. We find

$$L_1 = \sqrt{h_1^2 + x^2} \quad \text{and} \quad L_2 = \sqrt{h_2^2 + (\ell - x)^2}.$$

Using the formula $\text{distance} = (\text{velocity})(\text{time})$, one can verify that the time it takes for light to travel from $(0, h_1)$ to $(\ell, -h_2)$ is given by

$$T = T(x) = \frac{1}{c_1} \sqrt{h_1^2 + x^2} + \frac{1}{c_2} \sqrt{h_2^2 + (\ell - x)^2}.$$

The time is an extremum for the value of x that satisfies

$$\frac{dT}{dx} = \frac{1}{c_1} \frac{x}{\sqrt{h_1^2 + x^2}} - \frac{1}{c_2} \frac{\ell - x}{\sqrt{h_2^2 + (\ell - x)^2}} = 0.$$

This requires $\frac{1}{c_1} \frac{x}{\sqrt{h_1^2 + x^2}} = \frac{1}{c_2} \frac{\ell - x}{\sqrt{h_2^2 + (\ell - x)^2}}$ which can be written in terms of the angles θ_1 and θ_2 illustrated in the above figure. We find

$$\frac{1}{c_1} \sin \theta_1 = \frac{1}{c_2} \sin \theta_2.$$

This is Snell's law of refraction. One can verify, using the second derivative test, that this solution gives a minimum value for the time of travel. One can verify the second derivative simplifies to

$$\frac{d^2T}{dx^2} = \frac{1}{c_1} \frac{h_1^2}{\sqrt{h_1^2 + x^2}} + \frac{1}{c_2} \frac{h_2^2}{\sqrt{h_2^2 + (\ell - x)^2}}.$$

The second derivative is always positive and so the critical point corresponds to a minimum time. ■

Functions of two variables

Let $z = z(x, y)$ denote a function of x and y that is defined everywhere in a domain R of the x, y -plane. Further we assume that this function is continuous with derivatives that are also continuous. Here z can be thought of as the height of a continuous surface above the x, y -plane. The function $z = z(x, y)$ is said to have a relative or local maximum value at a point (x_0, y_0) in the interior of the region R if

$$z(x_0, y_0) \geq z(x, y) \tag{2.5}$$

for all points (x, y) in the delta neighborhood $N_\delta = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 \leq \delta^2\}$, of the point (x_0, y_0) , where δ is a small positive number. If the inequality in equation (2.5) holds for

all points (x, y) interior to the region R and for points (x, y) on the boundary ∂R of the region R , then $z = z(x_0, y_0)$ is called an absolute maximum value over the region R .

Similarly, the function $z = z(x, y)$ is said to have a relative or local minimum at a point (x_0, y_0) interior to the region R if

$$z(x_0, y_0) \leq z(x, y) \quad (2.6)$$

for all points (x, y) in the delta neighborhood N_δ . Again, if the inequality in equation (2.6) holds for all points (x, y) interior to the region R and for points (x, y) on the boundary ∂R of the region R , then $z(x_0, y_0)$ is called an absolute minimum value over the region R . Boundary points of the region R must be tested separately for maximum and minimum values.

A function $z = z(x, y)$ can also be thought of as representing a scalar field associated with the points (x, y) within a region R . The directional derivative of z in the direction of a unit vector $\hat{\mathbf{e}}_\alpha = \cos \alpha \hat{\mathbf{e}}_1 + \sin \alpha \hat{\mathbf{e}}_2$ and evaluated at a point (x_0, y_0) is given by

$$\frac{dz}{ds} = \text{grad } z \cdot \hat{\mathbf{e}}_\alpha = \left(\frac{\partial z}{\partial x} \hat{\mathbf{e}}_1 + \frac{\partial z}{\partial y} \hat{\mathbf{e}}_2 \right) \cdot (\cos \alpha \hat{\mathbf{e}}_1 + \sin \alpha \hat{\mathbf{e}}_2) = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \sin \alpha \quad (2.7)$$

where all derivatives are evaluated at the point (x_0, y_0) . Points on the surface $z = z(x, y)$ that correspond to stationary points are those points where the directional derivative is zero for all directions α . Therefore, at a stationary point we will have

$$\frac{\partial z}{\partial x} = 0, \quad \text{and} \quad \frac{\partial z}{\partial y} = 0. \quad (2.8)$$

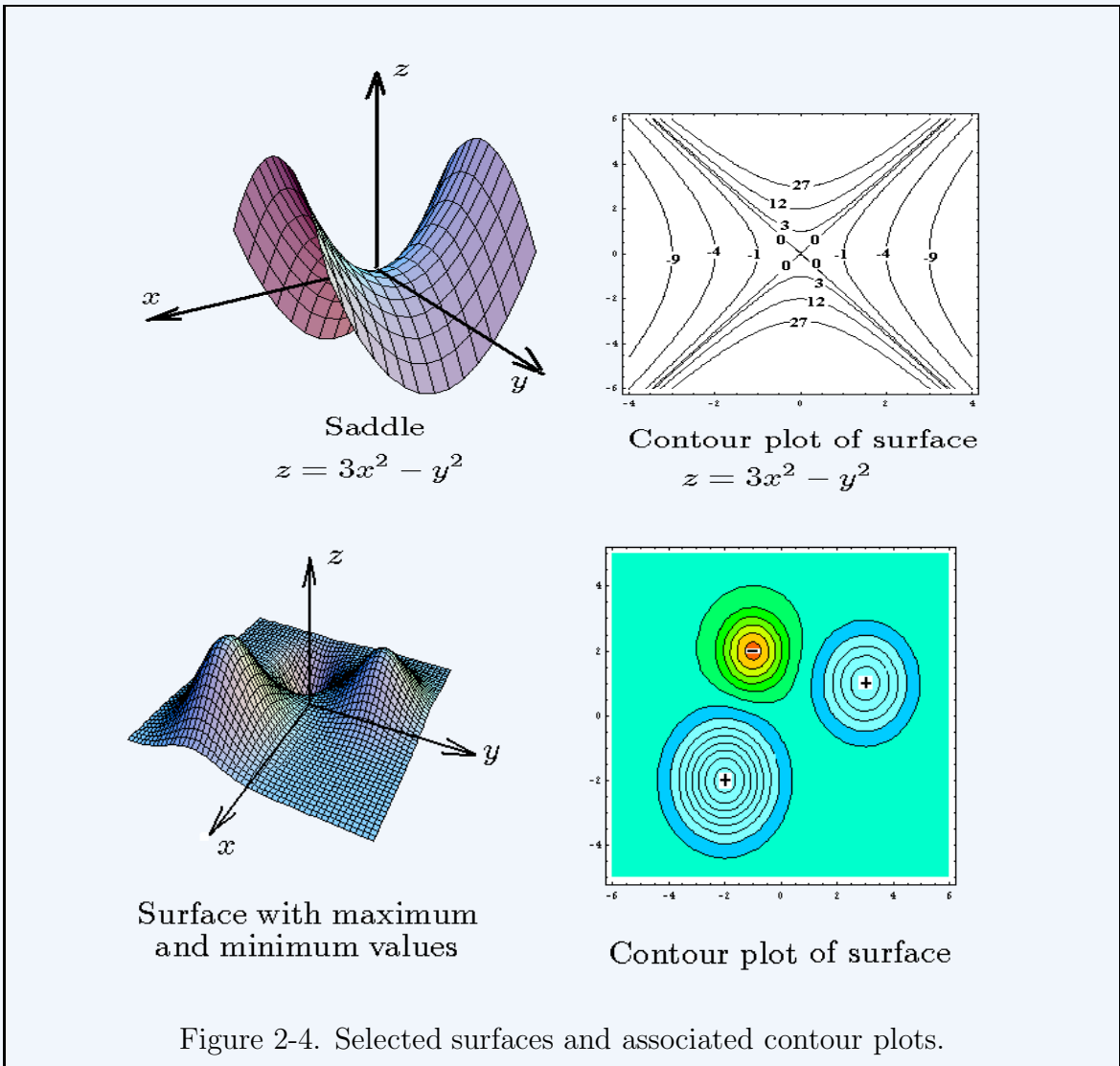
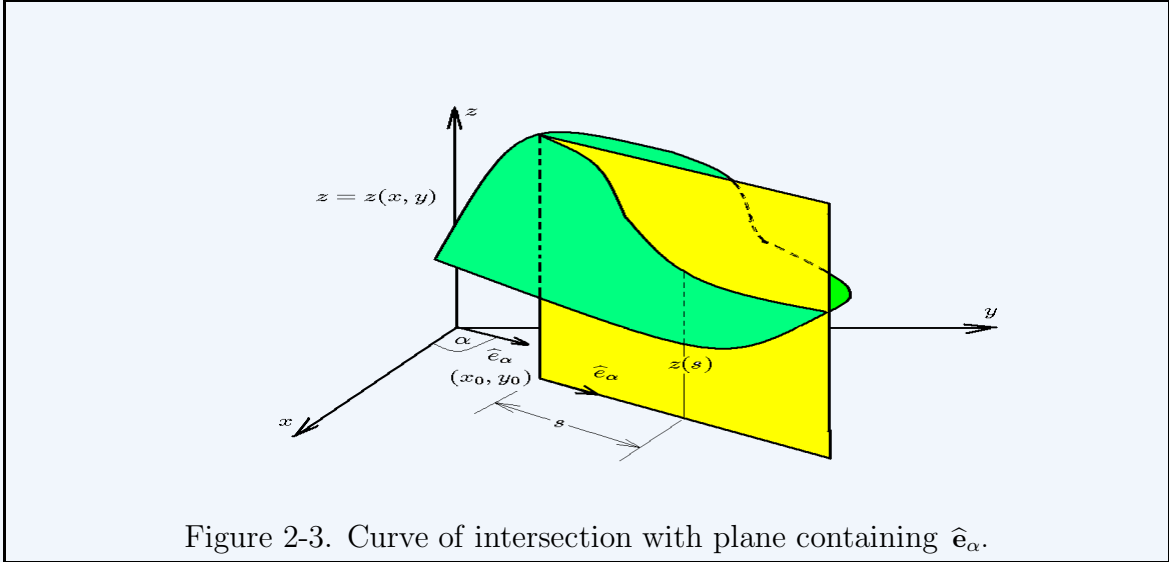
Stationary points are those points where the tangent plane to the surface is parallel to the x, y -plane. Here we assume that points (x, y) used to define $z = z(x, y)$ are restricted to a region R of the x, y -plane. We convert the problem of analyzing stationary points, for determining maximum and minimum values for functions of two variables, to a familiar one dimensional problem as follows. If (x_0, y_0) is a stationary point to be tested, then slide the free vector $\hat{\mathbf{e}}_\alpha$ to the point (x_0, y_0) and construct a plane normal to the plane $z = 0$, such that this plane contains the vector $\hat{\mathbf{e}}_\alpha$. The constructed plane intersects the given surface in a curve that can be represented by the equation

$$z = z(s) = z(x_0 + s \cos \alpha, y_0 + s \sin \alpha) \quad (2.9)$$

where s represents distance in the direction $\hat{\mathbf{e}}_\alpha$. We can now analyze the change in z along the curve of intersection for all directions α . Methods from calculus can now be applied to analyze maximum and minimum values associated with the curve of intersection formed by the plane and surface. The situation is illustrated in the figure 2-3.

At a stationary point we must have $\frac{dz}{ds} = 0$ for all directions α . In addition the second directional derivative

$$\begin{aligned} \frac{d^2 z}{ds^2} &= \text{grad} \left(\frac{dz}{ds} \right) \cdot \hat{\mathbf{e}}_\alpha \\ \frac{d^2 z}{ds^2} &= \left[\left(\frac{\partial^2 z}{\partial x^2} \cos \alpha + \frac{\partial^2 z}{\partial y \partial x} \sin \alpha \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial^2 z}{\partial x \partial y} \cos \alpha + \frac{\partial^2 z}{\partial y^2} \sin \alpha \right) \hat{\mathbf{e}}_2 \right] \cdot [\cos \alpha \hat{\mathbf{e}}_1 + \sin \alpha \hat{\mathbf{e}}_2] \quad (2.10) \\ \frac{d^2 z}{ds^2} &= \frac{\partial^2 z}{\partial x^2} \cos^2 \alpha + 2 \frac{\partial^2 z}{\partial x \partial y} \sin \alpha \cos \alpha + \frac{\partial^2 z}{\partial y^2} \sin^2 \alpha \end{aligned}$$



can be evaluated at the stationary point. If this second directional derivative is positive for all directions α , then the stationary point corresponds to a relative minimum. If the second directional derivative is negative for all directions α , then the stationary point corresponds to a relative maximum of the function $z = z(x, y)$ at the stationary point.

A function $z = z(x, y)$ is said to have a saddle point at a stationary point (x_0, y_0) if there exists a delta neighborhood N_δ of the point (x_0, y_0) such that for some points $(x, y) \in N_\delta$ we have $z(x, y) > z(x_0, y_0)$ and for other points $(x, y) \in N_\delta$ we have $z(x, y) < z(x_0, y_0)$.

If graphical software is available it is sometimes advantageous to plot graphs of the surfaces being tested in order to display where maximum and minimum values occur. One can also examine numerous level curves called contour plots that represent the intersection of the surface $z = z(x, y)$ with the plane $z = c = \text{constant}$ for selected values of the constant c . The figure 2-4 illustrates some sketches of surfaces and the corresponding level curves associated with the surfaces.

Analysis of second directional derivative

To analyze a stationary value associated with a function $z = z(x, y)$ one must be able to analyze the second directional derivative given by equation (2.10). Let

$$A = \frac{\partial^2 z}{\partial x^2}, \quad B = \frac{\partial^2 z}{\partial x \partial y}, \quad C = \frac{\partial^2 z}{\partial y^2} \quad (2.11)$$

denote the second derivatives in equation (2.10) evaluated at a stationary point (x_0, y_0) . The second directional derivative given by equation (2.10) can then be written in a more tractable form for analysis purposes. We write equation (2.10) in the form

$$\left. \frac{d^2 z}{ds^2} \right|_0 = A \cos^2 \alpha + 2B \cos \alpha \sin \alpha + C \sin^2 \alpha \quad (2.12)$$

and then factor out the leading term followed by a completing the square operation on the first two terms to obtain

$$\begin{aligned} \left. \frac{d^2 z}{ds^2} \right|_0 &= A \left[\cos^2 \alpha + 2 \frac{B}{A} \cos \alpha \sin \alpha + \frac{C}{A} \sin^2 \alpha \right] \\ &= A \left[\left(\cos \alpha + \frac{B}{A} \sin \alpha \right)^2 + \frac{(AC - B^2)}{A^2} \sin^2 \alpha \right] \end{aligned} \quad (2.13)$$

Case 1: Assume that $(AC - B^2) = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = 0$, then in those directions α that satisfy $\cos \alpha + \frac{B}{A} \sin \alpha = 0$, the second directional derivative vanishes. For all other values of α the second directional derivative is of constant sign that is the same sign as A . If the above condition is satisfied, then the second directional derivative test fails. Note that if $AC - B^2 = 0$ there may be a local minimum, a local maximum or neither maximum or minimum at the point being tested, hence the test is inconclusive.

Case 2: Assume that $(AC - B^2) = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 < 0$, then the second directional derivative is not of constant sign. It assumes different signs in different directions α . In particular, if $\alpha = 0$ we have $\left. \frac{d^2 z}{ds^2} \right|_0 = A$ and for α satisfying $\cos \alpha + \frac{B}{A} \sin \alpha = 0$, we have $\left. \frac{d^2 z}{ds^2} \right|_0 = \frac{A(AC - B^2)}{A^2} \sin^2 \alpha$. Hence if

$A > 0$, then $A(AC - B^2)$ is negative and if $A < 0$, then $A(AC - B^2)$ is positive. This shows the second directional derivative is of nonconstant sign. In this situation the stationary point (x_0, y_0) is said to correspond to a saddle point.

Case 3: Assume that $(AC - B^2) = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, then the second directional derivative is of constant sign, which is the sign of A .

(i) If $A > 0$, then $\frac{d^2 z}{ds^2} > 0$ so that the curve $z = z(s)$ is concave upward for all directions α and consequently the stationary point corresponds to a relative minimum.

(ii) If $A < 0$, then $\frac{d^2 z}{ds^2} < 0$ so that the curve $z = z(s)$ is concave downward for all directions α and consequently the stationary point corresponds to a relative maximum.

Generalization

The analysis of a function having maximum and minimum values can be approached by way of a Taylor series expansion and quadratic forms. For a function of one-variable a Taylor series expansion can be written

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2!}f''(\xi)(\Delta x)^2 \quad (2.14)$$

where the last term is the Lagrange remainder term. Now if $f'(x_0) = 0$, then the difference $f(x_0 + \Delta x) - f(x_0) > 0$ if $f''(\xi) > 0$ and $f(x_0 + \Delta x) - f(x_0) < 0$ if $f''(\xi) < 0$. These inequalities give the definition of a maximum and minimum value at x_0 . Observe that if $f''(x)$ is continuous in a δ -neighborhood of x_0 , then δ can be selected small enough so that the conditions $f''(\xi) > 0$ and $f''(\xi) < 0$ can be replaced by the conditions $f''(x_0) > 0$ and $f''(x_0) < 0$ as the tests for minimum and maximum values respectively. That is, if $f''(x)$ is continuous, then there exists a δ -neighborhood of x_0 where $f''(x_0)$ and $f''(\xi)$ have the same sign everywhere in the neighborhood.

For functions of two-variables we have the Taylor series expansion

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) = & f(x_0, y_0) + \frac{\partial f}{\partial x} \Big|_0 \Delta x + \frac{\partial f}{\partial y} \Big|_0 \Delta y \\ & + \frac{1}{2!} \left(\frac{\partial^2 f}{\partial x^2} (\Delta x)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f}{\partial y^2} (\Delta y)^2 \right)_{(\xi, \eta)} \end{aligned} \quad (2.15)$$

where the subscript 0 denotes that the derivatives are to be evaluated at the point $P_0 = (x_0, y_0)$. Now if $\frac{\partial f}{\partial x} \Big|_0 = 0$ and $\frac{\partial f}{\partial y} \Big|_0 = 0$, then the equation (2.15) can be written in matrix notation

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \frac{1}{2!} [\Delta x, \Delta y] \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}_{(\xi, \eta)} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \quad (2.16)$$

where the right-hand side of equation (2.16) contains a quadratic form evaluated at the point (ξ, η) . The matrix of the quadratic form is called the Hessian matrix. If the Hessian matrix of the quadratic form is positive definite, then

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) > 0,$$