6.5 Homogeneous Systems

$\rightarrow$ **EXAMPLE 6** from Lecture on 6.2 (Example 8 p. 246)

Let A be an $m \times n$ matrix.
$W = \{\overrightarrow{x} \mid A\overrightarrow{x} = \overrightarrow{0}_{R^m}\}$ is a subspace of R^n called

- the **solution space** of $A\overrightarrow{x} = \overrightarrow{0}_{R^m}$, or
- the **null space** of A.

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dim(null space of A) =nullity A

EXAMPLE 1

Find a basis for $W = \{ \overrightarrow{x} \mid A\overrightarrow{x} = \overrightarrow{0} \}$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 3 & 5 & 0 & 6 & 2 \\ 2 & 3 & 2 & 5 & 2 \end{pmatrix}.$$

The augmented matrix of the homogeneous system has the reduced row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3 & -1 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution:

$$
\begin{aligned}
x_4 &= s \text{ (arbitrary)} \\
x_5 &= t \text{ (arbitrary)} \\
x_1 &= 3s + t \\
x_2 &= -3s - t \\
x_3 &= -s - \frac{1}{2}t
\end{aligned}
$$

can be expressed in a vector form:

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3s + t \\ -3s - t \\ -s - \frac{1}{2}t \\ s \\ t \end{pmatrix}
$$

$$
= s \begin{pmatrix} 3 \\ -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}
$$

The vectors $\overrightarrow{v_1} = (3, -3, -1, 1, 0)$ and $\overrightarrow{v_2} = (1, -1, -\frac{1}{2}, 0, 1)$

- span W and
- are linearly independent.

Therefore, $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ forms a basis for W.
The nullity of A is 2.

$\rightarrow$ p. 277: Procedure for finding a basis for the solution space of a homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$.

- Solve the given system using Gauss-Jordan reduction.
- If the solution is unique

$$\overrightarrow{x} = \overrightarrow{0}$$

  then the solution space is $\{\overrightarrow{0}\}$ and has dimension 0.
- If some of the unknowns have arbitrary values $s_1, \ldots, s_p$ (since the corresponding columns in the reduced row echelon form do not contain leading entries), then express the solution as

$$\overrightarrow{x} = s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_p\overrightarrow{v_p}$$

  In this case, $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}\}$ forms a basis for the solution space. The space has dimension p.

$\rightarrow$ Example 1 p.278