Chapter 1

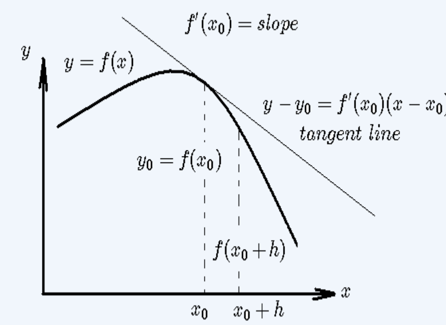
Preliminary Concepts

The prerequisite mathematical background for the material presented in this text are basic concepts from the subject areas of calculus, advanced calculus, differential equations and partial differential equations. The material selected for presentation in this first chapter is a combination of background preliminaries for review purposes together with an introduction of new concepts associated with the background material. We begin by examining selected fundamentals from the subject area of calculus.

Functions and derivatives

A real function f of a single real variable x is a rule which defines a one-to-one correspondence between elements x in a set A and image elements y in a set B. A real function f of a single variable x can be represented using the notation $y = f(x)$. The set A is called the domain of the function f and elements $x \in A$ are real numbers for which the image element $y = f(x) \in B$ is also a real quantity.

The set of image elements $y \in B$, as x varies over all elements in the domain, constitutes the range of the function f. An element x belonging to the domain of the function is called an independent variable and the corresponding image element y, in the range of f, is called a dependent variable. Functions can be represented graphically by plotting a set of points (x, y) on a Cartesian set of axes. A function $f(x)$ is continuous[‡] at a point x_0 if (i) $f(x_0)$ is defined, and (ii) $\lim_{x\to x_0} f(x)$ exists and (iii) $\lim_{x\to x_0} f(x) = f(x_0)$. A function is called discontinuous at a point if it is not continuous at the point.



Let $y = f(x)$ represent a function of a single real variable x. The derivative of this function, evaluated at a point x_0 is denoted using the notation $\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0)$ and is defined by the limiting process

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1.1}$$

if this limit exists. The derivative at a point x_0 can also be written as the limiting process

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1.2}$$

by making the substitution $h = x - x_0$ in the equation (1.1). The derivative $\frac{dy}{dx}$ evaluated at a point x_0 denotes the slope of the tangent line to the curve at the point $(x_0, f(x_0))$ on the

[‡] A more formal definition of continuity is that $f(x)$ is continuous for $x = x_0$ if for every small $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$, whenever $|x - x_0| < \delta$. A function is called continuous over an interval if it is continuous at all points within the interval.

curve $y = f(x)$. The equation of this tangent line is $y - y_0 = f'(x_0)(x - x_0)$. Higher ordered derivatives are defined as derivatives of lower ordered derivatives. For example, the second derivative $\frac{d^2y}{dx^2}$ is defined as the derivative of a first derivative or $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

Note that primes $'$ are used to denote differentiation with respect to the argument of the function. When the number of primes becomes too large we switch to an index in parenthesizes. For example, if $y = y(x)$ denotes a function which has derivatives through the nth order, then these derivatives are denoted

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \frac{d^3y}{dx^3} = y''', \quad \frac{d^4y}{dx^4} = y^{(4)}, \quad \ldots, \quad \frac{d^ny}{dx^n} = y^{(n)} \tag{1.3}$$

A δ-neighborhood of a point x_0 on the x-axis is defined as the set of points x satisfying $\{\, x \,\big|\, |x - x_0| < \delta \,\}$. This δ-neighborhood can also be written as the set of points x satisfying the inequality $x_0 - \delta < x < x_0 + \delta$.

Rolle's theorem

Let $y = f(x)$ denote a function of a single real variable x, which is continuous and differentiable at all points within an interval $a \le x \le b$. The Rolle's theorem, which is associated with continuous functions with continuous derivatives, states that if the height of the curve is the same at the points $x = a$ and $x = b$, then there exists at least one point $\xi \in (a, b)$ where the derivative satisfies the condition $f'(\xi) = 0$. This can be interpreted geometrically that the tangent line to the curve represented by $y = f(x)$, at the point $(\xi, f(\xi))$, is horizontal with zero slope. A representative situation illustrating the Rolle's theorem is given in the figure 1-1.

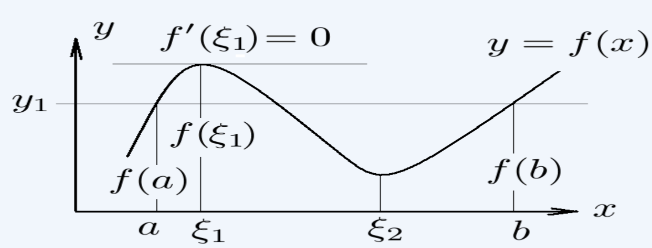
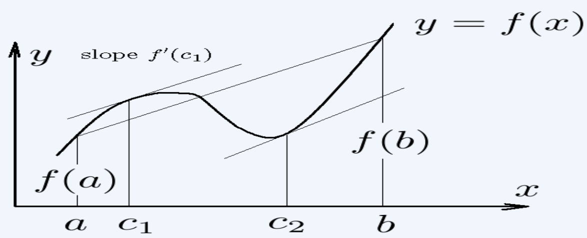


Figure 1-1. A situation illustrating the Rolle's theorem.

Mean value theorem

The mean value theorem is associated with functions $y = f(x)$ which are continuous and differentiable at all points within an interval $a \le x \le b$. Recall that a line passing through two different points $(a, f(a))$ and $(b, f(b))$ on a given curve is called a secant line. The mean value theorem states that if a function is continuous, then there exists at least one point $c \in (a, b)$ where the slope of the curve $y = f(x)$ at $x = c$ is the same as the slope of the secant line through the end points $(a, f(a))$ and $(b, f(b))$. The mean value theorem can be written $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some value c satisfying $a < c < b$. A representative situation illustrating the mean value theorem is given in the figure 1-2.



$$f'(c_1) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c_2) = \frac{f(b) - f(a)}{b - a}$$

Figure 1-2. A situation illustrating the mean value theorem.

A function f of two real variables x and y is a rule which assigns a one-to-one correspondence between an ordered pair of real numbers (x, y) from a set A and an image element z in a set B. The image element is denoted by the notation $z = f(x, y)$ where (x, y) represents a point in a domain A and z represents a point in the range B. Here x and y are real quantities and f represents a rule for producing a real image point z. The quantities x and y are called the independent variables of the function and z is called the dependent variable of the function.

A δ-neighborhood of a point (x_0, y_0), lying in the x, y-plane, is defined as the set of points (x, y) inside a small circle with center (x_0, y_0) and radius δ. All points (x, y) lying in a δ-neighborhood of the point (x_0, y_0) must satisfy the inequality

$$(x - x_0)^2 + (y - y_0)^2 < \delta^2. \tag{1.4}$$

A function $f(x, y)$ is continuous‡ at a point (x_0, y_0) if (i) $f(x_0, y_0)$ exists, (ii) $\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y)$ exists, and (iii) $\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0)$ and this limit must exist and be independent of the method that $x \to x_0$ and $y \to y_0$. If $f(x, y)$ is continuous for all points in a region R, then $f(x, y)$ is said to be continuous over the region R. A set of points R is called an open set, if every point $(x, y) \in R$ has some δ-neighborhood which lies entirely within the set R. If, in addition, the set R has the property that any two arbitrary distinct points $(x_0, y_0) \in R$ and $(x_1, y_1) \in R$ can be joined by connected line segments, which lie within the region R, then the set R is called a connected open set or a domain. The boundary of the region R is denoted ∂R and represents a set of points with the following property. A point $(x, y) \in \partial R$ is called a boundary point of the region R if every δ-neighborhood of the point (x, y) contains at least one point within the region R and at least one point not in the region R. A closed region R is one that contains its boundary points.

The partial derivatives of a function $z = z(x, y)$ with respect to x and y are defined

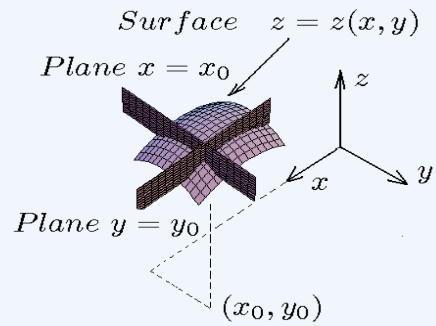
$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{z(x + \Delta x, y) - z(x, y)}{\Delta x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{z(x, y + \Delta y) - z(x, y)}{\Delta y} \tag{1.5}$$

if these limits exist.

‡ A more formal definition of continuity is that $f(x, y)$ is continuous at a point (x_0, y_0) if (i) it is defined at this point and (ii) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points (x, y) in a δ-neighborhood of (x_0, y_0) the relation $|f(x, y) - f(x_0, y_0)| < \epsilon$ is satisfied.

The function $z = z(x, y)$ represents a surface and points on this surface can be described by the position vector $\vec{r} = \vec{r}(x, y) = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z(x, y)\,\widehat{\mathbf{e}}_3$. The curves $\vec{r} = \vec{r}(x_0, y)$ and $\vec{r} = \vec{r}(x, y_0)$ represent curves on this surface called coordinate curves. These curves can be viewed as the intersection of the planes $y = y_0$ and $x = x_0$ with the surface $z = z(x, y)$. At the point (x_0, y_0, z_0) of the surface, where $z_0 = z(x_0, y_0)$, the vectors $\frac{\partial \vec{r}}{\partial x}\Big|_{(x_0,y_0,z_0)}$ and $\frac{\partial \vec{r}}{\partial y}\Big|_{(x_0,y_0,z_0)}$ represent tangent vectors to the surface curves $\vec{r} = \vec{r}(x, y_0)$ and $\vec{r} = \vec{r}(x_0, y)$ respectively. A normal vector $\vec{N}$ to the surface $z = z(x, y)$, at the point (x_0, y_0), can be calculated from the cross product $\vec{N} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}$ evaluated at the point (x_0, y_0, z_0). Note that the vector $-\vec{N}$ is also normal to the surface.

A surface is called a smooth surface if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are well defined continuous functions over the surface. This implies that a smooth surface has a well defined normal at each point on the surface. A sketch of the planes $x = x_0$ and $y = y_0$ intersecting the surface $z = z(x, y)$ shows the coordinate surface curves. The partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, evaluated at the point of intersection of the coordinate curves, represent the slopes of these coordinate curves at the point of intersection.



The locus of points (x, y) which satisfy $z(x, y) = constant$ produces a curve on the surface where z has a constant level and so such curves are called level curves. By selecting various constants $c_1, c_2, c_3, \ldots$ one can construct level curves $z(x, y) = c_i$ for $i = 1, 2, \ldots, n$. These level curves represent an intersection of the surface $z = z(x, y)$ with the planes $z = c_i = constant$, for $i = 1, 2, \ldots$. There exists many graphical packages for computer usage which will graph these level curves. Some of the more advanced graphical packages make it possible to sketch both the given surface and the level curves associated with the surface. These packages can be extremely helpful in illustrating the surface represented by a given function. A representative sketch of a set of level curves or contour plot associated with a specific surface is illustrated in the figure 1-3.

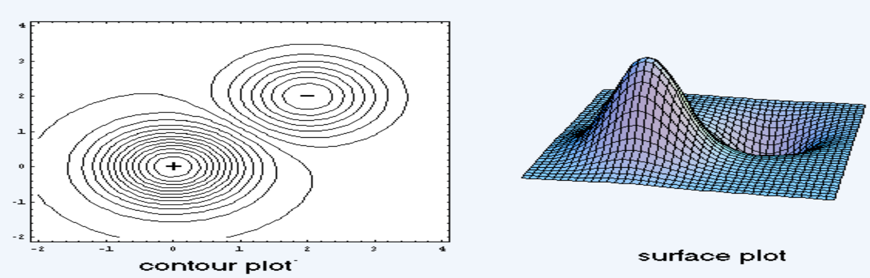


Figure 1-3. Contour plot and sketch of the function

$$z = z(z, y) = 5\exp\left[-x^2 - y^2\right] - 3\exp\left[-(x-2)^2 - (y-2)^2\right]$$

Higher ordered derivatives

Higher ordered partial derivatives are defined as derivatives of derivatives. For example, if the first ordered partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiated with respect to x and y there results four possible second ordered derivatives. These four possibilities are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2},\quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y},\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\partial x},\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}$$

Here the notation $\frac{\partial^2 z}{\partial x\partial y}$ denotes first a differentiation with respect to x which is to be followed by a differentiation with respect to y. In the case that all the derivatives are continuous over a common domain of definition, then the mixed partial derivatives are equal so that

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}. \tag{1.6}$$

This result is known as Clairaut's theorem. Higher ordered mixed derivatives have a similar property. For example, the third partial derivatives of the function z can be denoted

$$\frac{\partial^3 z}{\partial x^3}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right),\quad \frac{\partial^3 z}{\partial y^2\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y\partial x}\right),\quad \text{etc.} \tag{1.7}$$

Using Clairaut's theorem one can say that if all the mixed third partial derivatives are defined and continuous over a common domain of definition then it can be shown that

$$\frac{\partial^3 z}{\partial x\partial y^2}=\frac{\partial^3 z}{\partial y\partial x\partial y}=\frac{\partial^3 z}{\partial y^2\partial x}\qquad\text{and}\qquad \frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial^3 z}{\partial x\partial y\partial x}=\frac{\partial^3 z}{\partial x^2\partial y}$$

with similar results applying to mixed higher derivatives. The above concepts can be generalized to functions of n-real variables.

Partial derivatives are often times represented using a subscript notation. For example, if $z=z(x,y)$, then the first and second partial derivatives can be represented by the following subscript notation

$$z_x=\frac{\partial z}{\partial x},\quad z_y=\frac{\partial z}{\partial y},\quad z_{xx}=\frac{\partial^2 z}{\partial x^2},\quad z_{xy}=\frac{\partial^2 z}{\partial x\partial y},\qquad z_{yy}=\frac{\partial^2 z}{\partial y^2} \tag{1.8}$$

As another example, if $F=F(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y})=F(x,y,w,w_x,w_y)$, and we treat each variable as being independent, then the first partial derivatives of F can be represented

$$F_x=\frac{\partial F}{\partial x},\quad F_y=\frac{\partial F}{\partial y},\quad F_w=\frac{\partial F}{\partial w},\quad F_{w_x}=\frac{\partial F}{\partial w_x},\quad F_{w_y}=\frac{\partial F}{\partial w_y} \tag{1.9}$$

Curves in space

A set of parametric equations of the form

$$x=x(t),\quad y=y(t),\quad z=z(t)\quad \text{with parameter } t \tag{1.10}$$

defines a space curve C. The position vector to a general point on the curve C is written as

$$\vec{r}=\vec{r}(t)=x(t)\,\widehat{\mathbf{e}}_1+y(t)\,\widehat{\mathbf{e}}_2+z(t)\,\widehat{\mathbf{e}}_3. \tag{1.11}$$

where $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$, $\widehat{\mathbf{e}}_3$ are unit basis vectors in the directions of the x, y and z axes respectively. These basis vectors are sometimes written as the $\hat{i}$, $\hat{j}$, $\hat{k}$ unit vectors in the directions of the x, y and z axes. Consider a point P_1 on the curve C which is defined by the parametric value $t = t_1$ and having the position vector defined by $\vec{r}_1 = \vec{r}(t_1)$. The tangent vector to the curve C at the point P_1 is given by

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\widehat{\mathbf{e}}_1 + \frac{dy}{dt}\,\widehat{\mathbf{e}}_2 + \frac{dz}{dt}\,\widehat{\mathbf{e}}_3\bigg|_{t=t_1} = x'(t_1)\,\widehat{\mathbf{e}}_1 + y'(t_1)\,\widehat{\mathbf{e}}_2 + z'(t_1)\,\widehat{\mathbf{e}}_3 \tag{1.12}$$

Let ds denote an element of arc length along the curve C. An element of arc length squared is written

$$ds^2 = d\vec{r}\cdot d\vec{r}, \quad \text{or} \quad \left|\,\frac{d\vec{r}}{dt}\,\right| = \frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} \tag{1.13}$$

Hence, one can write a unit tangent vector to a point on the curve C from the relation

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\,\frac{d\vec{r}}{dt}\,\right|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds} \tag{1.14}$$

where s denotes arc length along the curve. When equation (1.14) is evaluated at the parametric value $t = t_1$ one obtains the unit tangent vector $\vec{T}$ to the point P_1.

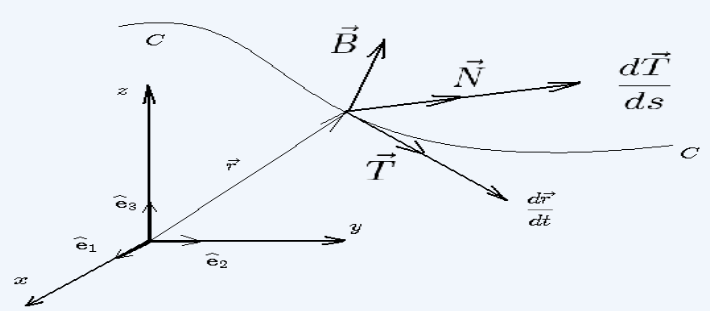


Figure 1-4. A space curve C with moving trihedral.

The rate of change of the unit tangent vector $\vec{T}$ with respect to arc length s along the curve C gives a vector which is normal to the unit tangent vector. The reason for this is that at any point on the curve C the unit tangent vector $\vec{T}$ satisfies $\vec{T}\cdot\vec{T} = 1$ and, consequently, if one differentiates this relation with respect to arc length s, there results

$$\frac{d}{ds}\left(\vec{T}\cdot\vec{T}\right) = \frac{d}{ds}(1) \;\Rightarrow\; \vec{T}\cdot\frac{d\vec{T}}{dt} + \frac{d\vec{T}}{ds}\cdot\vec{T} = 2\vec{T}\cdot\frac{d\vec{T}}{ds} = 0, \qquad \text{or} \qquad \vec{T}\cdot\frac{d\vec{T}}{ds} = 0$$

This last equation tells us that $\frac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$ since their dot product is zero. From the infinite number of vectors perpendicular to $\vec{T}$, the unit vector in the direction $\frac{d\vec{T}}{ds}$ is given the special name of principal unit normal to the curve C at the point P_1, and this principal unit normal is denoted by the symbol $\vec{N}$. At a point on the curve C, where the unit tangent is constructed, one can write

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}, \qquad \vec{N}\cdot\vec{N} = 1, \qquad \frac{d\vec{T}}{ds}\cdot\frac{d\vec{T}}{ds} = \kappa^2 \tag{1.15}$$

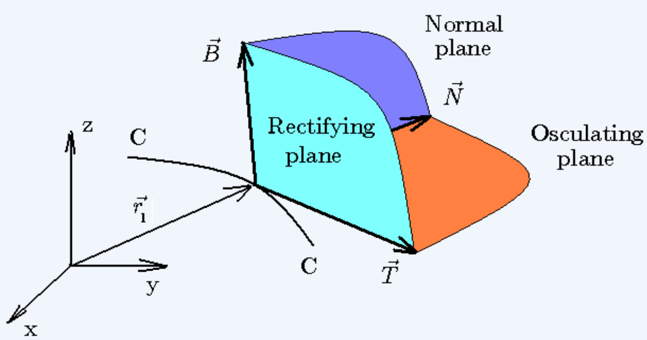
where κ is a scalar called the curvature of the curve C at a specific point where $\vec{T}$ is constructed. The quantity $\rho = 1/\kappa$ is called the radius of curvature of the curve C at this point. The unit vector $\vec{B}$ defined by $\vec{B} = \vec{T}\times\vec{N}$ is called the binormal vector to the curve C and is perpendicular to both $\vec{T}$ and $\vec{N}$. The three unit vectors $\vec{T}$, $\vec{N}$, $\vec{B}$ form a right-handed localized coordinate system at each point along the curve C and is often called a moving trihedral as the arc length s changes. A nominal situation is illustrated in the figure 1-4.

The vectors $\vec{T}$, $\vec{N}$, $\vec{B}$ satisfy the Frenet-Serret formulas

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}, \qquad \frac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T}, \qquad \frac{d\vec{B}}{ds} = -\tau\vec{N} \tag{1.16}$$

where τ is a scalar called the torsion and the quantity $\sigma = 1/\tau$ is called the radius of torsion. The torsion τ is a measure of the twisting of a space curve out of a plane. If $\tau = 0$, then the curve C is a plane curve. The curvature κ and radius of curvature ρ measure the turning of the curve C in relation to a localized circle which just touches the curve C at a point. If the curvature $\kappa = 0$, then the curve C is a straight line.

The plane containing the unit tangent vector $\vec{T}$ and the principal normal vector $\vec{N}$ is called the osculating plane. The plane containing the unit vectors $\vec{B}$ and $\vec{N}$ is perpendicular to the unit tangent vector and is called the normal plane to the curve. The plane containing the unit vectors $\vec{B}$ and $\vec{T}$ which is perpendicular to the unit principal normal vector is called the rectifying plane.



It is an easy exercise to show that the equation of the tangent line to the space curve C at the point P_1 is given by $(\vec{r}-\vec{r}_1)\times\vec{T} = \vec{0}$. Another easy exercise is to show the equations of the osculating, normal and rectifying planes constructed at the point P_1 are given by

Osculating plane	Normal plane	Rectifying plane	
$(\vec{r}-\vec{r}_1)\cdot\vec{B} = 0$	$(\vec{r}-\vec{r}_1)\cdot\vec{T} = 0$	$(\vec{r}-\vec{r}_1)\cdot\vec{N} = 0$	(1.17)

where $\vec{r} = x\,\widehat{\mathbf{e}}_1 + y\,\widehat{\mathbf{e}}_2 + z\,\widehat{\mathbf{e}}_3$ is the position vector to a variable point on the line or plane being constructed.

Curvilinear coordinates

Parametric equations of the form

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v), \tag{1.18}$$

which involve two independent parameters u and v, are used to define a surface. If one can solve for u and v from two of the equations, then the results can be substituted into the third equation to obtain the surface in the form $F(x,y,z) = 0$. The special case where equations (1.18) have the form

$$x = u, \qquad y = v, \qquad z = z(u,v) \tag{1.19}$$

produces the surface in the form $z = z(x,y)$.

The position vector

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\widehat{\mathbf{e}}_1 + y(u,v)\,\widehat{\mathbf{e}}_2 + z(u,v)\,\widehat{\mathbf{e}}_3 \tag{1.20}$$

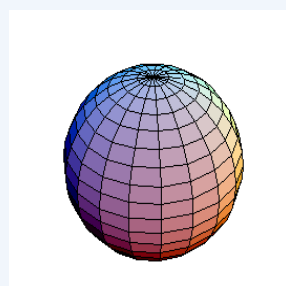
defines a general point on the surface. By setting $u = u_1 = \textit{constant}$, one obtains a curve on the surface given by

$$\vec{r}(u_1,v) = x(u_1,v)\,\widehat{\mathbf{e}}_1 + y(u_1,v)\,\widehat{\mathbf{e}}_2 + z(u_1,v)\,\widehat{\mathbf{e}}_3, \quad v \text{ is parameter} \tag{1.21}$$

Similarly, if one sets the parameter $v = v_1 = \textit{constant}$ one obtains the curves

$$\vec{r}(u,v_1) = x(u,v_1)\,\widehat{\mathbf{e}}_1 + y(u,v_1)\,\widehat{\mathbf{e}}_2 + z(u,v_1)\,\widehat{\mathbf{e}}_3, \quad u \text{ is parameter} \tag{1.22}$$

These curves are called coordinate curves.
If one selects equally spaced constant values $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ for the parametric values in the above curves, then the surface will be covered by a two parameter family of coordinate curves. A point on the surface can then characterized by assigning values to the parameters u and v and the set of points (u,v) are referred to as curvilinear coordinates of points on the surface. An example is illustrated in the accompanying figure.



Coordinate curves for unit sphere
$x = \sin u \cos v,\ \ y = \sin u \sin v,\ \ z = \cos u$

The partial derivatives $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ represent tangent vectors to the coordinate curves at a point (u,v) on the surface and consequently a normal vector $\vec{\mathcal{N}}$ to the surface is given by

$$\vec{\mathcal{N}} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$$

so that a unit normal vector $\widehat{n}$ to the surface is given by

$$\widehat{n} = \pm \frac{\vec{\mathcal{N}}}{|\vec{\mathcal{N}}|} = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{|\,\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\,|} \tag{1.23}$$

Note that there are always two unit normals at any point on a surface. These unit normals are given by $\widehat{n}$ and $-\widehat{n}$. If the surface is a closed surface then there is an outward pointing and inward pointing unit normal at each point on the surface. Therefore, you must select which unit normal you want.

The differential $d\vec{r}$ of the position vector $\vec{r}=\vec{r}(u,v)$ is written

$$d\vec{r}=\frac{\partial\vec{r}}{\partial u}\,du+\frac{\partial\vec{r}}{\partial v}\,dv \tag{1.24}$$

so that the square of an element of arc length on the surface is given by

$$ds^2=d\vec{r}\cdot d\vec{r}=\frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial u}\,du^2+2\frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial v}\,du\,dv+\frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial v}\,dv^2$$

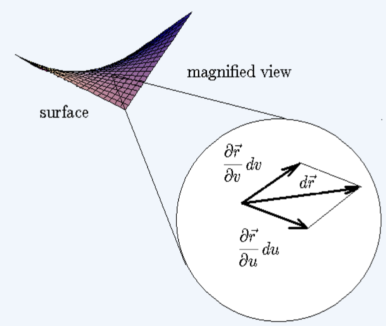
This is often written in the form

$$ds^2=E\,du^2+2F\,du\,dv+G\,dv^2 \tag{1.25}$$

where

$$E=\frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial u},\qquad F=\frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial v},\qquad G=\frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial v} \tag{1.26}$$

The differential $d\vec{r}$ defines a vector lying in the tangent plane to the surface and can be thought of as the diagonal of an elemental area parallelogram on the surface having sides $\frac{\partial\vec{r}}{\partial u}\,du$ and $\frac{\partial\vec{r}}{\partial v}\,dv$. The area of this elemental parallelogram is given by



$$d\sigma=\left|\,\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\,\right|\,du\,dv=\sqrt{\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)\cdot\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)}\,du\,dv$$

One can employ the vector identity $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$ to represent the element of surface area in the form

$$d\sigma=\sqrt{EG-F^2}\,du\,dv \tag{1.27}$$

In the special case the surface is defined by the parametric equations having the form $x=u,\ y=v, z=z(u,v)=z(x,y)$, then the element of surface area reduces to

$$d\sigma=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy \tag{1.28}$$

Using this same idea but changing symbols around, a surface defined by the parametric equations $x=u,\ y=y(u,v)=y(x,z),\ z=v$, has an element of surface area in the form

$$d\sigma=\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dz\,dx \tag{1.29}$$

Similarly, a surface defined by $x = x(u,v) = x(y,z)$, $y = u$, $z = v$, has the element of surface area

$$d\sigma = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz \tag{1.30}$$

If the surface is given in the implicit form $F(x,y,z) = 0$, then one can construct a normal vector to the surface from the equation $\vec{\mathcal{N}} = \operatorname{grad} F = \nabla F$, with unit normal vector $\hat{n} = \pm\frac{\nabla F}{|\nabla F|}$, because the gradient vector evaluated at a surface point is perpendicular to the surface defined by $F(x,y,z) = 0$. To show this, write the differential of F as follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz = \left(\frac{\partial F}{\partial x}\,\widehat{\mathbf{e}}_1 + \frac{\partial F}{\partial y}\,\widehat{\mathbf{e}}_2 + \frac{\partial F}{\partial z}\,\widehat{\mathbf{e}}_3\right)\cdot(dx\,\widehat{\mathbf{e}}_1 + dy\,\widehat{\mathbf{e}}_2 + dz\,\widehat{\mathbf{e}}_3) = \operatorname{grad} F\cdot d\vec{r} = 0.$$

This shows that the vector $\operatorname{grad} F$ is perpendicular to the vector $d\vec{r}$ which lies in the tangent plane to the surface, and so, must be perpendicular to the surface at the surface point of evaluation. In the special case $F = z(x,y) - z = 0$ defines the surface, then a unit normal vector to the surface is given by

$$\widehat{n} = \frac{\frac{\partial z}{\partial x}\,\widehat{\mathbf{e}}_1 + \frac{\partial z}{\partial y}\,\widehat{\mathbf{e}}_2 - \widehat{\mathbf{e}}_3}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \tag{1.31}$$

and then the surface area given by equation (1.28) can be written in the form

$$d\sigma = \frac{dx\,dy}{|\,\widehat{n}\cdot\widehat{\mathbf{e}}_3\,|} \tag{1.32}$$

In a similar manner it can be shown that the surface elements given by equations (1.29) and (1.30) can be written in the alternative forms

$$d\sigma = \frac{dx\,dz}{|\,\widehat{n}\cdot\widehat{\mathbf{e}}_2\,|} \qquad \text{and} \qquad d\sigma = \frac{dy\,dz}{|\,\widehat{n}\cdot\widehat{\mathbf{e}}_1\,|} \tag{1.33}$$

where $\widehat{n}$ is a unit normal to the surface. These equations have the physical interpretation of representing the projections of the surface element $d\sigma$ onto the x-y, x-z or y-z planes.

Integration

Integration is sometimes referred to as an anti-derivative. If you know a differentiation formula, then you can immediately obtain an integration formula. That is, if

$$\frac{dF(x)}{dx} = f(x), \quad \text{then} \quad \int f(x)\,dx = F(x) + C, \tag{1.34}$$

where C is a constant of integration. In the use of definite integrals the above relations are written

$$\text{if} \quad \frac{dF(x)}{dx} = f(x), \quad \text{then} \quad \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a). \tag{1.35}$$

If we replace the upper limit of integration by a variable quantity x, then equation (1.35) can be written as

$$F(x) = F(a) + \int_a^x f(x)\,dx, \quad \text{with} \quad \frac{dF(x)}{dx} = f(x).$$