6.8 Orthonormal Bases

DEF $\to$ p.306

A set of vectors $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}$ in R^n is said to be **orthogonal** if

(a) $\overrightarrow{v_i} \bullet \overrightarrow{v_j} = 0$ for all $i \neq j$.

S is said to be **orthonormal** if in addition to the condition (a), it also satisfies

(b) $\overrightarrow{v_i} \bullet \overrightarrow{v_i} = 1$ for all $i = 1, \ldots, k$.

EXAMPLE 1 $S = \{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\}$ is an orthonormal set in R^3, since

(a) $\overrightarrow{i} \bullet \overrightarrow{j} = \overrightarrow{j} \bullet \overrightarrow{k} = \overrightarrow{i} \bullet \overrightarrow{k} = 0$, and

(b) $\overrightarrow{i} \bullet \overrightarrow{i} = \overrightarrow{j} \bullet \overrightarrow{j} = \overrightarrow{k} \bullet \overrightarrow{k} = 1$.

EXAMPLE 2 $S = \{(1,-1,2,3),(-1,1,1,0)\}$ is an orthogonal set in R^4, since

(a) $(1,-1,2,3) \bullet (-1,1,1,0) = 0.$

However, S is not orthonormal since (b) does not hold: $(1,-1,2,3) \bullet (1,-1,2,3) = 15 \neq 1.$

TH 6.16 $\rightarrow$ p.307

Every orthogonal set of nonzero vectors is linearly independent.

DEF $\rightarrow$ p.307

An orthogonal (orthonormal) set of vectors in a vector space V that forms a basis for V is called an **orthogonal** (**orthonormal**) **basis** for V.

TH 6.17 → p.308

If $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ is an orthonormal basis for R^n then for every vector $\overrightarrow{v}$ in R^n:

$$\overrightarrow{v} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_n\overrightarrow{v_n}$$

where $c_i = \overrightarrow{v} \bullet \overrightarrow{v_i}$ for all i.

Proof

Since S is a basis, then we can write

$$\overrightarrow{v} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_n\overrightarrow{v_n}$$

Multiplying both sides by $\overrightarrow{v_i}$ yields

$$\overrightarrow{v} \bullet \overrightarrow{v_i} = c_1(\overrightarrow{v_1} \bullet \overrightarrow{v_i}) + c_2(\overrightarrow{v_2} \bullet \overrightarrow{v_i}) + \cdots + c_n(\overrightarrow{v_n} \bullet \overrightarrow{v_i})$$

Since S is orthogonal then only the i-th term on the right hand side can be nonzero:

$$\overrightarrow{v} \bullet \overrightarrow{v_i} = c_i(\overrightarrow{v_i} \bullet \overrightarrow{v_i})$$

Because S is orthonormal,

$$c_i = \overrightarrow{v} \bullet \overrightarrow{v_i}$$

TH 6.18 **Gram-Schmidt Process** → p.308

Let W be a nonzero subspace of R^n with a basis $S = \{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$. Then an orthonormal basis T for W can be determined by the following process:

(1) Determine an orthogonal basis for W: $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$

$$\vec{v_1} = \vec{u_1}$$

$$\vec{v_2} = \vec{u_2} - \frac{\vec{u_2} \bullet \vec{v_1}}{\vec{v_1} \bullet \vec{v_1}} \vec{v_1}$$

$$\vdots$$

$$\vec{v_i} = \vec{u_i} - \frac{\vec{u_i} \bullet \vec{v_1}}{\vec{v_1} \bullet \vec{v_1}} \vec{v_1} - \cdots - \frac{\vec{u_i} \bullet \overrightarrow{v_{i-1}}}{\overrightarrow{v_{i-1}} \bullet \overrightarrow{v_{i-1}}} \overrightarrow{v_{i-1}}$$

$$\vdots$$

(2) Determine $T = \{\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k}\}$, an orthonormal basis for W by normalizing each of the orthogonal vectors $\vec{v_i}$:

$$\vec{w_i} = \frac{\vec{v_i}}{\left\| \vec{v_i} \right\|} \text{ for all } i$$

EXAMPLE 3

Apply the Gram-Schmidt Process to the set

$$\{\underbrace{(1,1,0)}_{\overrightarrow{u_1}}, \underbrace{(1,2,0)}_{\overrightarrow{u_2}}, \underbrace{(0,1,2)}_{\overrightarrow{u_3}}\}.$$

(1)

$$\overrightarrow{v_1} = \overrightarrow{u_1} = (1,1,0)$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2}\bullet\overrightarrow{v_1}}{\overrightarrow{v_1}\bullet\overrightarrow{v_1}}\overrightarrow{v_1}$$

$$= (1,2,0) - \frac{(1,2,0)\bullet(1,1,0)}{(1,1,0)\bullet(1,1,0)}(1,1,0)$$

$$= (1,2,0) - \frac{3}{2}(1,1,0)$$

$$= (-\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

$$\overrightarrow{v_3} = \overrightarrow{u_3} - \frac{\overrightarrow{u_3}\bullet\overrightarrow{v_1}}{\overrightarrow{v_1}\bullet\overrightarrow{v_1}}\overrightarrow{v_1} - \frac{\overrightarrow{u_3}\bullet\overrightarrow{v_2}}{\overrightarrow{v_2}\bullet\overrightarrow{v_2}}\overrightarrow{v_2}$$

$$= (0,1,2) - \frac{(0,1,2)\bullet(1,1,0)}{(1,1,0)\bullet(1,1,0)}(1,1,0)$$

$$- \frac{(0,1,2)\bullet(-\frac{1}{2},\frac{1}{2},0)}{(-\frac{1}{2},\frac{1}{2},0)\bullet(-\frac{1}{2},\frac{1}{2},0)}(-\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

$$= (0,1,2) - \frac{1}{2}(1,1,0) - (-\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

$$= (0,0,2).$$

(2) $\overrightarrow{w_1} = \dfrac{\overrightarrow{v_1}}{\left\|\overrightarrow{v_1}\right\|} = \dfrac{(1,1,0)}{\sqrt{1+1+0}}$

$= (\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0) = (\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, 0)$

$\overrightarrow{w_2} = \dfrac{\overrightarrow{v_2}}{\left\|\overrightarrow{v_2}\right\|} = \dfrac{(-\frac{1}{2},\frac{1}{2},0)}{\sqrt{\frac{1}{4}+\frac{1}{4}+0}}$

$= \sqrt{2}(-\dfrac{1}{2}, \dfrac{1}{2}, 0) = (\dfrac{-\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, 0)$

$\overrightarrow{w_3} = \dfrac{\overrightarrow{v_3}}{\left\|\overrightarrow{v_3}\right\|} = \dfrac{(0,0,2)}{\sqrt{0+0+4}} = (0, 0, 1)$

We obtained the orthonormal basis

$T = \{(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, 0), (\dfrac{-\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, 0), (0, 0, 1)\}.$

EXAMPLE 4 Verify **TH 6.17** for $\overrightarrow{v} = (2, -1, 3)$ and the orthonormal basis T obtained above.

$c_1 = \overrightarrow{v} \bullet \overrightarrow{w_1} = (2, -1, 3) \bullet (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) = \frac{\sqrt{2}}{2}$

$c_2 = \overrightarrow{v} \bullet \overrightarrow{w_2} = (2, -1, 3) \bullet (\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) = \frac{-3\sqrt{2}}{2}$

$c_3 = \overrightarrow{v} \bullet \overrightarrow{w_3} = (2, -1, 3) \bullet (0, 0, 1) = 3$

Verify:

$\frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) - \frac{3\sqrt{2}}{2}(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) + 3(0, 0, 1)$

$= (\frac{1}{2}, \frac{1}{2}, 0) + (\frac{3}{2}, \frac{-3}{2}, 0) + (0, 0, 3)$

$= (2, -1, 3)$ ✓