3.1 Determinants - Definition and Properties

(→ p. 140, 141)

• A sequence \( j_1, j_2, \ldots, j_n \) made up of (rearranged) integers from the set \( S_n = \{1, 2, \ldots, n\} \) is called a \textit{permutation} of \( S_n \).

• There are \( n! \) permutations of \( S_n \).

• **Inversion**: \( j_r > j_s \) when \( r < s \).

• A permutation is called \textbf{odd (even)} if it has an odd (even) number of inversions.
### $n = 2$

<table>
<thead>
<tr>
<th>$j_1j_2$</th>
<th># of inversions</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0</td>
<td>even</td>
</tr>
<tr>
<td>21</td>
<td>1</td>
<td>odd</td>
</tr>
</tbody>
</table>

### $n = 3$

<table>
<thead>
<tr>
<th>$j_1j_2j_3$</th>
<th># of inversions</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>0</td>
<td>even</td>
</tr>
<tr>
<td>132</td>
<td>1</td>
<td>odd</td>
</tr>
<tr>
<td>213</td>
<td>1</td>
<td>odd</td>
</tr>
<tr>
<td>231</td>
<td>2</td>
<td>even</td>
</tr>
<tr>
<td>312</td>
<td>2</td>
<td>even</td>
</tr>
<tr>
<td>321</td>
<td>3</td>
<td>odd</td>
</tr>
</tbody>
</table>
The determinant of an $n \times n$ matrix $A$ is

$$\det(A) = |A| = \sum \ (\pm)a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

for all permutations of $S_n$

where the sign of each term is determined as follows:

• for an even permutation, the sign is $+$, and
• for an odd permutation, the sign is $-$. 


\textbf{Example} $n = 2$ : $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|A| = +a_{11}a_{22} - a_{12}a_{21}$$

e.g., $\begin{vmatrix} -3 & -1 \\ 3 & 2 \end{vmatrix} = (-3)(2) - (-1)(3) = -3$
**EXAMPLE**

$n = 3 : A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$|A| = +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

e.g.,

$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

$|A| = (-1)(2)(1) + (1)(1)(1) + (0)(1)(-1)$

$- (0)(2)(1) - (-1)(1)(-1) - (1)(1)(1)$

$= -3$
Property (→p.143)
If we interchange two numbers in the permutation $j_1, j_2, \ldots, j_n$ then the number of inversions is either increased or decreased by an odd number.

\[ \text{TH 3.1} \] (→p.143) $|A^T| = |A|$

\[ \text{TH 3.2} \] (→p.144)
If $B$ is obtained from $A$ by interchanging two rows (or columns) then

$$|B| = -|A|$$
**TH 3.3** (→p.145)
If two rows (or columns) of $A$ are equal then

$$|A| = 0$$

**TH 3.4** (→p.145)
If $A$ contains a zero row (or zero column) then

$$|A| = 0$$

**TH 3.5** (→p.145)
If $B$ is obtained from $A$ by multiplying a row (or column) by a number $c$ then

$$|B| = c|A|$$
**TH 3.6** (→p.146)
If \( B \) is obtained from \( A \) by adding a multiple of one row (or column) to another then

\[
|B| = |A|
\]

**TH 3.7** (→p.146)
If \( A \) is an \( n \times n \) upper triangular (or lower triangular) matrix then

\[
|A| = a_{11}a_{22}\cdots a_{nn}
\]
**EXAMPLE**

Find the determinant of

\[
A = \begin{pmatrix}
0 & 1 & 2 \\
2 & 3 & -2 \\
3 & -1 & 1 \\
\end{pmatrix}
\]

Transform \( A \) to \( B \) by \( r_1 \leftrightarrow r_2 \).

**TH 3.2** \( \Rightarrow \) \( \det(B) = -\det(A) \).

\[
B = \begin{pmatrix}
2 & 3 & -2 \\
0 & 1 & 2 \\
3 & -1 & 1 \\
\end{pmatrix}
\]

Transform \( B \) to \( C \) by \( \frac{1}{2} r_1 \rightarrow r_1 \).

**TH 3.5** \( \Rightarrow \) \( \det(C) = \frac{1}{2} \det(B) \).

\[
C = \begin{pmatrix}
1 & \frac{3}{2} & -1 \\
0 & 1 & 2 \\
3 & -1 & 1 \\
\end{pmatrix}
\]

Transform \( C \) to \( D \) by \( r_3 - 3r_1 \rightarrow r_3 \).

**TH 3.6** \( \Rightarrow \) \( \det(D) = \det(C) \).
Transform $D$ to $E$ by $r_3 + \frac{11}{2}r_2 \rightarrow r_3$.

**TH 3.6** $\Rightarrow \det(E) = \det(D)$.

$$D = \begin{pmatrix} 1 & \frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & -\frac{11}{2} & 4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & \frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 15 \end{pmatrix}$$

Matrix $E$ is upper triangular

**TH 3.7** $\Rightarrow \det(E) = (1)(1)(15) = 15$.

$$15 = \det(E) = \det(D) = \det(C) = \frac{1}{2} \det(B) = \frac{1}{2}(-\det(A))$$

Therefore

$$\det(A) = -30.$$
TH 3.8 (→p.148) If $A$ and $B$ are $n \times n$ matrices then
\[
\det(AB) = \det(A) \det(B)
\]

COROLLARY 3.2 (→p.149)
\[
\det(A^{-1}) = \frac{1}{\det(A)}.
\]