Chapter 10

Fourier, Laplace and Z-transforms

Integral transforms

The figure 10-1 illustrates a mathematical operation where a function $f(t)$ undergoes a change to produce a new type of function $F(s)$. The operation depicted is that of an operator box where the input function $f(t)$ entering the box gets multiplied by a kernel function $K(s,t)$ and then the result is integrated with respect to t from $-T$ to T and then T is allowed to increase without bound. The output from the box is a function $F(s)$ called the integral transform of the function $f(t)$. The transform variable s is usually a complex variable $s = x + iy$ and selected such that the improper integral defining the integral transform exists. Consequently, the output function $F(s)$ is a function of a complex variable. By varying the kernel function and limits of integration a variety of integral transform operators can be constructed. In this chapter we investigate the Fourier[1], Laplace[2] and Z-transforms and illustrate some of their properties in relation to complex variable theory.

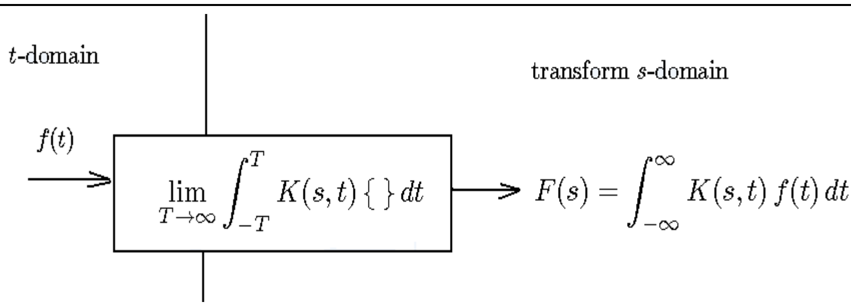

Figure 10-1. An integral transform

In general, a transformation or operator $L\{\ \}$ is called a linear operator if for all functions $f(t)$ and $g(t)$ and for all constants c_1, c_2 the operator has the property that

$$L\{c_1 f(t) + c_2 g(t)\} = c_1\, L\{f(t)\} + c_2\, L\{g(t)\} \tag{10.1}$$

Thus, a linear operator satisfies the property that the transformation of a linear combination of two functions produces the same linear combination of the transformed functions. Let us investigate the integral transforms known as the Fourier transform which can be viewed as a limiting case of a Fourier series.

[1] Jean Baptiste Joseph Fourier (1768-1830) French mathematician
[2] Pierre Simon De Laplace (1749-1827) French mathematician

Fourier series

The Fourier series representation of a function $F(x)$ over the interval $-L \le x \le L$ is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \tag{10.2}$$

where

$$\begin{aligned} a_0 =& \frac{1}{L}\int_{-L}^{L} F(x)\,dx \\ a_n =& \frac{1}{L}\int_{-L}^{L} F(x)\cos\frac{n\pi x}{L}\,dx \\ b_n =& \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{n\pi x}{L}\,dx \end{aligned} \tag{10.3}$$

This series can be converted to complex form by making the substitutions

$$\begin{aligned} \cos\frac{n\pi x}{L} =& \frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right) \\ \sin\frac{n\pi x}{L} =& \frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right) \end{aligned} \tag{10.4}$$

to obtain

$$F(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/L} \tag{10.5}$$

where

$$\begin{aligned} c_0 =& \frac{a_0}{2} &&= \frac{1}{2L}\int_{-L}^{L} F(\xi)\,d\xi \\ c_n =& \frac{1}{2}(a_n - ib_n) &&= \frac{1}{2L}\int_{-L}^{L} F(\xi)e^{-in\pi\xi/L}\,d\xi \\ c_{-n} =& \frac{1}{2}(a_n + ib_n) &&= \frac{1}{2L}\int_{-L}^{L} F(\xi)e^{in\pi\xi/L}\,d\xi \end{aligned} \tag{10.6}$$

In the last summation of equation (10.5) replace n by $-n$ to obtain

$$F(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=-1}^{-\infty} c_n e^{in\pi x/L}$$

which produces the complex form for the Fourier series as

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \qquad \text{where} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} F(\xi)e^{-in\pi\xi/L} \tag{10.7}$$

The magnitudes of the various coefficients $|c_n|$ represent a discrete amplitude spectrum of $F(x)$ and can be plotted versus frequency.

The Fourier transform can be considered as a limiting case of the Fourier series as $L \to \infty$. In equations (10.7) make the substitutions $\omega = \frac{n\pi}{L}$ with $\Delta\omega = \frac{\pi}{L}$ and assume that $c_n = \frac{1}{2L} f(\omega)$ as $L \to \infty$. Then in the limit as L increases without bound the equations (10.7) can be written

$$F(x) = \lim_{L\to\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(\omega)e^{i\omega x}\,\Delta\omega, \qquad f(\omega) = \lim_{L\to\infty}\int_{-L}^{L} F(x)e^{-i\omega x}\,dx \tag{10.8}$$

The summation term becomes an integration in the limit as $L \to \infty$ and the equations (10.8) are written as the Fourier transform pair

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{i\omega x}\,d\omega, \qquad f(\omega) = \int_{-\infty}^{\infty} F(x)e^{-i\omega x}\,dx \tag{10.9}$$

Interchanging the dummy symbols x and ω gives the alternative representation

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega, \qquad F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx \tag{10.10}$$

Fourier transforms

Consider the Fourier integral formula from the theory of Fourier series and integrals. One form for the Fourier integral formula is

$$f(x) = \int_{-\infty}^{\infty} e^{-i\omega x}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,e^{i\omega\xi}\,d\xi\right]d\omega \tag{10.11}$$

The Fourier integral formula given by equation (10.11) can be written[3] as a transform pair by defining the Fourier exponential transform and inverse transform as follows.

$$\mathcal{F}_e\{f(x); x \to \omega\} = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{i\omega x}\,dx \tag{10.12}$$

$$\mathcal{F}_e^{-1}\{F(\omega); \omega \to x\} = f(x) = \int_{-\infty}^{\infty} F(\omega)\,e^{-i\omega x}\,d\omega \tag{10.13}$$

These transformations can be viewed as the operators illustrated in the figure 10-2.

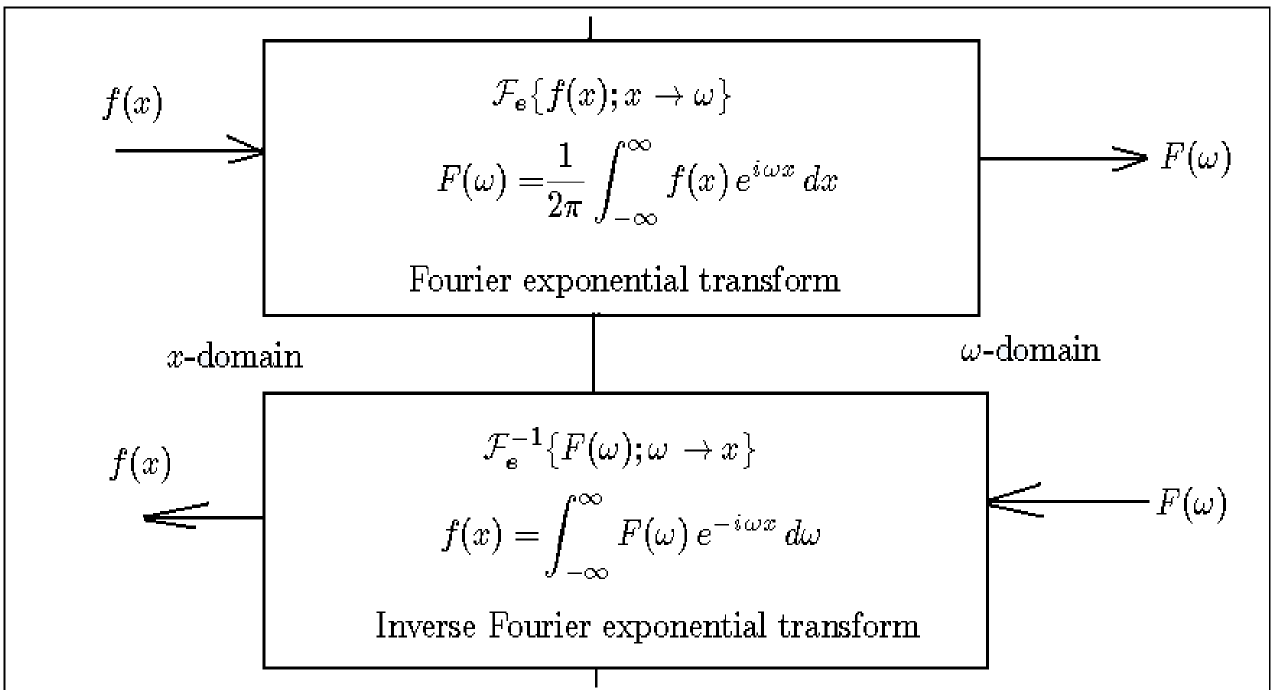

Figure 10-2. The Fourier exponential transform and its inverse transform

[3] The Fourier exponential transform can be defined in several different ways.

In figure 10-2 the function $f(x)$ gets multiplied by the kernel function $\frac{1}{2\pi}e^{i\omega x}$ and then integrated with respect to x from $-T$ to T in the limit as T increases without bound. Whenever this improper integral exists it is called the Fourier transform of $f(x)$. The transformation is denoted $\mathcal{F}_e\{f(x); x \to \omega\} = F(\omega)$. When the Fourier transform function $F(\omega)$ is multiplied by the kernel function $e^{-i\omega x}$ and then integrated with respect to ω from $-T$ to T and T is allowed to increase without bound, then these operations either reproduce the original function $f(x)$ or some equivalent representation of the original function $f(x)$ and so it is called the inverse Fourier exponential transform and denoted using the notation $\mathcal{F}_e^{-1}\{F(\omega); \omega \to x\} = f(x)$. The table 10-1 lists some important operational properties associated with the Fourier exponential transform and its inverse while the table 10-2 is a short table of Fourier exponential transforms obtained using methods from complex variable theory.

Operational properties

The following is a summary of the operational properties listed in the table 10-1. The first property listed states that the columns in a Fourier exponential transform table can be interchanged if proper changes of variables are made. In particular, note that upon replacing ω by $-x$ and x by ω simultaneously in the definitions given by the equations (10.12) and (10.13) we obtain

$$\begin{aligned} 2\pi F(-x) &= \int_{-\infty}^{\infty} f(\omega)e^{-ix\omega}\,d\omega = \mathcal{F}_e^{-1}\{f(\omega)\} \\ f(\omega) &= \int_{\infty}^{-\infty} F(-x)e^{i\omega x}\,(-dx) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, F(-x)e^{i\omega x}\,dx = \mathcal{F}_e\{2\pi\, F(-x)\} \end{aligned} \tag{10.14}$$

The equations (10.14) illustrate the Fourier exponential transform column interchange property given as the first entry of 10-1. The entries labeled 1, 2 and 5, 6 in the table 10-2 are examples of this interchange property.

The second property in table 10-1 follows from the definition of the Fourier exponential transform since the integral operator is a linear operator. The third property in table 10-1 is a differentiation property of the Fourier exponential transform. If $\mathcal{F}_e\{f(x)\} = F(\omega)$, then $\mathcal{F}_e\{f'(x)\} = -i\omega F(\omega)$. This property follows from the definition

$$\mathcal{F}_e\{f'(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)e^{i\omega x}\,dx$$

which we integrate by parts to obtain

$$\mathcal{F}_e\{f'(x)\} = \frac{1}{2\pi}\left[f(x)e^{i\omega x}\right]_{-\infty}^{\infty} - (i\omega)\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = -i\omega\mathcal{F}_e\{f(x)\} = -i\omega F(\omega).$$

Here we assume $\lim_{x\to\pm\infty} f(x) = 0$ to obtain the above result.

The fourth and fifth properties in table 10-1 follow from the property 3. Here each time a differentiation is performed in the x-domain it corresponds to a multiplication by $-i\omega$ in the transform ω-domain.

Table 10-1. Fourier Exponential Transform Properties

$f(x)=\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega=\mathcal{F}_e^{-1}\{F(\omega)\}$ | $F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx=\mathcal{F}_e\{f(x)\}$

	$f(x)=\mathcal{F}_e^{-1}\{F(\omega)\}$	$F(\omega)=\mathcal{F}_e\{f(x)\}$	Comments
1.	$2\pi F(-x)$	$f(\omega)$	Column interchange
2.	$c_1 f(x)+c_2 g(x)$	$c_1 F(\omega)+c_2 G(\omega)$	Linearity property
3.	$f'(x)$	$-i\omega F(\omega)$	Derivative property
4.	$f''(x)$	$(-i\omega)^2 F(\omega)$	
5.	$f^{(n)}(x)$	$(-i\omega)^n F(\omega)$	
6.	$f(x-\alpha)$	$e^{i\omega\alpha}F(\omega)$	Shift property
7.	$xf(x)$	$-i\frac{dF}{d\omega}=-iF'(\omega)$	Multiplication by x property
8.	$x^n f(x)$	$(-i)^n\frac{d^nF(\omega)}{d\omega^n}$	
9.	$f^{*}g=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)g(x-\tau)\,d\tau$	$F(\omega)G(\omega)$	Convolution property
10.	$f(x)g(x)$	$\int_{-\infty}^{\infty} F(\omega-\tau)G(\tau)\,d\tau$	
11.	$\delta(x-x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	Dirac delta function
12.	$f(ax),\quad a>0$	$\frac{1}{a}F\left(\frac{\omega}{a}\right)$	Scaling property
13.	$f(ax)e^{ibx},\ a>0$	$\frac{1}{a}F\left(\frac{\omega+b}{a}\right)$	Shift and scaling
14.	$f(ax)\cos bx$	$\frac{1}{2a}\left[F\left(\frac{\omega+b}{a}\right)+F\left(\frac{\omega-b}{a}\right)\right]$	
15.	$f(ax)\sin bx$	$\frac{1}{2ia}\left[F\left(\frac{\omega+b}{a}\right)-F\left(\frac{\omega-b}{a}\right)\right]$	
16.	$f(x)e^{iax}$	$F(\omega+a)$	Shift property