6.4 Basis and Dimension

DEF ($\rightarrow$ p. 263)

A set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ of vectors in a vector space V is a **basis** for V if

(1) S spans V and

(2) S is linearly independent.

EXAMPLE 1 (→ **EXAMPLE 1** from the previous lecture)

Let $S = \{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\} = \{(1,0,0),(0,1,0),(0,0,1)\}$

(1) we've shown that S spans R^3

(2) $c_1 \overrightarrow{i} + c_2 \overrightarrow{j} + c_3 \overrightarrow{k} = \overrightarrow{0}$ corresponds to the homogeneous system with the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The solution is unique: $c_1 = c_2 = c_3 = 0$ (the trivial solution).

Answer: S is a basis for R^3.

→ Example 1 p. 263 ($n = 2,$ general n)

EXAMPLE 2

Is the set $S = \{(1,1),(1,-1)\}$ a basis for R^2?

(1) Does S span R^2?

Solve $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{array}{rcrcl} 1c_1 & + & 1c_2 & = & x \\ 1c_1 & - & 1c_2 & = & y \end{array}$$

$$\left(\begin{array}{rr|r} 1 & 1 & x \\ 1 & -1 & y \end{array}\right) \xrightarrow{r_2 - r_1 \to r_2} \left(\begin{array}{rr|r} 1 & 1 & x \\ 0 & -2 & y - x \end{array}\right)$$

This system is consistent for every x and y, therefore $\underline{S \text{ spans } R^2}$.

(2) Is S linearly independent?

Solve $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{array}{rcrcl} 1c_1 & + & 1c_2 & = & 0 \\ 1c_1 & - & 1c_2 & = & 0 \end{array}$$

$$\left(\begin{array}{rr|r} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right) \xrightarrow{r_2 - r_1 \to r_2} \left(\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array} \right)$$

The system has a unique solution $c_1 = c_2 = 0$ (trivial solution).

Therefore $\underline{S \text{ is linearly independent}}$.

Consequently, $\underline{S \text{ is a basis for } R^2}$.

EXAMPLE 3 Is $S = \{(1,2,3),(0,1,2),(-1,0,1)\}$ a basis for R^3?

It was already shown ($\rightarrow$ **EXAMPLE 3** from the previous lecture) that S does not span R^3.

Therefore S is not a basis for R^3.

EXAMPLE 4 Is $S = \{(1,0),(0,1),(-2,5)\}$ a basis for R^2?

It was already shown ($\rightarrow$ **EXAMPLE 4** from the previous lecture) that S is linearly dependent.

Therefore S is not a basis for R^2.

$\rightarrow$ Example 2 p.263.

EXAMPLE 5

$$S = \{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\overrightarrow{v_1}}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\overrightarrow{v_2}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\overrightarrow{v_3}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\overrightarrow{v_4}} \}$$

is a basis for the vector space M_{22}.

(1) $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + c_3\overrightarrow{v_3} + c_4\overrightarrow{v_4} = \overrightarrow{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

is equivalent to:

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is consistent for every $a, b, c,$ and d.
Therefore S spans M_{22}.

(2) $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + c_3\overrightarrow{v_3} + c_4\overrightarrow{v_4} = \overrightarrow{0}$
is equivalent to:

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The system has only the trivial solution $\Rightarrow$ S is linearly independent.

Consequently, S is a basis for M_{22}.

EXAMPLE 6 Is $S = \{1, t, t^2, t^3\}$ a basis for P_3?

(1) $c_1(1) + c_2(t) + c_3(t^2) + c_4(t^3)$
$$= a + bt + ct^2 + dt^3$$
has a solution for every $a, b, c,$ and d :
$c_1 = a, c_2 = b, c_3 = c, c_4 = d.$
Therefore S spans P_3.

(2) $c_1(1) + c_2(t) + c_3(t^2) + c_4(t^3) = 0$
can only be solved by
$c_1 = c_2 = c_3 = c_4 = 0.$
Therefore S is linearly independent.

Consequently, S is a basis for P_3.

$\rightarrow$ Example 3 p.264.

THEOREM (→ Th. 6.5 p. 265)

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ be a set of nonzero vectors in a vector space V. The following statements are equivalent:

(A) S is a basis for V,

(B) every vector in V can be expressed as a linear combination of the vectors in S in a **unique** way.

Proof (A)⇒(B)

- Every vector in V can be expressed as a linear combination of vectors in S because S spans V.
- Suppose $\overrightarrow{v}$ can be represented as a linear combination of vectors in S in **two ways**:

$$\overrightarrow{v} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}$$

$$\overrightarrow{v} = d_1\overrightarrow{v_1} + \cdots + d_k\overrightarrow{v_k}$$

Subtract:

$$\overrightarrow{0} = (c_1 - d_1)\overrightarrow{v_1} + \cdots + (c_k - d_k)\overrightarrow{v_k}$$

Since S is linearly independent, then

$$c_1 - d_1 = \cdots = c_k - d_k = 0$$

so that

$$c_1 = d_1$$
$$\vdots$$
$$c_k = d_k$$

The representation is **unique**.

Proof (B)⇒(A)

- (B)⇒ Every vector in *V* is in span *S*.
- **Zero vector** in *V* can be represented in a **unique** way as a linear combination of vectors in *S*:

$$\overrightarrow{0} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}$$

This unique way must be: $c_1 = \cdots = c_k = 0$.
Therefore *S* is linearly independent.

Consequently, *S* is a basis for *V*.

Back to **EXAMPLE 2**:

$S = \{(1,1),(1,-1)\}$

Instead of showing that

- $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} = \overrightarrow{v}$ has a solution, and
- $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} = \overrightarrow{0}$ has a unique solution,

we can show

- $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} = \overrightarrow{v}$ has a unique solution.

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \end{array}\right) \xrightarrow{r_2 - r_1 \to r_2} \left(\begin{array}{cc|c} 1 & 1 & x \\ 0 & -2 & y - x \end{array}\right)$$

Unique solution for every x and $y \Rightarrow S$ is a basis for R^2.

TH 6.6 ($\rightarrow$ p. 266)

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ be a set of nonzero vectors in a vector space V. Some subset of S is a basis for $W = \text{span}\ S$.

$\rightarrow$ Procedure p. 268

EXAMPLE 7 Find a basis for

$$\text{span}\{\underbrace{(1,2,3)}_{\vec{v_1}}, \underbrace{(-1,-2,-3)}_{\vec{v_2}}, \underbrace{(0,1,1)}_{\vec{v_3}}, \underbrace{(1,1,2)}_{\vec{v_4}}\}.$$

Set $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + c_3\overrightarrow{v_3} + c_4\overrightarrow{v_4} = \overrightarrow{0}$.

The corresponding system has augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 3 & -3 & 1 & 2 & 0 \end{array}\right)$$

which is equivalent
$(r_2 - 2r_1 \to r_2; r_3 - 3r_1 \to r_3; r_3 - r_2 \to r_3)$ to

$$\left(\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Can set c_2 and c_4 arbitrary. For example

- If $c_2 = 1, c_4 = 0$ then $\overrightarrow{v_2}$ can be expressed as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$.
- If $c_2 = 0, c_4 = 1$ then $\overrightarrow{v_4}$ can be expressed as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$.

Therefore, every vector in span S can be expressed as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$. Also note that $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$ are linearly independent. Consequently, they form a basis for span S.

Summarizing: The vectors corresponding to the columns with leading entries form a basis for W.

Different initial ordering of vectors, e.g., $\{\overrightarrow{v_2}, \overrightarrow{v_1}, \overrightarrow{v_3}, \overrightarrow{v_4}\}$ may change the basis obtained by the procedure above (in this case: $\overrightarrow{v_2}, \overrightarrow{v_3}$).

TH 6.7 (→ p. 269)

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ span V and let $T = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots \overrightarrow{w_n}\}$ be a linearly independent set of vectors in V. Then $n \leq k$.

COROLLARY 6.1 (→ p. 270)

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ and $T = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots \overrightarrow{w_n}\}$ both be bases for V. Then $n = k$.

DEF (→ p. 270)

The dimension of a vector space V, denoted dim V, is the number of vectors in a basis for V.

$\dim(\{\overrightarrow{0}\}) = 0.$

- $\dim(R^n) = n$ (→ Example 6 p. 270)
- $\dim(P_n) = n + 1$ (→ Example 7 p. 270)
- $\dim(M_{mn}) = mn$

TH 6.8 (→ p. 271)

If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there exists a basis T for V, which contains S.

EXAMPLE 8 (→ Example 9 p. 271)

Find a basis for R^4 that contains the vectors $\overrightarrow{v_1} = (1, 0, 1, 0)$ and $\overrightarrow{v_2} = (-1, 1, -1, 0)$.

Solution:

The natural basis for R^4:

$$\{\underbrace{(1,0,0,0)}_{\overrightarrow{e_1}}, \underbrace{(0,1,0,0)}_{\overrightarrow{e_2}}, \underbrace{(0,0,1,0)}_{\overrightarrow{e_3}}, \underbrace{(0,0,0,1)}_{\overrightarrow{e_4}}\}$$

Follow the procedure of **EXAMPLE 7** to determine a basis of span$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}\}$.

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

has the reduced row echelon form:

$$\left(\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

Answer: $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{e_1}, \overrightarrow{e_4}\}$.

TH 6.9 (→ p. 272)

Let V be an n-dimensional vector space, and let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_n}\}$ be a set of n vectors in V.

(a) If S is linearly independent then it is a basis for V.

(b) If S spans V then it is a basis for V.