4.2 n-Vectors

- Definition of an n**-vector** ($\rightarrow$ p. 10): an $n \times 1$ or $1 \times n$ matrix, e.g.,

$$\overrightarrow{u} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \overrightarrow{v} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- $a_1, \ldots a_n$: **components** of $\overrightarrow{u}$
- Set of all n-vectors is denoted by R^n.
- $\overrightarrow{u}$ and $\overrightarrow{v}$ are **equal** if $u_i = v_i$ for all $i = 1, \ldots, n$.

- **Sum** ($\rightarrow$ p. 185)

$$\overrightarrow{u} + \overrightarrow{v} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

- **Scalar multiple** ($\rightarrow$ p. 186)

$$c\,\overrightarrow{u} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}$$

$\boxed{\textbf{TH 4.2}}$ ($\rightarrow$ p. 186) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^n. Also, let c and d be scalars (real numbers). Then

(α) $\vec{u} + \vec{v}$ is in R^n.

- (**a**) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- (**b**) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
- (**c**) There exists $\vec{0}$ in R^n such that $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$.
- (**d**) For every $\vec{u}$ in R^n there exists $-\vec{u}$ in R^n such that $\vec{u} + (-\vec{u}) = \vec{0}$.

(β) $c\vec{u}$ is in R^n .

- (**e**) $c(\vec{u} + \vec{v}) = (c\vec{u}) + (c\vec{v})$.
- (**f**) $(c + d)\vec{u} = (c\vec{u}) + (d\vec{u})$.
- (**g**) $c(d\vec{u}) = (cd)\vec{u}$.
- (**h**) $1\vec{u} = \vec{u}$.

- **Length** ($\rightarrow$ p. 190)

$$\left\| \overrightarrow{u} \right\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

- **Dot product** ($\rightarrow$ p. 17)

$$\overrightarrow{u} \bullet \overrightarrow{v} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$\boxed{\texttt{TH 3.4}}$ ($\rightarrow$ p. 192) Properties of Dot Product

- $\overrightarrow{u} \bullet \overrightarrow{u} \geq 0$; $\overrightarrow{u} \bullet \overrightarrow{u} = 0 \Leftrightarrow \overrightarrow{u} = \overrightarrow{0}$
- $\overrightarrow{u} \bullet \overrightarrow{v} = \overrightarrow{v} \bullet \overrightarrow{u}$
- $(\overrightarrow{u} + \overrightarrow{v}) \bullet \overrightarrow{w} = \overrightarrow{u} \bullet \overrightarrow{w} + \overrightarrow{v} \bullet \overrightarrow{w}$
- $(c\overrightarrow{u}) \bullet \overrightarrow{v} = \overrightarrow{u} \bullet (c\overrightarrow{v}) = c(\overrightarrow{u} \bullet \overrightarrow{v})$

$\boxed{\texttt{TH 4.4}}$ ($\rightarrow$ p. 192) **Cauchy-Schwarz Inequality**

If $\overrightarrow{u}$ and $\overrightarrow{v}$ are vectors in R^n then

$$\left|\overrightarrow{u}\bullet\overrightarrow{v}\right| \leq \left\|\overrightarrow{u}\right\|\left\|\overrightarrow{v}\right\|$$

Outline of the Proof

If $\overrightarrow{u} = \overrightarrow{0}$ then $\left\|\overrightarrow{u}\right\| = 0$ therefore $\overrightarrow{u}\bullet\overrightarrow{v} = 0$.

The inequality is satisfied.

Assume $\overrightarrow{u} \neq \overrightarrow{0}$, and let r be a scalar. Then

$$\begin{aligned} 0 &\leq (r\overrightarrow{u}+\overrightarrow{v})\bullet(r\overrightarrow{u}+\overrightarrow{v}) \\ &= (r\overrightarrow{u})\bullet(r\overrightarrow{u}) + (r\overrightarrow{u})\bullet\overrightarrow{v} + \overrightarrow{v}\bullet(r\overrightarrow{u}) + \overrightarrow{v}\bullet\overrightarrow{v} \\ &= r^2\,\underbrace{(\overrightarrow{u}\bullet\overrightarrow{u})}_{a} + r\,\underbrace{(2(\overrightarrow{u}\bullet\overrightarrow{v}))}_{b} + \underbrace{\overrightarrow{v}\bullet\overrightarrow{v}}_{c} \end{aligned}$$

$p(r) = ar^2 + br + c$ is nonnegative for all r,

therefore $b^2 - 4ac \leq 0$.

- **Angle** θ between $\vec{u}$ and $\vec{v}$ ($\rightarrow$ p. 193)

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$\boxed{\textbf{DEF}}$ ($\rightarrow$ p. 194)

- $\vec{u}$ and $\vec{v}$ are **orthogonal** if
$$\vec{u} \bullet \vec{v} = 0,$$
- $\vec{u}$ and $\vec{v}$ are **parallel** if
$$|\vec{u} \bullet \vec{v}| = \|\vec{u}\| \, \|\vec{v}\|,$$
- $\vec{u}$ and $\vec{v}$ are **in the same direction** if
$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \, \|\vec{v}\|.$$

$\boxed{\textbf{TH 4.5}}$ (→ p. 194) **Triangle Inequality**

If $\vec{u}$ and $\vec{v}$ are vectors in R^n then

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

Outline of the proof:

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v}) \\ &= \|\vec{u}\|^2 + 2(\vec{u}\bullet\vec{v}) + \|\vec{v}\|^2\end{aligned}$$

Apply Cauchy-Schwarz Inequality (Th 4.4) to conclude the proof.

$\boxed{\textbf{DEF}}$ (→ p. 195) **Unit vector** in the direction of $\vec{u}$

$$\frac{1}{\|\vec{u}\|}\vec{u}$$