6.1 Vector Spaces

$\boxed{\textbf{DEF}}$ **Vector Space** (→ p. 238)

A set V with operations $\oplus$ and $\odot$ such that:

(α) $\vec{u} \oplus \vec{v}$ is in V for all $\vec{u}$ and $\vec{v}$ in V.

- **(a)** $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ for all $\vec{u}$ and $\vec{v}$ in V
- **(b)** $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$ for all $\vec{u}, \vec{v},$ and $\vec{w}$ in V
- **(c)** There exists $\vec{0}$ in V such that $\vec{u} \oplus \vec{0} = \vec{0} \oplus \vec{u} = \vec{u}$ for all $\vec{u}$ in V
- **(d)** For every $\vec{u}$ in V there exists $-\vec{u}$ in V such that $\vec{u} \oplus -\vec{u} = \vec{0}$

(β) $c \odot \vec{u}$ is in V for all $\vec{u}$ in V and numbers c.

- **(e)** $c \odot (\vec{u} \oplus \vec{v}) = (c \odot \vec{u}) \oplus (c \odot \vec{v})$ for all c in R and $\vec{u}$ and $\vec{v}$ in V.
- **(f)** $(c + d) \odot \vec{u} = (c \odot \vec{u}) \oplus (d \odot \vec{u})$ for all c and d in R and $\vec{u}$ in V.
- **(g)** $c \odot (d \odot \vec{u}) = (cd) \odot \vec{u}$ for all c and d in R and $\vec{u}$ in V.
- **(h)** $1 \odot \vec{u} = \vec{u}$ for all $\vec{u}$ in V.

We say that V is **closed under the operation** $\oplus$ if it satisfies property (α).
Likewise, V is **closed under** $\odot$ if it satisfies (β).

$\boxed{\texttt{EXAMPLE 1}}$ ($\rightarrow$ Example 1 p.239)

$V = R^n$ with

$\oplus$ - the usual operation of vector addition

$\odot$ - the usual operation of scalar multiplication

satisfies properties (α), (a)-(d), (β), (e)-(h).

($\rightarrow$ Theorem 4.2 p.186)

Therefore it is a vector space.

EXAMPLE 2 Let V be a set of all ordered pairs of numbers of the form $(x, -x)$, and let $\oplus$ and $\odot$ be the usual operations in R^2.

Verify (α): taking arbitrary vectors in V:

$\overrightarrow{u} = (x, -x)$ and $\overrightarrow{v} = (x', -x')$, we have

$\overrightarrow{u} \oplus \overrightarrow{v} = (x, -x) \oplus (x', -x') = (x + x', -x - x')$

Since the second element is the negative of the first, the vector is in V .

Property holds (V is closed under $\oplus$.)

Verify (a):

$LHS = \overrightarrow{u} \oplus \overrightarrow{v} = (x, -x) \oplus (x', -x')$

$= (x + x', -x - x')$

$RHS = \overrightarrow{v} \oplus \overrightarrow{u} = (x', -x') \oplus (x, -x)$

$= (x' + x, -x' - x)$

$LHS = RHS$ for all $\overrightarrow{u}$ and $\overrightarrow{v}$ in V.

Property holds.

The other properties can be shown to hold, too.

V with $\oplus$ and $\odot$ is a vector space.

$\boxed{\textbf{EXAMPLE 3}}$ Let V be a set of all ordered pairs of numbers of the form (x, y), and let $\oplus$ and $\odot$ be defined as follows

$(x, y) \oplus (x', y') = (x + x', y')$ and

$c \odot (x, y) = (cx, cy)$.

Obviously (α) holds.

Verify (a):

$LHS = \overrightarrow{u} \oplus \overrightarrow{v} = (x, y) \oplus (x', y')$

$= (x + x', y')$

$RHS = \overrightarrow{v} \oplus \overrightarrow{u} = (x', y') \oplus (x, y)$

$= (x' + x, y)$

Generally, $LHS \neq RHS$.

Property <u>does not hold</u>.

Consequently,

<u>V with $\oplus$ and $\odot$ is not a vector space.</u>

Examples of vector spaces from the Text:

$\longrightarrow$ Example 4 p. 240
$V = M_{23}$ - set of all 2×3 matrices
$\oplus, \odot$ - the usual operations of matrix addition and scalar multiplication

$\longrightarrow$ Example 5 p. 240
V - set of all real-valued functions defined on $[a, b]$ with $\oplus$ and $\odot$ defined as follows:

$$(f \oplus g)(x) = f(x) \oplus g(x)$$

$$(c \odot f)(x) = cf(x)$$

$\longrightarrow$ Example 8 p. 241
$V = P_n$ - set of all polynomials of degree n or less with $\oplus$ and $\odot$ defined as above.

Note that a set of all polynomials of degree exactly n is <u>not</u> a vector space.

THEOREM 6.1 → p. 242

Let V be a vector space.

(a) $0\vec{u} = \vec{0}$ for every $\vec{u}$ in V.

(b) $c\vec{0} = \vec{0}$ for every c in R.

(c) If $c\vec{u} = \vec{0}$ then $c = 0$ or $\vec{u} = \vec{0}$.

(d) $(-1)\vec{u} = -\vec{u}$ for every $\vec{u}$ in V.