6.3 Linear Independence

DEF (→ p. 253)

Let \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \) be a set of vectors in a vector space \( V \). We say that \( S \) spans \( V \) (or \( V \) is spanned by \( S \)) if every vector in \( V \) is a linear combination of vectors in \( S \).

To check if \( S \) spans \( V \)

1. choose an arbitrary vector \( \mathbf{v} \) in \( V \),
2. check if \( \mathbf{v} \) is in span \( S \). If so, \( S \) spans \( V \). Otherwise it does not.
EXAMPLE 1

Does the set \( S = \{(1,0,0), (0,1,0), (0,0,1)\} \) span \( \mathbb{R}^3 \)?

Let \( \vec{v} = (x,y,z) \) - arbitrary vector in \( \mathbb{R}^3 \).

Can we find \( c_1, c_2, c_3 \) such that
\[
\begin{align*}
c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{align*}
\]

Solution: \( c_1 = x, c_2 = y, c_3 = z \).

Answer: \( S \) spans \( \mathbb{R}^3 \).
EXAMPLE 2

Does the set \( S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\} \) span \( R^3 \)?

Solve

\[
\begin{align*}
1c_1 + 2c_2 + 3c_3 &= (1, 2, 3) \\
0c_1 + 1c_2 + 2c_3 &= (0, 1, 2) \\
-2c_1 + 0c_2 + 1c_3 &= (-2, 0, 1)
\end{align*}
\]

\[
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}
+
\begin{pmatrix}
c_2 \\
1c_2 + 0c_3
\end{pmatrix}
+
\begin{pmatrix}
c_3 \\
0c_1 + 2c_2 + 1c_3
\end{pmatrix}
= \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

\[
\begin{align*}
1c_1 &+ 0c_2 - 2c_3 = x \\
2c_1 &+ 1c_2 + 0c_3 = y \\
3c_1 &+ 2c_2 + 1c_3 = z
\end{align*}
\]
\[
\begin{pmatrix}
1 & 0 & -2 & | & x \\
2 & 1 & 0 & | & y \\
3 & 2 & 1 & | & z
\end{pmatrix}
\]

\[
r_2 - 2r_1 \rightarrow r_2 \\
r_3 - 3r_1 \rightarrow r_3
\]

\[
\begin{pmatrix}
1 & 0 & -2 & | & x \\
0 & 1 & 4 & | & y - 2x \\
0 & 2 & 7 & | & z - 3x
\end{pmatrix}
\]

\[
r_3 - 2r_2 \rightarrow r_3
\]

\[
\begin{pmatrix}
1 & 0 & -2 & | & x \\
0 & 1 & 4 & | & y - 2x \\
0 & 0 & -1 & | & z - 2y + x
\end{pmatrix}
\]

This system is consistent for every \(x, y,\) and \(z,\) therefore \(S\) spans \(R^3\).
EXAMPLE 3 Does the set 
\[ S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\} \] span \( \mathbb{R}^3 \)?

Setting up a system, we get the augmented matrix:
\[
\begin{pmatrix}
1 & 0 & -1 & | & x \\
2 & 1 & 0 & | & y \\
3 & 2 & 1 & | & z
\end{pmatrix}
\]

with the reduced row echelon form:
\[
\begin{pmatrix}
1 & 0 & -1 & | & 2y - z \\
0 & 1 & 2 & | & -3y + 2z \\
0 & 0 & 0 & | & x - 2y + z
\end{pmatrix}
\]

There exist values of \( x, y, \) and \( z \) for which the system has no solution (e.g., \( x = 1, y = z = 0 \)). Consequently, \( S \) does not span \( \mathbb{R}^3 \).

Also, recall that EXAMPLE 8 from the lecture on the previous section showed that \((1, 2, -1)\) is not in span\{\((1, 2, 3), (0, 1, 2), (-1, 0, 1)\)\}. 
**DEF** Linear Independence (→ p. 256)

$S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \}$ is **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_k$, not all of which are zero, such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0} \quad (*)$$

Otherwise (i.e., if the only way for $(*)$ to hold is if $c_1 = c_2 = \cdots = c_k = 0$) we say $S$ is **linearly independent**.

To check if $S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \}$ is linearly independent, begin by setting up the homogeneous system corresponding to $(*)$

- if the system has the **trivial solution only** then $S$ is **linearly independent**,  
- if the system has **many solutions** then $S$ is **linearly dependent**.
**EXAMPLE 4** Determine whether the set

\[ S = \{(1, 0), (0, 1), (-2, 5)\} \] is linearly dependent or linearly independent.

\[
c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

can be rewritten as the homogeneous system

\[
\begin{align*}
1c_1 & + 0c_2 & - 2c_3 & = 0 \\
0c_1 & + 1c_2 & + 5c_3 & = 0
\end{align*}
\]

with the augmented matrix

\[
\begin{pmatrix}
1 & 0 & -2 & | & 0 \\
0 & 1 & 5 & | & 0
\end{pmatrix}
\]

The system has many solutions \( \Rightarrow S \) is **linearly dependent**.
EXAMPLE 5 Determine whether the set
\[ S = \{(1,0,-1,0),(1,1,0,2),(0,3,1,-2),(0,1,-1,2)\} \]
is linearly dependent or linearly independent.

The homogeneous system has the augmented matrix
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & | & 0 \\
0 & 1 & 3 & 1 & | & 0 \\
-1 & 0 & 1 & -1 & | & 0 \\
0 & 2 & -2 & 2 & | & 0
\end{pmatrix}
\]
with a row echelon form:
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & | & 0 \\
0 & 1 & 3 & 1 & | & 0 \\
0 & 0 & 1 & 1 & | & 0 \\
0 & 0 & 0 & 1 & | & 0
\end{pmatrix}
\]
The system has only the trivial solution \( \Rightarrow S \) is 
linearly independent.
Examples 7, 8, 9, 10, 11 p. 256-258

Example 12 p. 258: Every set of vectors containing the zero vector is linearly dependent.

(e.g., \( c_1 \overrightarrow{v_1} + c_2 \overrightarrow{v_2} + \cdots + c_{k-1} \overrightarrow{v_{k-1}} + c_k \overrightarrow{0} = \overrightarrow{0} \) can be solved by \( c_1 = \cdots = c_{k-1} = 0 \) and \( c_k \) being any nonzero number)

Discussion under Example 12 p.258:
Let \( S_1 \) and \( S_2 \) be sets of vectors in a vector space \( V \) such that \( S_1 \subseteq S_2 \) (\( S_1 \) is a subset of \( S_2 \)). Then
- \( S_1 \) is linearly dependent \( \Rightarrow \) \( S_2 \) is linearly dependent,
- \( S_2 \) is linearly independent \( \Rightarrow \) \( S_1 \) is linearly independent.

**TH 6.4** (→ p. 259)
Nonzero vectors \( \overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k} \) in a vector space \( V \) are linearly dependent if and only if at least one of the vectors, \( \overrightarrow{v_j} \) can be expressed as a linear combination of the preceding vectors \( \overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_{j-1}} \).