Chapter 10

Fourier, Laplace and Z-transforms

Integral transforms

The figure 10-1 illustrates a mathematical operation where a function $f(t)$ undergoes a change to produce a new type of function $F(s)$. The operation depicted is that of an operator box where the input function $f(t)$ entering the box gets multiplied by a kernel function $K(s,t)$ and then the result is integrated with respect to $t$ from $-T$ to $T$ and then $T$ is allowed to increase without bound. The output from the box is a function $F(s)$ called the integral transform of the function $f(t)$. The transform variable $s$ is usually a complex variable $s = x + iy$ and selected such that the improper integral defining the integral transform exists. Consequently, the output function $F(s)$ is a function of a complex variable. By varying the kernel function and limits of integration a variety of integral transform operators can be constructed. In this chapter we investigate the Fourier\(^1\), Laplace\(^2\) and Z-transforms and illustrate some of their properties in relation to complex variable theory.

![Figure 10-1. An integral transform](image)

In general, a transformation or operator $L \{ \}$ is called a linear operator if for all functions $f(t)$ and $g(t)$ and for all constants $c_1, c_2$ the operator has the property that

$$L \{ c_1 f(t) + c_2 g(t) \} = c_1 L \{ f(t) \} + c_2 L \{ g(t) \}$$

(10.1)

Thus, a linear operator satisfies the property that the transformation of a linear combination of two functions produces the same linear combination of the transformed functions. Let us investigate the integral transforms known as the Fourier transform which can be viewed as a limiting case of a Fourier series.

---

\(^1\) Jean Baptiste Joseph Fourier (1768-1830) French mathematician

\(^2\) Pierre Simon De Laplace (1749-1827) French mathematician
Fourier series

The Fourier series representation of a function \( F(x) \) over the interval \(-L \leq x \leq L\) is given by

\[
F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \tag{10.2}
\]

where

\[
a_0 = \frac{1}{L} \int_{-L}^{L} F(x) \, dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} F(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} F(x) \sin \frac{n\pi x}{L} \, dx
\]

This series can be converted to complex form by making the substitutions

\[
\cos \frac{n\pi x}{L} = \frac{1}{2} \left( e^{in\pi x/L} + e^{-in\pi x/L} \right)
\]

\[
\sin \frac{n\pi x}{L} = \frac{1}{2i} \left( e^{in\pi x/L} - e^{-in\pi x/L} \right) \tag{10.4}
\]

to obtain

\[
F(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/L} \tag{10.5}
\]

where

\[
c_0 = \frac{a_0}{2} = \frac{1}{2L} \int_{-L}^{L} F(\xi) \, d\xi
\]

\[
c_n = \frac{1}{2} (a_n - ib_n) = \frac{1}{2L} \int_{-L}^{L} F(\xi) e^{-in\pi \xi/L} \, d\xi
\]

\[
c_{-n} = \frac{1}{2} (a_n + ib_n) = \frac{1}{2L} \int_{-L}^{L} F(\xi) e^{in\pi \xi/L} \, d\xi \tag{10.6}
\]

In the last summation of equation (10.5) replace \( n \) by \(-n\) to obtain

\[
F(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=-\infty}^{-1} c_n e^{in\pi x/L}
\]

which produces the complex form for the Fourier series as

\[
F(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad \text{where} \quad c_n = \frac{1}{2L} \int_{-L}^{L} F(\xi) e^{-in\pi \xi/L} \tag{10.7}
\]

The magnitudes of the various coefficients \(|c_n|\) represent a discrete amplitude spectrum of \( F(x) \) and can be plotted versus frequency.

The Fourier transform can be considered as a limiting case of the Fourier series as \( L \to \infty \). In equations (10.7) make the substitutions \( \omega = \frac{n\pi}{L} \) with \( \Delta \omega = \frac{\pi}{L} \) and assume that \( c_n = \frac{1}{\pi} f(\omega) \) as \( L \to \infty \). Then in the limit as \( L \) increases without bound the equations (10.7) can be written

\[
F(x) = \lim_{L \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f(\omega) e^{i\omega x} \Delta \omega, \quad f(\omega) = \lim_{L \to \infty} \int_{-L}^{L} F(x) e^{-i\omega x} \, dx \tag{10.8}
\]
The summation term becomes an integration in the limit as \( L \to \infty \) and the equations (10.8) are written as the Fourier transform pair

\[
F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{i\omega x} d\omega, \quad f(\omega) = \int_{-\infty}^{\infty} F(x) e^{-i\omega x} dx
\] (10.9)

Interchanging the dummy symbols \( x \) and \( \omega \) gives the alternative representation

\[
f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega, \quad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx
\] (10.10)

### Fourier transforms

Consider the Fourier integral formula from the theory of Fourier series and integrals. One form for the Fourier integral formula is

\[
f(x) = \int_{-\infty}^{\infty} e^{-i\omega x} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{i\omega \xi} d\xi \right] d\omega
\] (10.11)

The Fourier integral formula given by equation (10.11) can be written\(^3\) as a transform pair by defining the Fourier exponential transform and inverse transform as follows.

\[
\mathcal{F}_e \{ f(x); x \to \omega \} = F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx
\] (10.12)

\[
\mathcal{F}_e^{-1} \{ F(\omega); \omega \to x \} = f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega
\] (10.13)

These transformations can be viewed as the operators illustrated in the figure 10-2.

---

\(^3\) The Fourier exponential transform can be defined in several different ways.
In figure 10-2 the function $f(x)$ gets multiplied by the kernel function $\frac{1}{2\pi}e^{i\omega x}$ and then integrated with respect to $x$ from $-T$ to $T$ in the limit as $T$ increases without bound. Whenever this improper integral exists it is called the Fourier transform of $f(x)$. The transformation is denoted $\mathcal{F}_e\{f(x); x \rightarrow \omega\} = F(\omega)$. When the Fourier transform function $F(\omega)$ is multiplied by the kernel function $e^{-i\omega x}$ and then integrated with respect to $\omega$ from $-T$ to $T$ and $T$ is allowed to increase without bound, then these operations either reproduce the original function $f(x)$ or some equivalent representation of the original function $f(x)$ and so it is called the inverse Fourier exponential transform and denoted using the notation $\mathcal{F}_e^{-1}\{F(\omega); \omega \rightarrow x\} = f(x)$. The table 10-1 lists some important operational properties associated with the Fourier exponential transform and its inverse while the table 10-2 is a short table of Fourier exponential transforms obtained using methods from complex variable theory.

**Operational properties**

The following is a summary of the operational properties listed in the table 10-1. The first property listed states that the columns in a Fourier exponential transform table can be interchanged if proper changes of variables are made. In particular, note that upon replacing $\omega$ by $-x$ and $x$ by $\omega$ simultaneously in the definitions given by the equations (10.12) and (10.13) we obtain

\[
2\pi F(-x) = \int_{-\infty}^{\infty} f(\omega)e^{-i\omega x} d\omega = \mathcal{F}_e^{-1}\{f(\omega)\}
\]

\[
f(\omega) = \int_{-\infty}^{\infty} F(-x)e^{i\omega x} (-dx) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi F(-x)e^{i\omega x} dx = \mathcal{F}_e\{2\pi F(-x)\}
\]

The equations (10.14) illustrate the Fourier exponential transform column interchange property given as the first entry of 10-1. The entries labeled 1, 2 and 5, 6 in the table 10-2 are examples of this interchange property.

The second property in table 10-1 follows from the definition of the Fourier exponential transform since the integral operator is a linear operator. The third property in table 10-1 is a differentiation property of the Fourier exponential transform. If $\mathcal{F}_e\{f(x)\} = F(\omega)$, then $\mathcal{F}_e\{f'(x)\} = -i\omega F(\omega)$. This property follows from the definition

\[
\mathcal{F}_e\{f'(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f'(x)e^{i\omega x} dx
\]

which we integrate by parts to obtain

\[
\mathcal{F}_e\{f'(x)\} = \frac{1}{2\pi} \left[ f(x)e^{i\omega x} \right]_{-\infty}^{\infty} - (i\omega) \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx = -i\omega \mathcal{F}_e\{f(x)\} = -i\omega F(\omega).
\]

Here we assume $\lim_{x \to \pm\infty} f(x) = 0$ to obtain the above result.

The fourth and fifth properties in table 10-1 follow from the property 3. Here each time a differentiation is performed in the $x$-domain it corresponds to a multiplication by $-i\omega$ in the transform $\omega$-domain.
### Table 10-1. Fourier Exponential Transform Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x} d\omega = \mathcal{F}^{-1}{F(\omega)}$</td>
<td>$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx = \mathcal{F}{f(x)}$</td>
<td></td>
</tr>
<tr>
<td>$f(x) = \mathcal{F}^{-1}{F(\omega)}$</td>
<td>$F(\omega) = \mathcal{F}{f(x)}$</td>
<td>Comments</td>
</tr>
<tr>
<td>1. $2\pi F(-x)$</td>
<td>$f(\omega)$</td>
<td>Column interchange</td>
</tr>
<tr>
<td>2. $c_1 f(x) + c_2 g(x)$</td>
<td>$c_1 F(\omega) + c_2 G(\omega)$</td>
<td>Linearity property</td>
</tr>
<tr>
<td>3. $f'(x)$</td>
<td>$-i\omega F(\omega)$</td>
<td>Derivative property</td>
</tr>
<tr>
<td>4. $f''(x)$</td>
<td>$(-i\omega)^2 F(\omega)$</td>
<td></td>
</tr>
<tr>
<td>5. $f^{(n)}(x)$</td>
<td>$(-i\omega)^n F(\omega)$</td>
<td></td>
</tr>
<tr>
<td>6. $f(x - \alpha)$</td>
<td>$e^{i\alpha \omega} F(\omega)$</td>
<td>Shift property</td>
</tr>
<tr>
<td>7. $xf(x)$</td>
<td>$-\frac{dF}{d\omega} = -iF'(\omega)$</td>
<td>Multiplication by $x$ property</td>
</tr>
<tr>
<td>8. $x^n f(x)$</td>
<td>$(-i)^n \frac{d^n F(\omega)}{d\omega^n}$</td>
<td></td>
</tr>
<tr>
<td>9. $f^* g = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau)g(x - \tau) d\tau$</td>
<td>$F(\omega)G(\omega)$</td>
<td>Convolution property</td>
</tr>
<tr>
<td>10. $f(x)g(x)$</td>
<td>$\int_{-\infty}^{\infty} F(\omega - \tau)G(\tau) d\tau$</td>
<td></td>
</tr>
<tr>
<td>11. $\delta(x - x_0)$</td>
<td>$\frac{1}{2\pi} e^{i\omega x_0}$</td>
<td>Dirac delta function</td>
</tr>
<tr>
<td>12. $f(ax)$, $a &gt; 0$</td>
<td>$\frac{1}{a} F\left(\frac{\omega}{a}\right)$</td>
<td>Scaling property</td>
</tr>
<tr>
<td>13. $f(ax)e^{ibx}$, $a &gt; 0$</td>
<td>$\frac{1}{a} F\left(\frac{\omega + b}{a}\right)$</td>
<td>Shift and scaling</td>
</tr>
<tr>
<td>14. $f(ax) \cos bx$</td>
<td>$\frac{1}{2a} \left[ F\left(\frac{\omega + b}{a}\right) + F\left(\frac{\omega - b}{a}\right) \right]$</td>
<td></td>
</tr>
<tr>
<td>15. $f(ax) \sin bx$</td>
<td>$\frac{1}{2ia} \left[ F\left(\frac{\omega + b}{a}\right) - F\left(\frac{\omega - b}{a}\right) \right]$</td>
<td></td>
</tr>
<tr>
<td>16. $f(x)e^{iax}$</td>
<td>$F(\omega + a)$</td>
<td>Shift property</td>
</tr>
</tbody>
</table>