## 8.2 Diagonalization

 $\mathbf{DEF} (\to p. 356)$ 

We say a matrix *B* is **similar** to a matrix *A* if there exists a nonsingular matrix *P* such that  $B = P^{-1}AP$ .

Properties of similarity ( $\rightarrow$  p. 356)

- **1.** A is similar to A.
- **2.** If *B* is similar to *A* then *A* is similar to *B*.
- **3.** If *A* is similar to *B* and *B* is similar to *C* then *A* is similar to *C*.

*Proof of the third property:* 

We assume *A* is similar to *B*, i.e. there exists *P* s.t.  $B = P^{-1}AP$ .

Likewise, *B* is similar to *C*, i.e. there exists *Q* s.t.  $C = Q^{-1}BQ$ .

We can write

$$C = Q^{-1}BQ$$
$$= Q^{-1}(P^{-1}AP)Q$$
$$= (Q^{-1}P^{-1})A(PQ)$$
$$= (PQ)^{-1}A(PQ)$$

Since there exists a matrix S = PQ such that  $C = S^{-1}AS$ , we conclude that *C* is similar to *A*.

Note: According to property 2., instead of saying "*A* is similar to *B*" or "*B* is similar to *A*" we can say: "*A* and *B* are similar".

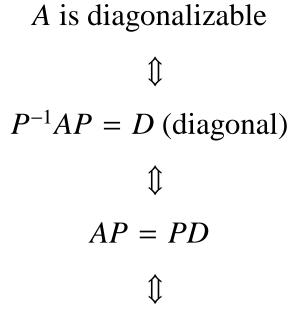
$$\mathbf{DEF} (\to p. 356)$$

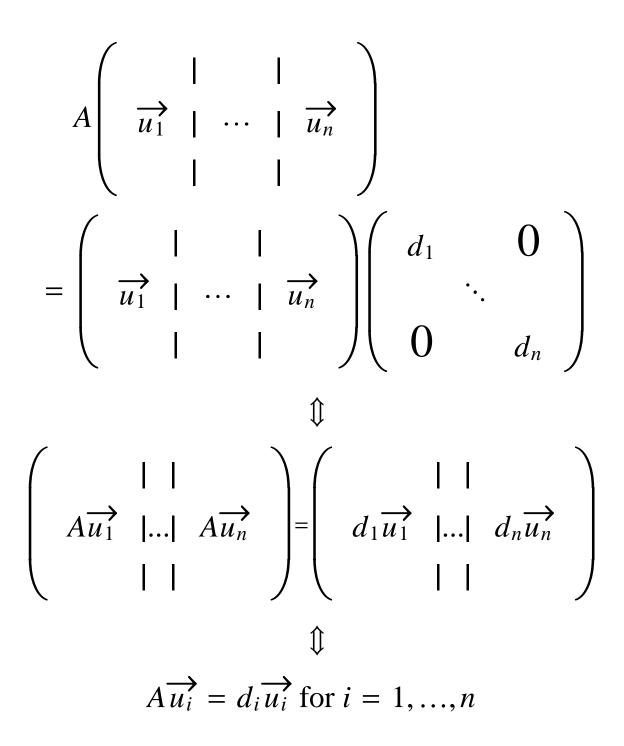
If a matrix A is similar to a diagonal matrix then A is said to be **diagonalizable** (i.e. A can be **diagonalized**.)

**TH** 8.3  $(\rightarrow p.357)$ 

An  $n \times n$  matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof





**THEOREM** If  $\lambda_1, \ldots, \lambda_k$  are distinct real

eigenvalues of the matrix A, then the corresponding eigenvectors  $\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}$  are linearly independent.

*Proof* Assume  $\overrightarrow{u_1}, ..., \overrightarrow{u_k}$  are linearly **dependent**. Then let  $\overrightarrow{u_j}$  be the first vector that can be expressed as a linear combination of preceding vectors:

$$\overrightarrow{u_j} = c_1 \overrightarrow{u_1} + \dots + c_{j-1} \overrightarrow{u_{j-1}} \qquad (\star)$$

Premultiplying both sides by A we get

$$\overrightarrow{Au_{j}} = c_{1}A\overrightarrow{u_{1}} + \dots + c_{j-1}A\overrightarrow{u_{j-1}}$$
$$\overrightarrow{\lambda_{j}u_{j}} = c_{1}\lambda_{1}\overrightarrow{u_{1}} + \dots + c_{j-1}\lambda_{j-1}\overrightarrow{u_{j-1}}$$

Multiplying both sides of ( $\star$ ) by  $\lambda_j$  yields

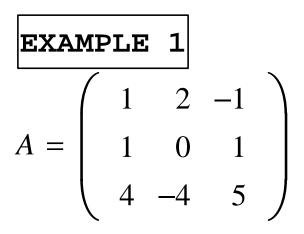
$$\lambda_j \overrightarrow{u_j} = c_1 \lambda_j \overrightarrow{u_1} + \dots + c_{j-1} \lambda_j \overrightarrow{u_{j-1}}$$

Subtract the last two equations:

 $\overrightarrow{0} = c_1(\lambda_1 - \lambda_j)\overrightarrow{u_1} + \dots + c_{j-1}(\lambda_{j-1} - \lambda_j)\overrightarrow{u_{j-1}}$ Since  $\overrightarrow{u_1}, \dots, \overrightarrow{u_{j-1}}$  are linearly independent and  $\lambda'$ s are distinct, this implies  $c_1 = \dots = c_{j-1} = 0$ . However, in this case (\*) would mean  $\overrightarrow{u_j} = \overrightarrow{0}$ . This is impossible since  $\overrightarrow{u_j}$  is an eigenvector.

**TH** 8.4 
$$(\rightarrow p. 357)$$

If all the roots of det( $\lambda I - A$ ) are real and distinct then *A* is diagonalizable.



In **EXAMPLE** 3 we found:

eigenvalue	eigenspace
1	$span\{(-1, 1, 2)\}$
2	$span\{(-2, 1, 4)\}$
3	$span\{(-1, 1, 4)\}$

Let 
$$P = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{pmatrix}$$
.  
Then  $P^{-1} = \begin{pmatrix} 0 & 2 & -\frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$  (verify!)  
 $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ 

**EXAMPLE 2** 
$$\rightarrow$$
 Example 5 p.360  
 $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ .  
 $det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & -2 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)^2.$ 

For the eigenvalue  $\underline{\lambda_1} = 1$ , the coefficient matrix of (\*):

$$\left(\begin{array}{rrrr}1 & 0 & -1\\0 & 0 & -2\\0 & 0 & 0\end{array}\right)$$
 has r.r.e.f. 
$$\left(\begin{array}{rrrr}1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{array}\right)$$

Solution:  $x_2 = s, x_1 = x_3 = 0$ ; Eigenspace: span{(0, 1, 0)}. Dimension of the eigenspace = 1 < 2 (multiplicity). *A* is **not diagonalizable**.

**EXAMPLE 3** 
$$\rightarrow$$
 Example 6 p.360  
 $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ .  
 $det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)^2$ .

For the eigenvalue  $\underline{\lambda_1} = 1$ , the coefficient matrix of (\*):

$$\left(\begin{array}{rrrr}1 & 0 & 0\\ 0 & 0 & 0\\ -1 & 0 & 0\end{array}\right)$$
 has r.r.e.f. 
$$\left(\begin{array}{rrrr}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right)$$

Solution:  $x_1 = 0, x_2 = s, x_3 = t$ . Eigenspace = span {(0, 1, 0), (0, 0, 1)}. Dimension = multiplicity.

A is diagonalizable.

To find *P* such that  $P^{-1}AP = A$ , consider  $\underline{\lambda_2} = 0$ :

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \text{has r.r.e.f.} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
  
Eigenspace = span{(-1,0,1)}.  
If  $P = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$  then  
 $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

If 
$$Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
 then  
$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

 $\rightarrow$  Procedure for diagonalizing A (p.361)