Chapter 11
Special Functions

We present only a small sampling of the numerous special functions of mathematics and mathematical physics which have been developed over the course of time. These special functions can be defined in a variety of ways. The Bateman\(^1\) project on higher transcendental functions, headed by A. Erdélyi\(^2\), points out that power series, generating functions, infinite products, indefinite and definite integrals, differential equations, difference equations, functional equations, trigonometric series, series of orthogonal functions, integral equations and recursion techniques are representative of the many methods employed to define and develop properties of special functions.

Many special functions of a complex variable can be defined by using either a series or an appropriate integral. We present selected special functions and study some of their properties when they are treated as functions of real variables and as functions of a complex variable. We begin by examining the gamma function.

The gamma function

The gamma function can be defined by the limit

\[
\Gamma(z) = \lim_{n \to \infty} \frac{n^n z^n}{n!} \frac{1}{(z + 1)(z + 2) \cdots (z + n)}
\]  

(11.1)

or it can be defined by the integral

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{Re} \{z\} > 0
\]  

(11.2)

The related integral

\[
\gamma(\alpha, z) = \int_0^z e^{-t} t^{\alpha-1} dt, \quad \text{Re} \{\alpha\} > 0
\]  

(11.3)

is known as the incomplete gamma function. A graph of the gamma function for real values of \(z\) is given in the figure 11-1. The gamma function is an analytic function of \(z\) with singularities at the points given by \(z = 0, -1, -2, \ldots\) which are simple poles. The function \(t^{z-1} = e^{(z-1)\log t}\) is multiple-valued when the variable \(t\) is complex and so a branch cut must be introduced to make the function single-valued. The branch cut is not unique and many different kinds of branch cuts can be introduced. The figure 11-2 illustrates two different branch cuts.

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\(^1\) Harry Bateman (1882-1946) English mathematician

One can write the equation (11.2) in the form

\[ \Gamma(z) = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \] (11.4)

The second integral in equation (11.4) is well behaved and so we leave it alone. In the first integral of equation (11.4) one can expand the function \( e^{-t} \) in a power series about \( t = 0 \) and the resulting series can then be integrated term by term to obtain the result

\[ \int_0^1 e^{-t} t^{z-1} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} \] (11.6)

which shows that the residue of \( \Gamma(z) \) at the simple pole \( z = -n \) is given by

\[ \text{Res} [\Gamma(z), -n] = \frac{(-1)^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots \] (11.6)

The relation between the definitions given by equations (11.1) and (11.2) can be derived as follows. If \( n \) is a positive integer and \( \text{Re}\{z\} > 0 \), then one can use repeated integration by parts to show

\[ \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \frac{n! n^z}{z(z+1)(z+2) \cdots (z+n)} \] (11.7)

and then in the limit as \( n \) increases without bound one can employ the definition of the exponential function

\[ \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n = e^{-t} \] (11.8)

to obtain

\[ \Gamma(z) = \lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \int_0^\infty e^{-t} t^{z-1} dt = \lim_{n \to \infty} \frac{n! n^z}{z(z+1)(z+2) \cdots (z+n)} \] (11.9)

### Properties of the gamma function

In the special case \( z = n \) is an integer with \( n > 1 \), one can write

\[ \Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt \] (11.10)

and use integration by parts to show

\[ \Gamma(n) = [-t^{n-1} e^{-t}]_{t=0}^\infty + (n-1) \int_0^\infty e^{-t} t^{n-2} dt \] (11.11)

In \( n > 1 \), then the term is brackets becomes zero when the limits are substituted. The equation (11.11) then reduces to

\[ \Gamma(n) = (n-1)\Gamma(n-1) \quad \text{or shifting the index } n \text{ up } \Gamma(n+1) = n\Gamma(n) \] (11.12)
By shifting the index $n$ down can show

\[
\Gamma(n - 1) = (n - 2)\Gamma(n - 2)
\]
\[
\Gamma(n - 2) = (n - 3)\Gamma(n - 3)
\]
\[
\vdots
\]

and consequently

\[
\Gamma(n) = n(n - 1)(n - 2)(n - 3) \cdots 3 \cdot 2 \cdot 1 \cdot \Gamma(1)
\] (11.13)

Substituting $n = 1$ in equation (11.10) gives

\[
\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1
\] (11.15)
producing

\[
\Gamma(n) = (n - 1)! \quad \text{or shifting the index} \quad \Gamma(n + 1) = n!
\] (11.16)

The relation given by equation (11.16) only holds when $n$ is a positive integer. If $n$ is replaced by $z$ in equation (11.10) then one can show using integration by parts that

\[
\Gamma(z + 1) = z\Gamma(z)
\] (11.17)

and for $n$ a positive integer

\[
\Gamma(z + n) = z(z + 1)(z + 2) \cdots (z + n - 1)\Gamma(z)
\] (11.18)
The gamma function can be represented as a complex integral. Consider the contour integral of the function \( f(z) = e^z z^{-\alpha} \) along the path illustrated in the figure 11-2(a) and observe that

\[
\lim_{\gamma \to 0} \lim_{R \to \infty} \lim_{\rho \to 0} \oint_{C_L} e^z z^{-\alpha} \, dz = \lim_{\rho \to 0} \int_{C_R} e^{\rho e^{i\theta}} (\rho e^{i\theta})^{-\alpha} e^{i\pi} \, d\theta + \int_{C} e^{\rho e^{i\theta}} (\rho e^{i\theta})^{-\alpha} e^{i\pi} \, d\theta + \int_{C} e^{\rho e^{i\theta}} (\rho e^{i\theta})^{-\alpha} e^{i\pi} \, d\theta
\]

On the circular portion \( C_\rho \) we have \( z = \rho e^{i\theta} \) and so one can write

\[
\lim_{\rho \to 0} \int_{C_\rho} e^z z^{-\alpha} \, dz = \lim_{\rho \to 0} \int_{-\pi}^{\pi} e^{\rho (\cos \theta + i \sin \theta)} (\rho^{1-\alpha} e^{i\theta (1-\alpha)}) i d\theta \leq \lim_{\rho \to 0} \rho^{1-\text{Re} \{\alpha\}} \int_{-\pi}^{\pi} e^{\rho \cos \theta} \, e^{\text{Im} \{\alpha\}} \, d\theta
\]

If \( \text{Re} \{\alpha\} > 1 \), then the right-hand side of the above equation approaches zero as \( \rho \) tends toward zero. Therefore,

\[
\lim_{\gamma \to 0} \lim_{R \to \infty} \lim_{\rho \to 0} \oint_{C_L} e^z z^{-\alpha} \, dz = (e^{i\pi \alpha} - e^{-i\pi \alpha}) \int_{0}^{\infty} x^{-\alpha} e^{-x} \, dx = 2i \Gamma(1 - \alpha) \sin \pi \alpha \quad (11.119)
\]
It will be shown in the next section that \( \Gamma(\alpha) \Gamma(1-\alpha) = \frac{\pi}{\sin \pi \alpha} \) for \( 0 < \alpha < 1 \) and so we use this result to simplify the equation (11.119) to the form

\[
\lim_{\gamma \to 0} \lim_{\rho \to 0} \lim_{R \to \infty} \int_{C_L} e^z z^{-\alpha} \, dz = \frac{2\pi i}{\Gamma(\alpha)}
\]  
(11.120)

Similarly, if we examine the integral \( \lim_{\gamma \to 0} \lim_{\rho \to 0} \lim_{R \to \infty} \int_{C_R} e^{-z} z^{\alpha-1} \, dz \) around the path \( C_R \) illustrated in the figure 11-2(b), then one can show

\[
\Gamma(\alpha) = \frac{1}{e^{i2\pi \alpha} - 1} \lim_{\gamma \to 0} \lim_{\rho \to 0} \lim_{R \to \infty} \int_{C_L} e^z z^{-\alpha} \, dz
\]  
(11.121)

This result is left as an exercise.

**The beta function**

The beta function is defined

\[
B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} \, dt, \quad \text{Re}\{p\} > 0, \quad \text{Re}\{q\} > 0
\]  
(11.22)

and the related function

\[
B_z(p, q) = \int_0^z t^{p-1} (1-t)^{q-1} \, dt, \quad \text{Re}\{p\} > 0, \quad \text{Re}\{q\} > 0
\]  
(11.23)

is known as the incomplete beta function. Interchanging \( p \) and \( q \) one can show that the beta function is symmetric in these quantities. One can make the substitution \( t = 1 - \tau \) in equation (11.22) to obtain

\[
B(p, q) = -\int_0^1 \tau^{p-1} (1-\tau)^{q-1} \, d\tau = \int_0^1 \tau^{q-1} (1-\tau)^{p-1} \, d\tau = B(q, p)
\]  
(11.24)

Consider the double integral

\[
I = \int_0^\infty \int_0^\infty e^{-xy} (xy)^{p-1} e^{-x} x^q \, dx \, dy
\]  
(11.25)

which can be evaluated in two different ways. If we express the integral (11.25) in the form

\[
I = \int_0^\infty \left[ \int_0^\infty e^{-x(y+p)} x^{p+q-1} y^{p-1} \, dx \right] dy
\]

there results

\[
I = \int_0^\infty y^{p-1} \frac{\Gamma(p+q)}{(1+y)^{p+q}} \, dy = \Gamma(p+q) B(p, q)
\]  
(11.26)

Alternatively, one can interchange the order of integration and write the integral (11.25) in the form

\[
I = \int_0^\infty \left[ \int_0^\infty e^{-xy} y^{p-1} \, dy \right] e^{-x} x^{p+q-1} \, dx
\]  
(11.27)