3.1 Determinants - Definition and Properties

($\rightarrow$ p. 140, 141)

- A sequence $j_1, j_2, \ldots, j_n$ made up of (rearranged) integers from the set $S_n = \{1, 2, \ldots, n\}$ is called a **permutation** of S_n.
- There are $n!$ permutations of S_n.
- **Inversion**: $j_r > j_s$ when $r < s$.
- A permutation is called **odd** (**even**) if it has an odd (even) number of inversions.

$\boxed{\textbf{EXAMPLES}}$ $n = 2$

$j_1 j_2$	# of inversions	
12	0	even
21	1	odd

$n = 3$

$j_1 j_2 j_3$	# of inversions	
123	0	even
132	1	odd
213	1	odd
231	2	even
312	2	even
321	3	odd

$\boxed{\textbf{DEF}}$ (→ p. 141)

The determinant of an $n \times n$ matrix A is

$$\det(A) = |A| = \sum_{\substack{\text{for all} \\ \text{permutations} \\ \text{of } S_n}} (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where the sign of each term is determined as follows:

- for an even permutation, the sign is +, and
- for an odd permutation, the sign is –.

$\boxed{\textbf{EXAMPLE}}$ $n = 2 : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|A| = +a_{11}a_{22} - a_{12}a_{21}$$

e.g., $\begin{vmatrix} -3 & -1 \\ 3 & 2 \end{vmatrix} = (-3)(2) - (-1)(3) = -3$

$\boxed{\textbf{EXAMPLE}}$ $n = 3 : A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\begin{aligned} |A| &= +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

e.g.,

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned} |A| &= (-1)(2)(1) + (1)(1)(1) + (0)(1)(-1) \\ &\quad - (0)(2)(1) - (-1)(1)(-1) - (1)(1)(1) \\ &= -3 \end{aligned}$$

Property (→p.143)

If we interchange two numbers in the permutation $j_1, j_2, \ldots, j_n$ then the number of inversions is either increased or decreased by an *odd* number.

$\boxed{\textbf{TH 3.1}}$ (→p.143) $|A^T| = |A|$

$\boxed{\textbf{TH 3.2}}$ (→p.144)

If B is obtained from A by interchanging two rows (or columns) then

$$|B| = -|A|$$

$\boxed{\textbf{TH 3.3}}$ ($\rightarrow$p.145)

If two rows (or columns) of A are equal then

$$|A| = 0$$

$\boxed{\textbf{TH 3.4}}$ ($\rightarrow$p.145)

If A contains a zero row (or zero column) then

$$|A| = 0$$

$\boxed{\textbf{TH 3.5}}$ ($\rightarrow$p.145)

If B is obtained from A by multiplying a row (or column) by a number c then

$$|B| = c|A|$$

$\boxed{\textbf{TH 3.6}}$ ($\to$p.146)

If B is obtained from A by adding a multiple of one row (or column) to another then

$$|B| = |A|$$

$\boxed{\textbf{TH 3.7}}$ ($\to$p.146)

If A is an $n \times n$ upper triangular (or lower triangular) matrix then

$$|A| = a_{11}a_{22}\cdots a_{nn}$$

$\boxed{\textbf{EXAMPLE}}$ Find the determinant of

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & -2 \\ 3 & -1 & 1 \end{pmatrix}$$

Transform A to B by $r_1 \leftrightarrow r_2$.

TH 3.2 $\Rightarrow \underline{\det(B) = -\det(A)}$.

$$B = \begin{pmatrix} 2 & 3 & -2 \\ 0 & 1 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

Transform B to C by $\frac{1}{2}r_1 \rightarrow r_1$.

TH 3.5 $\Rightarrow \underline{\det(C) = \frac{1}{2}\det(B)}$.

$$C = \begin{pmatrix} 1 & \frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

Transform C to D by $r_3 - 3r_1 \rightarrow r_3$.

TH 3.6 $\Rightarrow \underline{\det(D) = \det(C)}$.

$$D = \begin{pmatrix} 1 & \frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & -\frac{11}{2} & 4 \end{pmatrix}$$

Transform D to E by $r_3 + \frac{11}{2} r_2 \to r_3$.

TH 3.6 $\Rightarrow$ $\underline{\det(E) = \det(D)}$.

$$E = \begin{pmatrix} 1 & \frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 15 \end{pmatrix}$$

Matrix E is upper triangular

TH 3.7 $\Rightarrow$ $\underline{\det(E) = (1)(1)(15) = 15}$.

$$\begin{aligned} 15 &= \det(E) = \det(D) = \det(C) \\ &= \frac{1}{2}\det(B) = \frac{1}{2}(-\det(A)) \end{aligned}$$

Therefore

$$\det(A) = -30.$$

$\rightarrow$ Example 17 p. 148

$\boxed{\textbf{TH 3.8}}$ ($\rightarrow$p.148) If A and B are $n \times n$ matrices then

$$\det(AB) = \det(A)\det(B)$$

$\boxed{\textbf{COROLLARY 3.2}}$ ($\rightarrow$p.149)

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$