6.3 Linear Independence

$\boxed{\textbf{DEF}}$ ($\to$ p. 253)

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ be a set of vectors in a vector space V. We say that S spans V (or V is spanned by S) if every vector in V is a linear combination of vectors in S.

To check if S spans V

- choose an arbitrary vector $\overrightarrow{v}$ in V,
- check if $\overrightarrow{v}$ is in span S. If so, S spans V. Otherwise it does not.

$\boxed{\texttt{EXAMPLE 1}}$ ($\rightarrow$ Example 4 p. 255)

Does the set $S = \{\underbrace{(1,0,0)}_{\overrightarrow{i}}, \underbrace{(0,1,0)}_{\overrightarrow{j}}, \underbrace{(0,0,1)}_{\overrightarrow{k}}\}$ span R^3?

Let $\overrightarrow{v} = (x,y,z)$ - arbitrary vector in R^3.

Can we find c_1, c_2, c_3 such that $c_1\overrightarrow{i} + c_2\overrightarrow{j} + c_3\overrightarrow{k} = \overrightarrow{v}$?

$$c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solution: $c_1 = x, c_2 = y, c_3 = z$.

Answer: S spans R^3.

EXAMPLE 2

Does the set $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ span R^3?

Solve

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{array}{rcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & = & x \\ 2c_1 & + & 1c_2 & + & 0c_3 & = & y \\ 3c_1 & + & 2c_2 & + & 1c_3 & = & z \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & x \\ 2 & 1 & 0 & y \\ 3 & 2 & 1 & z \end{array}\right)$$

$$\xrightarrow[r_3-3r_1\to r_3]{r_2-2r_1\to r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & x \\ 0 & 1 & 4 & y-2x \\ 0 & 2 & 7 & z-3x \end{array}\right)$$

$$\xrightarrow{r_3-2r_2\to r_3} \left(\begin{array}{ccc|c} 1 & 0 & -2 & x \\ 0 & 1 & 4 & y-2x \\ 0 & 0 & -1 & z-2y+x \end{array}\right)$$

This system is consistent for every $x, y,$ and z, therefore S spans R^3.

$\boxed{\textbf{EXAMPLE 3}}$ Does the set
$S = \{(1,2,3),(0,1,2),(-1,0,1)\}$ span R^3?
Setting up a system, we get the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 2 & 1 & 0 & y \\ 3 & 2 & 1 & z \end{array}\right)$$

with the reduced row echelon form:

$$\left(\begin{array}{ccc|r} 1 & 0 & -1 & 2y - z \\ 0 & 1 & 2 & -3y + 2z \\ 0 & 0 & 0 & x - 2y + z \end{array}\right)$$

There exist values of $x, y,$ and z for which the system has no solution (e.g., $x = 1, y = z = 0$). Consequently, S does not span R^3.

Also, recall that **EXAMPLE 8** from the lecture on the previous section showed that $(1,2,-1)$ is not in $\text{span}\{(1,2,3),(0,1,2),(-1,0,1)\}$.

$\boxed{\textbf{DEF}}$ **Linear Independence** (→ p. 256)

$S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ is **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_k$, not all of which are zero, such that

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_k\overrightarrow{v_k} = \overrightarrow{0} \qquad (*)$$

Otherwise (i.e., if the only way for (*) to hold is if $c_1 = c_2 = \cdots = c_k = 0$) we say S is **linearly independent**.

To check if $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ is linearly independent, begin by setting up the homogeneous system corresponding to (*)

- if the system has the **trivial solution only** then S is **linearly independent**,
- if the system has **many solutions** then S is **linearly dependent**.

$\boxed{\texttt{EXAMPLE 4}}$ Determine whether the set $S = \{(1,0),(0,1),(-2,5)\}$ is linearly dependent or linearly independent.

$$c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

can be rewritten as the homogeneous system

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & - & 2c_3 & = & 0 \\ 0c_1 & + & 1c_2 & + & 5c_3 & = & 0 \end{array}$$

wih the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right)$$

The system has many solutions $\Rightarrow$ S is **linearly dependent**.

$\boxed{\textbf{EXAMPLE 5}}$ Determine whether the set $S = \{(1,0,-1,0),(1,1,0,2),(0,3,1,-2),(0,1,-1,2)\}$ is linearly dependent or linearly independent.

The homogeneous system has the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right)$$

with a row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

The system has only the trivial solution $\Rightarrow S$ is **linearly independent**.

$\rightarrow$ Examples 7, 8, 9, 10, 11 p. 256-258

$\rightarrow$ Example 12 p. 258: Every set of vectors containing the zero vector is linearly dependent.

(e.g., $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_{k-1}\overrightarrow{v_{k-1}} + c_k\overrightarrow{0} = \overrightarrow{0}$ can be solved by $c_1 = \cdots = c_{k-1} = 0$ and c_k being any **nonzero** number)

$\rightarrow$ Discussion under Example 12 p.258:
Let S_1 and S_2 be sets of vectors in a vector space V such that $S_1 \subseteq S_2$ (S_1 is a subset of S_2). Then

- $S_1$is linearly dependent $\Rightarrow$ S_2 is linearly dependent,
- $S_2$is linearly independent $\Rightarrow$ S_1 is linearly independent.

$\boxed{\textbf{TH 6.4}}$ ($\rightarrow$ p. 259)

Nonzero vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}$ in a vector space V are linearly dependent if and only if at least one of the vectors, $\overrightarrow{v_j}$ can be expressed as a linear combination of the preceding vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_{j-1}}$.