1.5. Solutions of Linear Systems of Equations

DEF (→ p. 45) **Elementary Row Operations**

(a) Interchange two rows

$$r_i \leftrightarrow r_j$$

(b) Multiply a row by a nonzero constant

$$cr_i \rightarrow r_i$$

(c) Add a multiple of one row to another

$$r_i + cr_j \rightarrow r_i$$

DEF (→ p. 46)

We say a matrix A is **row equivalent** to a matrix B if B can be obtained from A by performing a sequence of elementary row operations.

Properties of Row Equivalent Matrices ($\rightarrow$ p. 46)

(a) A is row equivalent to A,

(b) if A is row equivalent to B then B is row equivalent to A,

(c) if A is row equivalent to B and B is row equivalent to C then A is row equivalent to C.

Because of property (b), we can say ”A and B are row equivalent”.

TH 1.7 ($\rightarrow$ p. 50)

Linear systems whose augmented matrices are row equivalent have the same solutions.

EXAMPLE 1 Recall **EXAMPLE 1** from the Section 1.1 notes.

$$\begin{aligned} 2x - y &= -4 \\ -3x - 2y &= -1 \end{aligned} \qquad \left(\begin{array}{cc|c} 2 & -1 & -4 \\ -3 & -2 & -1 \end{array}\right)$$

$$\tfrac{1}{2}r_1 \to r_1$$

$$\begin{aligned} x - \tfrac{1}{2}y &= -2 \\ -3x - 2y &= -1 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -\tfrac{1}{2} & -2 \\ -3 & -2 & -1 \end{array}\right)$$

$$r_2 + 3r_1 \to r_2$$

$$\begin{aligned} x - \tfrac{1}{2}y &= -2 \\ -\tfrac{7}{2}y &= -7 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -\tfrac{1}{2} & -2 \\ 0 & -\tfrac{7}{2} & -7 \end{array}\right)$$

$$-\tfrac{2}{7}r_2 \to r_2$$

$$\begin{aligned} x - \tfrac{1}{2}y &= -2 \\ y &= 2 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -\tfrac{1}{2} & -2 \\ 0 & 1 & 2 \end{array}\right)$$

$$r_1 + \tfrac{1}{2}r_2 \to r_1$$

$$\begin{aligned} x &= -1 \\ y &= 2 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right)$$

DEF ($\rightarrow$ p. 44)

An $m \times n$ matrix is in **reduced row echelon form** if it satisfies the following properties:

(a) All zero rows, if any, are at the bottom of the matrix.

(b) The first nonzero entry in each row (if any) is a 1 - it is called the leading entry of the row.

(c) The leading entry in any row is to the left of the leading entry in the following row (if any).

(d) If a column contains a leading entry, all other entries in that column must be zero.

A matrix that satisfies properties (a), (b), and (c) is in **row echelon form**.

EXAMPLE 2 Which of the matrices below are in r.r.e.f. or r.e.f.?

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

EXAMPLE 3 ($\rightarrow$ Example 5 p. 47)

Use elementary row operations to transform the matrix

$$A = \begin{pmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{pmatrix}$$

to a row echelon form.

$r_1 \leftrightarrow r_3$

$$\begin{pmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{pmatrix}$$

$\frac{1}{2} r_1 \rightarrow r_1$

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{pmatrix}$$

$r_4 - 2r_1 \rightarrow r_4$

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{pmatrix}$$

$r_2 \leftrightarrow r_3$

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{pmatrix}$$

$\frac{1}{2}r_2 \rightarrow r_2$

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{pmatrix}$$

$r_4 + 2r_2 \rightarrow r_4$

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{pmatrix}$$

$r_4 - r_3 \rightarrow r_4$ (Shortcut!)

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\frac{1}{2}r_3 \rightarrow r_3$

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

TH 1.5 ($\rightarrow$ p. 46)

Every matrix is row equivalent to a matrix in row echelon form.

EXAMPLE 4 Solve the linear system

$$\begin{array}{rcrcrcrcl} & & 2y & + & 3z & - & 4w & = & 1 \\ & & & & 2z & + & 3w & = & 4 \\ 2x & + & 2y & - & 5z & + & 2w & = & 4 \\ 2x & & & - & 6z & + & 9w & = & 7 \end{array}$$

The augmented matrix of this system

$$\left(\begin{array}{rrrr|r} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{array}\right)$$

has been transformed to r.e.f. in **EXAMPLE 3**:

$$\left(\begin{array}{rrrr|r} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

which corresponds to the system:

$$
\begin{aligned}
x + y - \tfrac{5}{2}z + w &= 2 \\
y + \tfrac{3}{2}z - 2w &= \tfrac{1}{2} \\
z + \tfrac{3}{2}w &= 2 \\
0 &= 0
\end{aligned}
$$

Back substitution:

$$w = \textit{arbitrary}$$

$$z = 2 - \frac{3}{2}w$$

$$y = \frac{1}{2} - \frac{3}{2}z + 2w = \frac{1}{2} - \frac{3}{2}(2 - \frac{3}{2}w) + 2w$$

$$= \frac{1}{2} - 3 + \frac{9}{4}w + 2w = -\frac{5}{2} + \frac{17}{4}w$$

$$x = 2 - y + \frac{5}{2}z - w$$

$$= 2 - (-\frac{5}{2} + \frac{17}{4}w) + \frac{5}{2}(2 - \frac{3}{2}w) - w$$

$$= 2 + \frac{5}{2} - \frac{17}{4}w + 5 - \frac{15}{4}w - w = \frac{19}{2} - 9w$$

This illustrates solving a system by Gaussian elimination.

EXAMPLE 5 (→ Example 6 p.49)

Use elementary row operations to transform the matrix

$$A = \begin{pmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{pmatrix}$$

to a reduced row echelon form.

Continue from the r.e.f. obtained in **EXAMPLE 3**.

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$r_2 - \frac{3}{2} r_3 \to r_2$

$$\begin{pmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & 0 & -\frac{17}{4} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$r_1 + \frac{5}{2} r_3 \longrightarrow r_1$

$$\begin{pmatrix} 1 & 1 & 0 & \frac{19}{4} & 7 \\ 0 & 1 & 0 & -\frac{17}{4} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$r_1 - r_2 \longrightarrow r_1$

$$\begin{pmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

TH 1.6 ($\rightarrow$ p. 49)

Every matrix is row equivalent to a unique matrix in reduced row echelon form.

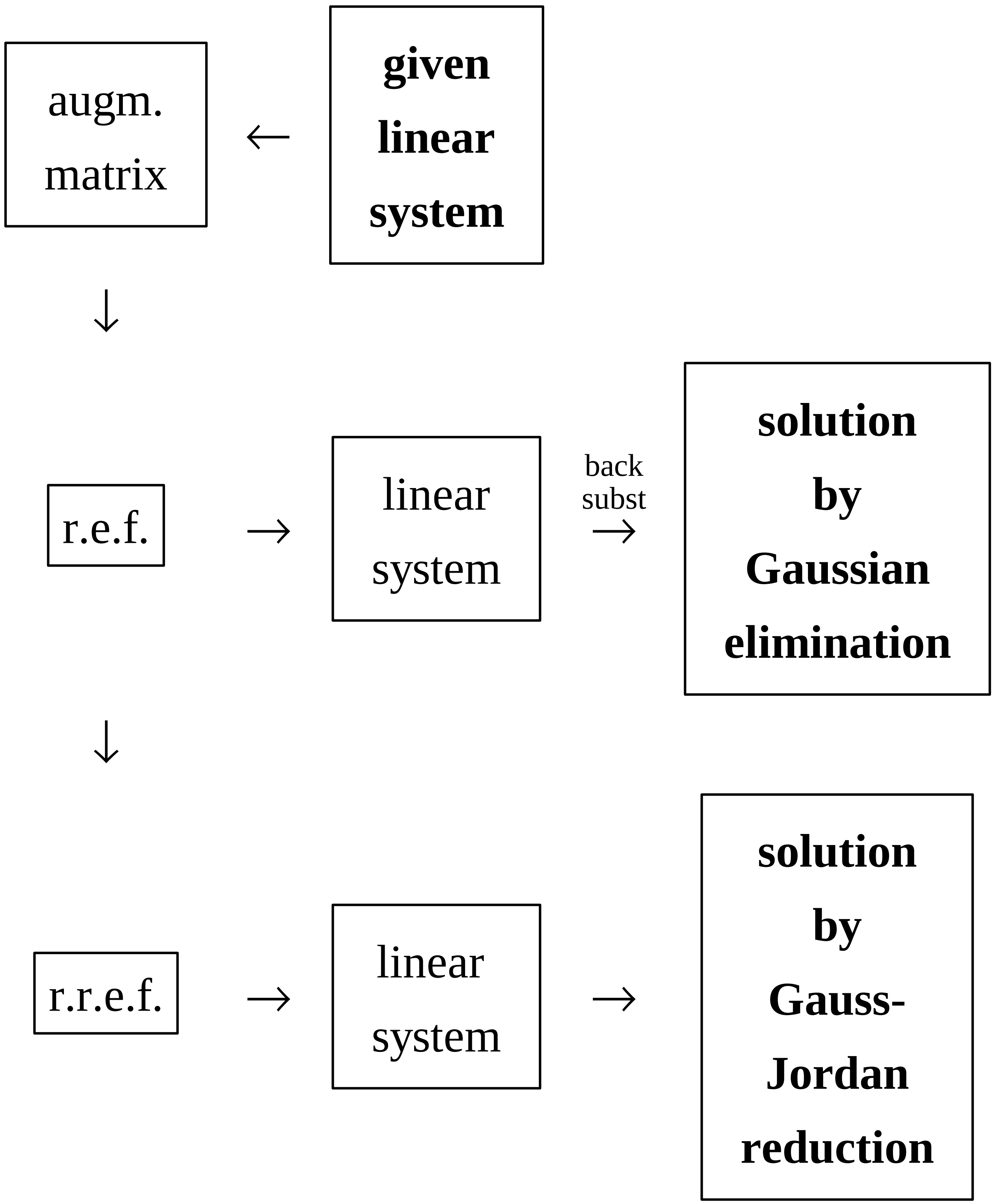
Recall **EXAMPLE 4** with the linear system:

$$\begin{array}{rcrcrcrcr} & & 2y & + & 3z & - & 4w & = & 1 \\ & & & & 2z & + & 3w & = & 4 \\ 2x & + & 2y & - & 5z & + & 2w & = & 4 \\ 2x & & & - & 6z & + & 9w & = & 7 \end{array}$$

Using the r.r.e.f. of the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution: w-arbitrary, $x = \frac{19}{2} - 9w$, $y = \frac{-5}{2} + \frac{17}{4}w$, $z = 2 - \frac{3}{2}w$.


augm. matrix
given linear system
r.e.f.
linear system
back subst
solution by Gaussian elimination
r.r.e.f.
linear system
solution by Gauss-Jordan reduction

EXAMPLE 6

$$\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -3 & -7 \\ 2 & -6 & 7 \end{array}\right)$$

$$r_2 - 2r_1 \to r_2$$

$$\begin{aligned} x - 3y &= -7 \\ 0 &= 21 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -3 & -7 \\ 0 & 0 & 21 \end{array}\right)$$

No solution

A linear system of equations either

(**a**) has exactly one solution, or
(**b**) has infinitely many solutions, or
(**c**) has no solution.

(a) or (b) **- consistent** system; (c) **- inconsistent** system

Homogeneous systems ($\rightarrow$p.56-58) are the linear systems whose right-hand side values are all zero:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Such a system either

(**a**) has exactly one solution:

$$x_1 = x_2 = \cdots = x_n = 0$$

called the **trivial solution**, or

(**b**) has infinitely many solutions: the trivial solution, as well as other, **nontrivial solutions**.

Note that all homogeneous systems are consistent.

$\rightarrow$ Th. 1.8 p.57