6.2 Subspaces

$\boxed{\texttt{DEF}}$ **Subspace** (→ p. 244)

If

- *W* is a nonempty subset of a vector space *V* and

- *W* is a vector space with respect to the operations in *V*,

then *W* is called a **subspace** of *V*.

$\boxed{\texttt{EXAMPLE 1}}$ (→Example 1 p. 245)

- Every vector space *V* contains a zero vector $\vec{0}$. The set $\{\vec{0}\}$ forms a subspace of *V*.

- *V* is a subspace of itself.

$\boxed{\textbf{EXAMPLE 2}}$ ($\rightarrow$ **EXAMPLE 2** from the lecture on Section 6.1)

V - set of all ordered pairs of numbers of the form $(x, -x)$
$\oplus$ and $\odot$ - the usual operations in R^2.

V is a nonempty (e.g. $(1, -1)$) subset of R^2.

V satisfies all ten properties of a vector space (α), (β), (a)-(h).

Therefore, V is a subspace of R^2.

$\boxed{\textbf{EXAMPLE 3}}$ ($\rightarrow$ Problem 1 p.243)

$V = \{(x, y) | \ x > 0, y > 0\}$.
$\oplus$ and $\odot$ - the usual operations in R^2

V is not closed under $\odot$, therefore it is not a vector space.

Consequently, V is not a subspace of R^2.

$\boxed{\textbf{THEOREM 6.2}}$ $\rightarrow$ p. 245

If W is a nonempty subset of a vector space V with operations $\oplus$ and $\odot$,

then W is a subspace of V if and only if W is closed under both $\oplus$ and $\odot$.

$\boxed{\textbf{EXAMPLE 4}}$ Let V be the set of pairs of numbers (x, y) such that $x = 2y$ with the usual operations in R^2. Determine whether V is a subspace of R^2.

- V is nonempty (e.g., (2,1))
- V is a subset of R^2.
- Let $\overrightarrow{u} = (x, y)$ and $\overrightarrow{v} = (x', y')$ be in V. Therefore, $x = 2y$ and $x' = 2y'$ so that we can write: $\overrightarrow{u} = (2y, y)$ and $\overrightarrow{v} = (2y', y')$.
  $\overrightarrow{u} + \overrightarrow{v} = (2y + 2y', y + y')$ satisfies the condition $2y + 2y' = 2(y + y')$ therefore $\overrightarrow{u} + \overrightarrow{v}$ is in V.
  Consequently, V is closed under the operation of vector addition.
- Let $\overrightarrow{u} = (x, y) = (2y, y)$ and c be a real number.
  $c\overrightarrow{u} = ((c)(2y), cy)$ satisfies the condition $(c)(2y) = 2(cy)$ therefore $c\overrightarrow{u}$ is in V.
  Consequently, V is closed under the operation of scalar multiplication.

Based on Theorem 6.2, we can conclude that V is a subspace of R^2.

$\boxed{\textbf{EXAMPLE 5}}$ Let V be the set of pairs of numbers (x,y) such that $x = 2+y$ with the usual operations in R^2. Determine whether V is a subspace of R^2.

- V is nonempty (e.g., (3,1))
- V is a subset of R^2.
- Let $\overrightarrow{u} = (x,y)$ and $\overrightarrow{v} = (x',y')$ be in V. Therefore, $x = 2+y$ and $x' = 2+y'$ so that we can write: $\overrightarrow{u} = (2+y,y)$ and $\overrightarrow{v} = (2+y',y')$.
  $\overrightarrow{u} + \overrightarrow{v} = (2+y+2+y',y+y')$ does not satisfy the condition $2+y+2+y' = 2+(y+y')$ therefore $\overrightarrow{u} + \overrightarrow{v}$ is not in V.
  Consequently, V is not closed under the operation of vector addition.

We can conclude that V is not a subspace of R^2.

$\rightarrow$ HW Problems 1-4 p. 250

EXAMPLE 6 (→Example 8 p. 246)

Let A be an $m \times n$ matrix. Define

$W = \{ \vec{x} \mid A\vec{x} = \vec{0}_{R^m} \}$ (solution set of the homogeneous system $A\vec{x} = \vec{0}_{R^m}$.)

- W is nonempty:

  The trivial solution, $\vec{0}_{R^n} \in W$.
- W is a subset of R^n.

(α) Let $\vec{u}$ and $\vec{v}$ be two elements of W, i.e.,

$A\vec{u} = A\vec{v} = \vec{0}_{R^m}$.

Their sum, $\vec{u} + \vec{v}$, satisfies:

$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0}_{R^m} + \vec{0}_{R^m} = \vec{0}_{R^m}$

therefore $\vec{u} + \vec{v} \in W$.

(β) Let c be a real number, and let $\vec{u}$ be an element of W, i.e., $A\vec{u} = \vec{0}_{R^m}$. The scalar multiple $c\vec{u}$ satisfies

$A(c\vec{u}) = c(A\vec{u}) = c(\vec{0}_{R^m}) = \vec{0}_{R^m}$.

W is a subspace of R^n. W is called the **solution space** of $A\vec{x} = \vec{0}_{R^m}$, or the **null space** of A.

$\boxed{\textbf{DEF}}$ **Linear Combination** (→ p. 247)

If $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}$ are vectors in a vector space V and $c_1, c_2, \ldots, c_k$ are real numbers then

$$\overrightarrow{v} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_k\overrightarrow{v_k}$$

is a vector in V called a **linear combination** of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}$.

$\boxed{\textbf{EXAMPLE 7}}$ Is $(1,3,1)$ a linear combination of $(0,1,2)$ and $(1,0,-5)$?

Are there real numbers c_1 and c_2 such that

$$c_1\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}?$$

This equation can be rewritten as a linear system:

$$\begin{array}{rcrcr} 0c_1 & + & 1c_2 & = & 1 \\ 1c_1 & + & 0c_2 & = & 3 \\ 2c_1 & - & 5c_2 & = & 1 \end{array}$$

which can be solved by Gauss-Jordan reduction

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & -5 & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 2 & -5 & 1 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_1 \to r_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \end{pmatrix}$$

$$\xrightarrow{r_3 + 5r_2 \to r_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: $c_1 = 3, c_2 = 1$.
Answer: $(1, 3, 1)$ **is** a linear combination of $(0, 1, 2)$ and $(1, 0, -5)$.

Check: $3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ ✓

EXAMPLE 8 Is $(1,2,-1)$ a linear combination of $(1,2,3),(0,1,2),$ and $(-1,0,1)$?

Are there real numbers $c_1, c_2,$ and c_3 such that

$$c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}?$$

This equation can be rewritten as a linear system:

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & - & 1c_1 & = & 1 \\ 2c_1 & + & 1c_2 & + & 0c_3 & = & 2 \\ 3c_1 & + & 2c_2 & + & 1c_3 & = & -1 \end{array}$$

which can be solved by Gauss-Jordan reduction

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & -2 \end{pmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 2 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow{r_3 - 3r_1 \to r_3} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 3 & -5 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2 \to r_3} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

The system has no solution.

Answer: $(1, 2, -1)$ **is not** a linear combination of $(1, 2, 3), (0, 1, 2),$ and $(-1, 0, 1)$.

$\boxed{\textbf{DEF}}$ **Span** ($\rightarrow$ p. 248)

If $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}$ is a set of vectors in a vector space $V,$ then the **set of all linear combinations** of the vectors in S is denoted by

$$\text{span } S = \text{span } \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}$$

Instead of asking:

is v a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}$?

we can ask:

is v in the span$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}$?

In the two recent examples:

- **EXAMPLE 7:** $(1, 3, 1)$ is in the span$\{(0, 1, 2), (1, 0, -5)\}$
- **EXAMPLE 8:** $(1, 2, -1)$ is not in the span$\{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$

$\boxed{\texttt{TH 6.3}}$ ($\rightarrow$ p. 249)

If S is a set of vectors in a vector space V then span S is a subspace of V.

Read the following Examples in the text:

$\rightarrow$Example 11 p. 248

$$\text{span}\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

$$= \{\begin{pmatrix} a & b & 0 \\ 0 & c & d \end{pmatrix} \mid a, b, c, d \in R\}$$

is a subspace of M_{23}

$\rightarrow$Example 12 p. 249

$\text{span}\{2t^2 + t + 2, t^2 - 2t, 5t^2 - 5t + 2, -t^2 - 3t - 2\}$
is a subspace of P_2.