8.2 Diagonalization

DEF (→ p. 356)

We say a matrix B is **similar** to a matrix A if there exists a nonsingular matrix P such that
$B = P^{-1}AP$.

Properties of similarity (→ p. 356)

1. A is similar to A.
2. If B is similar to A then A is similar to B.
3. If A is similar to B and B is similar to C then A is similar to C.

Proof of the third property:
We assume A is similar to B, i.e. there exists P s.t. $B = P^{-1}AP$.
Likewise, B is similar to C, i.e. there exists Q s.t. $C = Q^{-1}BQ$.
We can write

$$\begin{aligned} C &= Q^{-1}BQ \\ &= Q^{-1}(P^{-1}AP)Q \\ &= (Q^{-1}P^{-1})A(PQ) \\ &= (PQ)^{-1}A(PQ) \end{aligned}$$

Since there exists a matrix $S = PQ$ such that $C = S^{-1}AS$, we conclude that C is similar to A.

Note: According to property 2., instead of saying " A is similar to B" or " B is similar to A" we can say: " A and B are similar".

DEF (→ p. 356)

If a matrix A is similar to a diagonal matrix then A is said to be **diagonalizable** (i.e. A can be **diagonalized**.)

TH 8.3 (→ p. 357)

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof

$$A \text{ is diagonalizable}$$

$$\Updownarrow$$

$$P^{-1}AP = D \text{ (diagonal)}$$

$$\Updownarrow$$

$$AP = PD$$

$$\Updownarrow$$

$$A\begin{pmatrix} & | & & | & \\ \overrightarrow{u_1} & | & \cdots & | & \overrightarrow{u_n} \\ & | & & | & \end{pmatrix}$$

$$= \begin{pmatrix} & | & & | & \\ \overrightarrow{u_1} & | & \cdots & | & \overrightarrow{u_n} \\ & | & & | & \end{pmatrix}\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} & | & | & \\ A\overrightarrow{u_1} & | \ldots | & & A\overrightarrow{u_n} \\ & | & | & \end{pmatrix} = \begin{pmatrix} & | & | & \\ d_1\overrightarrow{u_1} & | \ldots | & & d_n\overrightarrow{u_n} \\ & | & | & \end{pmatrix}$$

$$\Updownarrow$$

$$A\overrightarrow{u_i} = d_i\overrightarrow{u_i} \text{ for } i = 1, \ldots, n$$

THEOREM If $\lambda_1, \ldots, \lambda_k$ are distinct real eigenvalues of the matrix A, then the corresponding eigenvectors $\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}$ are linearly independent.

Proof

Assume $\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}$ are linearly **dependent**.
Then let $\overrightarrow{u_j}$ be the first vector that can be expressed as a linear combination of preceding vectors:

$$\overrightarrow{u_j} = c_1\overrightarrow{u_1} + \cdots + c_{j-1}\overrightarrow{u_{j-1}} \qquad (\star)$$

Premultiplying both sides by A we get

$$A\overrightarrow{u_j} = c_1A\overrightarrow{u_1} + \cdots + c_{j-1}A\overrightarrow{u_{j-1}}$$

$$\lambda_j\overrightarrow{u_j} = c_1\lambda_1\overrightarrow{u_1} + \cdots + c_{j-1}\lambda_{j-1}\overrightarrow{u_{j-1}}$$

Multiplying both sides of $(\star)$ by λ_j yields

$$\lambda_j\overrightarrow{u_j} = c_1\lambda_j\overrightarrow{u_1} + \cdots + c_{j-1}\lambda_j\overrightarrow{u_{j-1}}$$

Subtract the last two equations:

$$\overrightarrow{0} = c_1(\lambda_1 - \lambda_j)\overrightarrow{u_1} + \cdots + c_{j-1}(\lambda_{j-1} - \lambda_j)\overrightarrow{u_{j-1}}$$

Since $\overrightarrow{u_1}, \ldots, \overrightarrow{u_{j-1}}$ are linearly independent and λ's are distinct, this implies $c_1 = \cdots = c_{j-1} = 0$.
However, in this case $(\star)$ would mean $\overrightarrow{u_j} = \overrightarrow{0}$.
This is impossible since $\overrightarrow{u_j}$ is an eigenvector.

TH 8.4 (→ p. 357)

If all the roots of $\det(\lambda I - A)$ are real and distinct then A is diagonalizable.

EXAMPLE 1

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

In **EXAMPLE 3** we found:

eigenvalue	eigenspace
1	$\text{span}\{(-1, 1, 2)\}$
2	$\text{span}\{(-2, 1, 4)\}$
3	$\text{span}\{(-1, 1, 4)\}$

Let $P = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{pmatrix}$.

Then $P^{-1} = \begin{pmatrix} 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$ (verify!)

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

EXAMPLE 2 $\rightarrow$ Example 5 p.360

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & -2 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)^2.$$

For the eigenvalue $\underline{\lambda_1 = 1}$, the coefficient matrix of (*):

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: $x_2 = s, x_1 = x_3 = 0$;
Eigenspace: $\text{span}\{(0, 1, 0)\}$.
Dimension of the eigenspace $= 1 < 2$ (multiplicity).
A is **not diagonalizable**.

EXAMPLE 3 → Example 6 p.360

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)^2.$$

For the eigenvalue $\underline{\lambda_1 = 1}$, the coefficient matrix of (*):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: $x_1 = 0, x_2 = s, x_3 = t$.
Eigenspace = span $\{(0,1,0),(0,0,1)\}$. Dimension = multiplicity.
A is diagonalizable.

To find P such that $P^{-1}AP = A$, consider $\underline{\lambda_2 = 0}$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenspace = span$\{(-1, 0, 1)\}$.

If $P = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ then

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ then

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\rightarrow$ Procedure for diagonalizing A (p.361)