8.3 Diagonalization of Symmetric Matrices

DEF $\rightarrow$ p.368

A is called an **orthogonal matrix** if $A^{-1} = A^T$.

TH 8.8 $\rightarrow$ p.369

A is **orthogonal** if and only if the column vectors of A form an **orthonormal** set.

Proof: Let $A = \begin{pmatrix} & | & & | & \\ \overrightarrow{u_1} & | & \cdots & | & \overrightarrow{u_n} \\ & | & & | & \end{pmatrix}$. Then

$$A \text{ is orthogonal}$$

$$\Updownarrow$$

$$A^{-1} = A^T$$

$$\Updownarrow$$

$$I_n = A^T A$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & \overrightarrow{u_1}^T & \\ - & - & - \\ & \vdots & \\ - & - & - \\ & \overrightarrow{u_n}^T & \end{pmatrix} \begin{pmatrix} & | \; | & \\ \overrightarrow{u_1} & |\ldots| & \overrightarrow{u_n} \\ & | \; | & \end{pmatrix}$$

$$\Updownarrow$$

$$\overrightarrow{u_i} \bullet \overrightarrow{u_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Updownarrow$$

The column vectors of A form an orthonormal set.

If the matrix A is symmetric then

- its eigenvalues are all real ($\rightarrow$ **`TH 8.6`** p. 366)
- eigenvectors corresponding to distinct eigenvalues are orthogonal ($\rightarrow$ **`TH 8.7`** p. 366)
- A is **orthogonally diagonalizable**, i.e. there exists an orthogonal matrix P such that $P^{-1}AP = D$, where D is diagonal. ($\rightarrow$ **`TH 8.9`** p. 369)

`EXAMPLE 1`

Orthogonally diagonalize $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 4$$

$$= \lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

For the eigenvalue $\underline{\lambda_1 = 3}$, the homogeneous system $(\lambda_1 I - A)\overrightarrow{u} = \overrightarrow{0}$ has the coefficient matrix:

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \text{ with r.r.e.f. } \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Solution: $x_2 = s$ (arbitrary); $x_1 = s$.
Eigenspace = span$\{(1,1)\}$
Normalize the eigenvector $\overrightarrow{u} = (1,1)$ for a column of P: $\frac{\overrightarrow{u}}{\|\overrightarrow{u}\|} = \frac{1}{\sqrt{2}}(1,1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

For the eigenvalue $\underline{\lambda_2 = -1}$, the homogeneous system $(\lambda_2 I - A)\overrightarrow{u} = \overrightarrow{0}$ has the coefficient matrix:

$$\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \text{ with r.r.e.f. } \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Solution: $x_2 = s$ (arbitrary); $x_1 = -s$.
Eigenspace = span$\{(-1,1)\}$
Normalize the eigenvector $\overrightarrow{u} = (-1,1)$ for a column of P: $\frac{\overrightarrow{u}}{\|\overrightarrow{u}\|} = \frac{1}{\sqrt{2}}(-1,1) = (\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Let $P = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$. Then

$$P^TAP = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

EXAMPLE 2 → Examples 1 p. 367 and 3 p. 368

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{pmatrix}.$$

$$P = \begin{pmatrix} 0 & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

EXAMPLE 3

Orthogonally diagonalize $A = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{pmatrix}$.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+4 & -2 & 2 \\ -2 & \lambda+7 & -4 \\ 2 & -4 & \lambda+7 \end{vmatrix}$$

$= \lambda^3 + 18\lambda^2 + 81\lambda + 108$

Integer factors of 108:

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm27, \pm36, \pm54, \pm108.$

λ value	$\det(\lambda I - A)$	eigenvalue?
1	$1 + 18 + 81 + 108 \neq 0$	No
-1	$-1 + 18 - 81 + 108 \neq 0$	No
2	$8 + 72 + 162 + 108 \neq 0$	No
-2	$-8 + 72 - 162 + 108 \neq 0$	No
3	$27 + 162 + 243 + 108 \neq 0$	No
-3	$-27 + 162 - 243 + 108 = 0$	Yes

Divide $\lambda^3 + 18\lambda^2 + 81\lambda + 108$ by $\lambda + 3$ to obtain:

$$\det(\lambda I - A) = (\lambda^2 + 15\lambda + 36)(\lambda + 3).$$

Use quadratic formula to find roots of $\lambda^2 + 15\lambda + 36$:

$$\lambda = \frac{-15 \pm \sqrt{15^2 - 4(1)(36)}}{2}$$
$$= \frac{-15 \pm \sqrt{225 - 144}}{2}$$
$$= \frac{-15 \pm 9}{2}$$

Therefore

$$\det(\lambda I - A) = (\lambda + 12)(\lambda + 3)^2$$

For the eigenvalue $\lambda_1 = -12$, the homogeneous system $(\lambda_1 I - A)\overrightarrow{u} = \overrightarrow{0}$ has the coefficient matrix:

$$\begin{pmatrix} -8 & -2 & 2 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{pmatrix} \text{ with r.r.e.f. } \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: $x_3 = s$ (arbitrary), $x_1 = \frac{1}{2}s; x_2 = -s$.

Eigenspace = span$\{(\frac{1}{2}, -1, 1)\}$ (or span$\{(1, -2, 2)\}$).
For columns of P we need orthonormal eigenvectors.
Normalize $\vec{v} = (1, -2, 2)$:

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{1+4+4}}(1, -2, 2) = (\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$$

For the eigenvalue $\lambda_2 = -3$, the homogeneous system $(\lambda_2 I - A)\vec{u} = \vec{0}$ has the coefficient matrix:

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{pmatrix} \text{ with r.r.e.f. } \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: $x_2 = s$ (arbitrary), $x_3 = t$ (arbitrary), $x_1 = 2s - 2t$.
Eigenspace = span$\{(2, 1, 0), (-2, 0, 1)\}$.

Use Gram-Schmidt process to obtain an orthonormal basis for the eigenspace.

$$\{\underbrace{(2,1,0)}_{\overrightarrow{u_1}},\underbrace{(-2,0,1)}_{\overrightarrow{u_2}}\}.$$

$$\overrightarrow{v_1} = \overrightarrow{u_1} = (2,1,0)$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2}\bullet\overrightarrow{v_1}}{\overrightarrow{v_1}\bullet\overrightarrow{v_1}}\overrightarrow{v_1}$$

$$= (-2,0,1) - \frac{(-2,0,1)\bullet(2,1,0)}{(2,1,0)\bullet(2,1,0)}(2,1,0)$$

$$= (-2,0,1) - \frac{-4}{5}(2,1,0)$$

$$= (-\frac{2}{5},\frac{4}{5},1)$$

$$\overrightarrow{w_1} = \frac{\overrightarrow{v_1}}{\left\|\overrightarrow{v_1}\right\|} = \frac{(2,1,0)}{\sqrt{4+1+0}} = (\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0)$$

$$= (\frac{2\sqrt{5}}{5},\frac{\sqrt{5}}{5},0)$$

$$\overrightarrow{w_2} = \frac{\overrightarrow{v_2}}{\left\|\overrightarrow{v_2}\right\|} = \frac{(-\frac{2}{5},\frac{4}{5},1)}{\sqrt{\frac{4}{25}+\frac{16}{25}+1}}$$

$$= \frac{\sqrt{5}}{3}(-\frac{2}{5},\frac{4}{5},1) = (\frac{-2\sqrt{5}}{15},\frac{4\sqrt{5}}{15},\frac{\sqrt{5}}{3})$$

Orthonormal basis for the eigenspace:

$$\{(\frac{2\sqrt{5}}{5},\frac{\sqrt{5}}{5},0),(\frac{-2\sqrt{5}}{15},\frac{4\sqrt{5}}{15},\frac{\sqrt{5}}{3})\}.$$

An orthogonal matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{5}}{5} & \frac{-2\sqrt{5}}{15} \\ \frac{-2}{3} & \frac{\sqrt{5}}{5} & \frac{4\sqrt{5}}{15} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix} \text{ satisfies}$$

$$P^T A P = \begin{pmatrix} -12 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

→ Example 4 p. 369