Chapter 1

Complex Numbers

Preliminary considerations

Historically, complex numbers were introduced to solve algebraic equations. It was observed that the algebraic equation $x^2 + 1 = 0$ possesses no real solutions and the notation $\pm \sqrt{-1}$ was used to represent the roots. It was the year 1545 when the mathematicians Girolamo Cardano\(^1\), Ferro Tartaglia\(^2\) and Rafel Bombeli\(^3\) first used the square root of negative numbers to represent roots of equations. In 1797 a Norwegian Surveyor Wessel\(^4\) reported to the Danish Academy of Science on the geometric interpretation of complex numbers of the form $z = a + ib$. Later in 1806 Robert Argand\(^5\) also produced an essay on the geometric interpretation of complex numbers. The Swiss mathematician Leonard Euler\(^6\) was the first to introduce the symbol $i$ to represent the imaginary component of a number. In 1831 Carl Fredrich Gauss\(^7\) formalized the work of Wessel and Argand and introduced numbers $z = x + iy$ where $i$ is a symbol used to represent a pure imaginary number having the property that $i^2 = -1$. Numbers of the form $z = x + iy$, where $x$ and $y$ are real numbers, were called complex numbers and many mathematicians have contributed to the development of the theory of complex numbers and functions associated with these numbers. In particular, much of the early work in the complex domain was done by the mathematicians Cauchy\(^8\), Weierstrass\(^9\) and Riemann\(^10\). For the last two hundred years many applications of complex numbers and complex functions have been developed in the areas of science and engineering. For example, the study areas of mappings, integration, differentiation, solutions of algebraic equations, solutions of ordinary differential equations, solutions of partial differential equations, summation of series, sequences, fractals and potential theory are just a few of the many application areas where one can find complex variable theory employed. You will find examples from many of these application areas presented throughout this textbook.

---

1 Girolamo Cardano (1501-1576) Italian- Doctorate in medicine in 1525.
2 Ferro Tartaglia (1500-1557) Italian self taught mathematician.
3 Rafel Bombeli (1526-1572) Italian self taught mathematician.
4 Casper Wessel (1745-1818) Norwegian surveyor.
6 Leonhard Euler (1707-1783) Swiss mathematician.
7 Karl Friedrich Gauss (1777-1855) German mathematician.
8 Augustin Louis Cauchy (1789-1857) French mathematician.
9 Karl Theodor Wilhelm Weierstrass (1815-1897) German mathematician.
10 Georg Friedrich Bernhard Riemann (1826-1866) German mathematician.
A complex number $z = x + iy$ can be interpreted as an ordered pair of real numbers $(x, y)$ and can be represented as a point in the x-y plane which is called the z-plane. The complex number $z = x + iy$ is illustrated in figure 1-1. Here $x$ is called the real part and $y$ is called the imaginary part of the complex number $z = x + iy$. The real part and imaginary parts of a complex number are often times denoted using the notations $\text{Re}\{\}$ for the real part and $\text{Im}\{\}$ for the imaginary part. For example, one can write $x = \text{Re}\{z\}$ and $y = \text{Im}\{z\}$. This notation can be employed for any complex quantity to emphasize the real part or imaginary part of the complex quantity. If $x$ and $y$ are allowed to vary, then the point $z$ becomes a variable and moves around the z-plane. In such circumstances the point $z$ is called a complex variable. In the special case $y = 0$, $z$ must move along the x-axis and in this case $z$ is called a real variable.

Complex numbers can also be interpreted as vectors as illustrated in the figure 1-1. The complex number $z_1 = x_1 + iy_1$ can be thought of as a vector with length $x_1$ for the x-component and length $y_1$ for the y-component of the vector. From the origin $(0,0)$ one can draw a straight line in the x-direction to the point $(x_1,0)$ and then draw a straight line from the point $(x_1,0)$ to the point $(x_1,y_1)$ which represents the terminus point of the vector $z_1 = x_1 + iy_1$. The straight line from the origin $(0,0)$ to the terminus point $(x_1,y_1)$ then represents the vector $z_1$.

Here $x_1$ and $y_1$ are the components of the vector and $\sqrt{x_1^2 + y_1^2}$ denotes the length of the vector. Vectors are to be treated as free vectors. This implies that the vector $z_1$ can be picked up and translated to some new origin as long as it keeps its same length and direction.

Stereographic projection

Consider a sphere with diameter of unit length which is tangent to the z-plane at $z = 0$ as illustrated in the accompanying figure. Let the point of tangency be called the south pole $S$ of the sphere with the north pole on top of the sphere as illustrated. Construct a straight line through the point $z = x + iy$ in the z-plane and the north pole of the sphere. This line intersects the sphere in a point $z'$. In this way one can construct a one-to-one correspondence between points in the z-plane and points $z'$ on the surface of the sphere. The point $N$ of the north pole corresponds to the “point” at
infinity, which we will have more to say about in a later section. The above correspondence of mapping points from the $z$-plane to points on the sphere is called a stereographic projection.

The sphere is often referred to as the Riemannian sphere.

**Properties of complex numbers**

The basic definitions and terminology associated with equality, addition, subtraction and multiplication of complex numbers are as follows.

**Equality of complex numbers:**

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if

$$\text{Re}\{z_1\} = \text{Re}\{z_2\} \quad \text{and} \quad \text{Im}\{z_1\} = \text{Im}\{z_2\}.$$

Equality is expressed by the equation $z_1 = z_2$.

This definition tells us that two complex numbers are equal if their real parts are equal and simultaneously their imaginary parts are equal. This same concept of equality can be applied to any two complex quantities $Q_1$ and $Q_2$. We say $Q_1$ and $Q_2$ are equal if their real parts are equal and their imaginary parts are equal.

**Addition of complex numbers:**

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are summed by adding their real parts and imaginary parts to obtain

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

The complex number $0 + i0$ or $0$ is known as the additive identity element and $-z$ is known as the additive inverse of $z$ or inverse of $z$ with respect to addition. That is, $z$ and $-z$ satisfy the equation $z + (-z) = 0$.

**Subtraction of complex numbers:**

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are subtracted by subtracting the real and imaginary parts to obtain

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

**Multiplication of complex numbers:**

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are multiplied by using the ordinary rules of algebra and $z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$

$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

where we have used the fact that $i^2 = -1$. The complex number $1 + i0$ or $1$ is known as the multiplicative identity element. If $z \neq 0$, then there exists a unique number $z_1$ such that $zz_1 = z_1z = 1$. The number $z_1$ is called the inverse of $z$ with respect to multiplication and is denoted by the symbol $z^{-1}$ or $1/z$. 
Algebraic properties of complex numbers

Complex numbers obey the following laws under addition and multiplication

<table>
<thead>
<tr>
<th></th>
<th>Addition</th>
<th>Multiplication</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative Law</td>
<td>$z_1 + z_2 = z_2 + z_1$</td>
<td>$z_1 z_2 = z_2 z_1$</td>
</tr>
<tr>
<td>Associative Law</td>
<td>$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$</td>
<td>$(z_1 z_2) z_3 = z_1 (z_2 z_3)$</td>
</tr>
<tr>
<td>Identity Element</td>
<td>$z + 0 + 0i = z$</td>
<td>$z \cdot 1 = z$</td>
</tr>
<tr>
<td>Inverse Element</td>
<td>$z + (-z) = 0 + 0i$</td>
<td>$z z^{-1} = z^{-1} z = 1, \ z \neq 0$</td>
</tr>
</tbody>
</table>

Complex numbers $z_1, z_2$ and $z_3$ also satisfy the distributive law

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

![Figure 1-2. Addition and subtraction of complex numbers $z_1$ and $z_2$ when treated like vectors.](image)

Using the above definitions we can define $-z$ as the multiplication of $z$ by the scalar $-1$ to obtain $(-1)z$. If $z$ is thought of as a vector, as illustrated in the figure 1-1, then multiplying $z$ by $(-1)$ has the effect of rotating the vector $z$ through $180^\circ$. Addition of complex numbers can be treated just like the parallelogram law of vector addition. For example, the addition of vector $z_2$ to $z_1$ is achieved by first drawing the vector $z_1$ and then moving the initial point or origin of the vector $z_2$ to the terminal point of the vector $z_1$. Subtraction can be thought of as the vector addition of a negative vector. Examples of these concepts are illustrated in the figure 1-2.
The symbol $i$ is used to denote an imaginary unit having the property
\[ i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = i^2 i^2 = (-1)(-1) = 1, \quad i^5 = i^4 i = i, \quad i^6 = i^4 i^2 = -1, \ldots \] (1.1)

Note that the imaginary unit $i$ is to be treated as an algebraic quantity whenever it occurs in an expression.

The representation $z = x + iy$ is called the rectangular form of the complex number $z$ and the notation $\mathbb{C}$ is used to denote all complex numbers having this form. From an examination of figure 1-1 we obtain the relation between rectangular coordinates $(x, y)$ and polar coordinates $(r, \theta)$ as
\[ x = r \cos \theta, \quad y = r \sin \theta, \quad \text{where} \quad r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x) \]

This gives us the polar form of a complex number and
\[ z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta). \] (1.2)

In the study of functions of a real variable in calculus one is introduced to the exponential function together with the sine and cosine functions. These functions were shown to have the Taylor series expansions
\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots \quad |x| < \infty \] (1.3)
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \quad |x| < \infty \] (1.4)
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \quad |x| < \infty \] (1.5)

If we formally replace $x$ in equation (1.3) by $i\theta$ one obtains after simplification
\[ e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^n \frac{\theta^{2n}}{(2n)!} + \cdots + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots \right) \right) \] (1.6)

and this result suggests that
\[ e^{i\theta} = \cos \theta + i \sin \theta \] (1.7)

Later we will show that this result is indeed true and find that the result is known as the Euler’s formula or Euler’s identity.

The Euler formula, given by equation (1.7), enables one to express a complex variable $z = x + iy$ in the polar form as
\[ z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = r e^{i\theta}. \] (1.8)

Here $r$ is called the modulus of $z$ or absolute value of $z$, written $\text{mod} \ z$ and $\theta$ is called the argument of $z$ and is written $\theta = \text{arg} \ z$. Using figure 1.1 one can verify that these quantities are given by the equations
\[ r = \text{mod} \ z = |z| = |x + iy| = \sqrt{x^2 + y^2}, \quad \theta = \text{arg} \ z = \tan^{-1}\left(\frac{y}{x}\right) \] (1.9)