8.1 Eigenvalues and Eigenvectors

DEF ($\rightarrow$ p. 343)

Let A be an $n \times n$ matrix. The scalar λ is called an **eigenvalue** of A if there exists a nonzero vector $\overrightarrow{v}$ such that

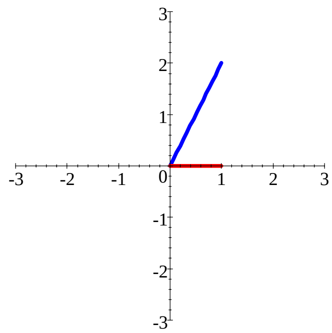
$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

$\overrightarrow{v}$ is called an **eigenvector** associated with the eigenvalue λ.

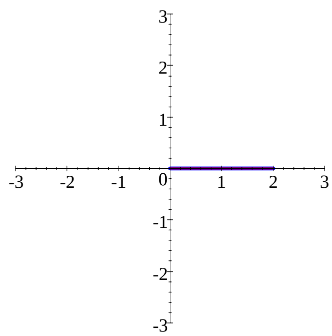
EXAMPLE 1 Consider the linear transformation

$$L\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- projection onto the x-axis $L(x,y) = (x,0)$ ($\rightarrow$ **EXAMPLE 1** of the Section 4.3 lecture)

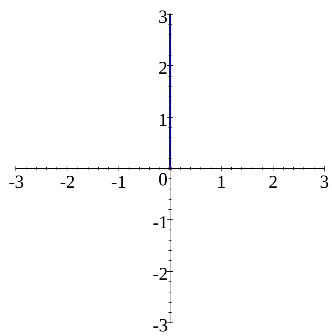


$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Eigenvalue: 1, eigenvector: $(2, 0)$.



$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalue: 0, eigenvector: $(0, 3)$.

Notice that if $\overrightarrow{v}$ is an eigenvector associated with the eigenvalue λ, then for any nonzero number c, $c\overrightarrow{v}$ is also an eigenvector associated with the same eigenvalue λ:

$$A(c\overrightarrow{v}) = c(A\overrightarrow{v}) = c(\lambda\overrightarrow{v}) = \lambda(c\overrightarrow{v})$$

Specifically, in last example, we have

- an eigenvalue $\lambda = 1$ and an associated eigenvector $(x, 0)$ (for any nonzero x), and
- an eigenvalue $\lambda = 0$ and an associated eigenvector $(0, y)$ (for any nonzero y).

We want to develop a general procedure for finding eigenvalues and eigenvectors.

If we had a scalar equation

$$ax = \lambda x$$

then we could solve for *x* as follows:

$$\lambda x - ax = 0$$

$$(\lambda - a)x = 0$$

The matrix equation

$$A\vec{v} = \lambda\vec{v}$$

can be rewritten as

$$\lambda\vec{v} - A\vec{v} = \vec{0}$$

but **not** as

$$(\underbrace{\lambda - A}_{\substack{\text{cannot} \\ \text{evaluate}}})\vec{v} = \vec{0}$$

Instead, write

$$\lambda I \vec{v} - A\vec{v} = \vec{0}$$

which yields

$$(\lambda I - A)\vec{v} = \vec{0} \qquad (*)$$

If $\vec{v}$ is an eigenvector of A, it **cannot** be a **zero** vector.

Therefore, such a $\vec{v}$ is a **nontrivial** solution of the homogeneous system (*).

For the system to have such solutions, its coefficient matrix, $\lambda I - A$, must be **singular**.

This is equivalent to

$$\det(\lambda I - A) = 0 \qquad (**)$$

→ Th. 8.2 p. 348

Procedure for finding eigenvalues and eigenvectors of a matrix.

(1) Find all values $\lambda = \lambda_1, \lambda_2, \ldots$ such that $\det(\lambda I - A) = 0$.

(2) For each eigenvalue λ_i found in (1), solve the system $(\lambda_i I - A)\overrightarrow{v} = \overrightarrow{0}$ for the corresponding $\overrightarrow{v}$.

Use the procedure for the matrix of **EXAMPLE 1**.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(1)

$$\lambda I - A = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda - 1)\lambda$$

$\det(\lambda I - A)$ is called the **characteristic polynomial**. The equation (**) is called the **characteristic equation**:

$$\det(\lambda I - A) = 0$$

$$(\lambda - 1)\lambda = 0$$

The eigenvalues are: $\lambda_1 = 1$ and $\lambda_2 = 0$.

(2) For $\lambda_1 = 1$, the homogeneous system (*) becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The coefficient matrix has the reduced row echelon form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$x_1 = s$ (arbitrary) and $x_2 = 0$. Therefore, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Eigenvectors associated with $\lambda_1 = 1$ are all scalar multiples of $(1, 0)$ (except for the zero vector).

Solution space: span$\{(1, 0)\}$ - **eigenspace** associated with the eigenvalue 1.

For $\lambda_2 = 0$, the homogeneous system (*) becomes

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The coefficient matrix has the reduced row echelon form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$x_2 = s$ (arbitrary) and $x_1 = 0$. Therefore, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eigenvectors associated with $\lambda_2 = 0$ are all scalar multiples of $(0, 1)$ (except for the zero vector).

Eigenspace: $\text{span}\{(0, 1)\}$.

EXAMPLE 2 Find all eigenvalues and a basis for each associated eigenspace for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

(1) The characteristic polynomial: $\det(\lambda I - A)$

$$= \det\begin{pmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{pmatrix}$$

$$= (\lambda - 1)(-1)^2 \det\begin{pmatrix} \lambda - 1 & -5 & 10 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 3 \end{pmatrix}$$

$$= (\lambda - 1)^2(\lambda - 2)(\lambda - 3)$$

Eigenvalues: $\lambda_1 = 1$ (multiplicity 2), $\lambda_2 = 2, \lambda_3 = 3$.

(2) For the eigenvalue $\underline{\lambda_1 = 1}$, the system (*) becomes

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The coefficient matrix has the reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_2 = s$ and $x_4 = t$ are arbitrary, while $x_1 = -2t$ and $x_3 = -2t$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Basis for the eigenspace:
$\{(0, 1, 0, 0), (-2, 0, -2, 1)\}$.

For the eigenvalue $\underline{\lambda_2 = 2}$, the system (*) becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The coefficient matrix has the reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_3 = s$ is arbitrary, while $x_1 = x_4 = 0$ and $x_2 = 5s$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}$$

Basis for the eigenspace: $\{(0, 5, 1, 0)\}$.

For the eigenvalue $\underline{\lambda_3 = 3}$, the basis for the eigenspace is $\{(0, -5, 0, 1)\}$. (Check!)

Equivalent conditions ($\rightarrow$ p. 348)

For any $n \times n$ matrix A, the following conditions are equivalent:

1. A is nonsingular.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. A is row equivalent to I_n.
4. For every $n \times 1$ matrix $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a unique solution.
5. $\det(A) \neq 0$.
6. rank $A = n$.
7. nullity $A = 0$.
8. The rows of A are linearly independent.
9. The columns of A are linearly independent.
10. Zero is not an eigenvalue of A.

$\rightarrow$ Th. 8.1 p.348

$\rightarrow$ Read the discussion on p.349 above Example 6:

The only *rational* roots of the characteristic polynomial

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

with *integer* coefficients $a_1, \ldots, a_n$ are integers that are among factors of a_n.

EXAMPLE 3 $\rightarrow$ Examples 5 p. 347 and 6 p.349

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}.$$

Characteristic polynomial:

$$\det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1 & -2 & 1 \\ -1 & \lambda & -1 \\ -4 & 4 & \lambda - 5 \end{vmatrix}$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

Rational eigenvalues must be among the numbers: $\pm 1, \pm 2, \pm 3,$ and ± 6.

Test $\lambda = -1$:

$$-1-6-11-6 \neq 0$$

$\lambda = -1$ is not an eigenvalue.
Test $\lambda = 1$:

$$1-6+11-6 = 0$$

$\lambda = 1$ is an eigenvalue.
Divide:

$$\begin{array}{r|l}
 & \lambda^2 - 5\lambda + 6 \\ \hline
\lambda - 1 & \lambda^3 - 6\lambda^2 + 11\lambda - 6 \\
 & \underline{-\lambda^3 + \lambda^2} \\
 & -5\lambda^2 + 11\lambda - 6 \\
 & \underline{5\lambda^2 - 5\lambda} \\
 & 6\lambda - 6 \\
 & \underline{-6\lambda + 6} \\
 & 0
\end{array}$$

Therefore

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda^2 - 5\lambda + 6)$$
$$= (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

The eigenvalues: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3.$

For the eigenvalue $\underline{\lambda_1 = 1}$, the coefficient matrix of (*):

$$\begin{pmatrix} 0 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & -4 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenspace = span $\{(\frac{-1}{2}, \frac{1}{2}, 1)\}$ (or span$\{(-1, 1, 2)\}$)

For the eigenvalue $\underline{\lambda_2 = 2}$, the coefficient matrix of (*):

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{-1}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenspace = span $\{(\frac{-1}{2}, \frac{1}{4}, 1)\}$ (or span$\{(-2, 1, 4)\}$)

For the eigenvalue $\underline{\lambda_3 = 3}$, the coefficient matrix of (*):

$$\begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ -4 & 4 & -2 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{-1}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenspace = span $\{(\frac{-1}{4}, \frac{1}{4}, 1)\}$ (or span$\{(-1, 1, 4)\}$)

EXAMPLE 4 → Example 7 p. 350

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

A has no real eigenvalues.

Geometric interpretation:
Example 10, p.208, introduced a linear transformation $L : R^2 \to R^2$ defined as

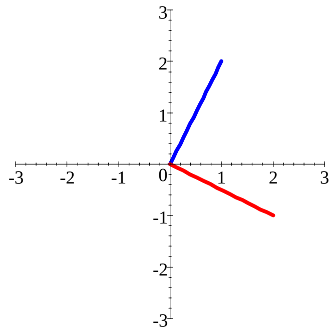
$$L\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

This is a **rotation** around the origin by the angle ϕ.

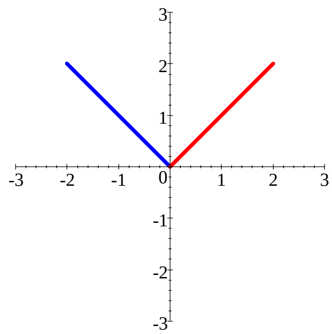
Taking $\phi = \frac{-\pi}{2}$ we obtain

$$L\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

For example,



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

No real eigenvalues can be ”seen” for this matrix.