6.6 The Rank

$\rightarrow$ **DEF** p. 283

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.

- The rows of A:

$$\overrightarrow{v_1} = (a_{11}, a_{12}, \ldots, a_{1n})$$
$$\vdots$$
$$\overrightarrow{v_m} = (a_{m1}, a_{m2}, \ldots, a_{mn})$$

  span a subspace of R^n, called the **row space** of A.

- The columns of A:

$$\overrightarrow{w_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, \overrightarrow{w_n} = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

span a subspace of R^m, called the **column space** of A.

TH 6.10 $\rightarrow$ p. 284

If the matrices A and B are row equivalent then their row spaces are equal.

EXAMPLE 1 Find the basis for the row space of

$$A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ -3 & 0 & 7 & -1 \\ 3 & 4 & 1 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}.$$

A is row equivalent to $B = \begin{pmatrix} \boxed{1} & 3 & 1 & 3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

The row space of both B and A is $\text{span}\{(1,3,1,3),(0,1,2,0),(0,0,1,-1)\}$. (Keep nonzero rows of B)

$\rightarrow$ Procedure (p.285)

DEF $\rightarrow$ p.285

- **row rank** of A = dim(row space of A)
- **column rank** of A = dim(column space of A)

The row rank of A of **EXAMPLE 1** is **3**.
To find the column rank of A, use the Procedure on p.285:

$$\begin{pmatrix} \boxed{1} & 3 & 1 & 3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \downarrow & \downarrow & \downarrow & \\ 1 & 3 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ -3 & 0 & 7 & -1 \\ 3 & 4 & 1 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}$$

The first three columns of A span the column space of A. Therefore, the column rank of A is **3**.

Question: Are the row and column ranks of a matrix always the same?

Answer: YES

TH 6.11 $\rightarrow$ p. 287

$$\text{row rank of } A = \text{ column rank of } A$$

DEF $\rightarrow$ p.249

$$\text{rank} A = \text{row rank of } A = \text{ column rank of } A$$

TH 6.12 $\rightarrow$ p. 288

If A is an $m \times n$ matrix then

$$\text{rank } A + \text{nullity } A = n$$

Equivalent conditions ($\rightarrow$ p.291)

For any $n \times n$ matrix A, the following conditions are equivalent:

1. A is nonsingular.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. A is row equivalent to I_n.
4. For every $n \times 1$ matrix $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a unique solution.
5. $\det(A) \neq 0$.
6. rank $A = n$.
7. nullity $A = 0$.
8. The rows of A are linearly independent.
9. The columns of A are linearly independent.