1.6. The Inverse of a Matrix

DEF ($\rightarrow$ p. 66)

An $n \times n$ matrix A is called **nonsingular** (**invertible**) if there exists a matrix B such that

$$AB = BA = I_n$$

B is called an inverse of A. If no matrix B exists, then A is called **singular** (**noninvertible**).

EXAMPLE 1

$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ satisfy

$AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Therefore B is an inverse of A.

$\boxed{\texttt{TH 1.9}}$ ($\to$ p. 67)

If A has an inverse, then the inverse is unique.

PROOF

By the definition, B is an inverse of A if

$$AB = I_n \qquad (*)$$

$$BA = I_n \qquad (**)$$

Assume C is also an inverse of A, so that it satisfies

$$AC = I_n \qquad (***)$$

$$CA = I_n \qquad (****)$$

We have

$$B = BI_n \overset{(**)}{=} B(AC) = (BA)C \overset{(***)}{=} I_nC = C$$

therefore the inverse is unique.

We denote the inverse of A by A^{-1}.

$\boxed{\textbf{EXAMPLE 2}}$ If $A = [r]$ is a 1×1 matrix, then

$$[r][\frac{1}{r}] = [\frac{1}{r}][r] = [1] = I_1$$

Therefore $[r]^{-1} = [\frac{1}{r}]$.

$\boxed{\textbf{EXAMPLE 3}}$ Find the inverse of

$$A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

$$x_{11} = -3, x_{21} = 1; \qquad x_{12} = -4, x_{22} = 1$$

Therefore $A^{-1} = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$. (Check!)

The augmented matrices of the systems in the example were

$$\left(\begin{array}{rr|r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right) \text{ and } \left(\begin{array}{rr|r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right)$$

leading by row operations to

$$\left(\begin{array}{rr|r} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right) \text{ and } \left(\begin{array}{rr|r} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right)$$

This could have been accomplished more compactly:

$$\left(\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{rr|rr} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

Procedure for finding the inverse of A ($\to$ p.71)

- Form an $n \times 2n$ matrix $[A|I_n]$ and transform it to r.r.e.f.: $[C|D]$.
- If $C = I_n$ then $A^{-1} = D$.
- If $C \neq I_n$ then A is singular.

$\boxed{\textbf{EXAMPLE 4}}$ Find the inverse of

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$r_2 - r_1 \to r_2$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$r_3 + 6r_1 \rightarrow r_3$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{pmatrix}$$

$r_3 + 4r_2 \rightarrow r_3$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{pmatrix}$$

$-1r_3 \rightarrow r_3$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{pmatrix}$$

$r_2 + r_3 \rightarrow r_2$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{pmatrix}$$

$r_1 + r_2 \rightarrow r_1$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right)$$

Answer: $A^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$. (Check!)

$\boxed{\textbf{EXAMPLE 5}}$ Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix}$$

$r_2 - 3r_1 \rightarrow r_2$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix}$$

$r_3 + 2r_1 \rightarrow r_3$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{pmatrix}$$

$r_3 + r_2 \rightarrow r_3$

$$\left(\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

A is singular.

TH 1.10 ($\rightarrow$ p. 68) Properties of the Inverse

If A and B are $n \times n$ nonsingular matrices then

- $(A^{-1})^{-1} = A$,
- $(AB)^{-1} = B^{-1}A^{-1}$,
- $(A^T)^{-1} = (A^{-1})^T$.

($\rightarrow$ Corollary 1.2 p.69)

TH 1.11 ($\rightarrow$ p. 69)

If A and B are $n \times n$ matrices then

$$AB = I_n \Leftrightarrow BA = I_n$$

Equivalent conditions ($\rightarrow$ p.76)

For any $n \times n$ matrix A, the following conditions are equivalent:

1. A is nonsingular.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. A is row equivalent to I_n.
4. For every $n \times 1$ matrix $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a unique solution.