1.3. Dot Product and Matrix Multiplication

DEF ($\rightarrow$ p. 17) The **dot product** of n-vectors:
$\overrightarrow{u} = (a_1, \ldots, a_n)$ and $\overrightarrow{v} = (b_1, \ldots, b_n)$ is

$$\overrightarrow{u} \bullet \overrightarrow{v} = a_1 b_1 + \cdots + a_n b_n$$

(regardless of whether the vectors are written as rows or columns).

DEF ($\rightarrow$ p. 18)

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix then the product of A and B is the $m \times p$ matrix $C = [c_{ij}]$ such that

$$\begin{aligned} c_{ij} &= row_i(A) \bullet col_j(B) \\ &= a_{i1} b_{1j} + \cdots + a_{in} b_{nj} \end{aligned}$$

EXAMPLE 1

$$\begin{pmatrix} 4 & -1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 8 & 0 \\ 6 & -2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 34 & -3 \\ 30 & -10 & 15 \end{pmatrix}$$

$\begin{pmatrix} 1 & 8 & 0 \\ 6 & -2 & 3 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 0 & 5 \end{pmatrix}$ cannot be multiplied.

As demonstrated above, *in general* $AB \neq BA$.
For *some* matrices A and B, we have $AB = BA$ e.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Such matrices A and B are said to **commute**.

→ Read pp.21-22: The Matrix-Vector Product Written in Terms of Columns

→ Read pp.27-28: The Summation Notation

Recall a linear system of m equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system can be written in the **matrix form**

$$A\overrightarrow{x} = \overrightarrow{b}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$\overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \overrightarrow{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- The $m \times n$ matrix A is called the coefficient matrix of the system.
- The $m \times (n + 1)$ matrix $[A|\overrightarrow{b}]$ is called the augmented matrix of the system.

EXAMPLE 2

The system

$$\begin{aligned} 2x - \ \ y &= -4 \\ -3x - 2y &= -1 \end{aligned}$$

of from **EXAMPLE 1** the Section 1.1 lecture notes has

- the coefficient matrix $A = \begin{pmatrix} 2 & -1 \\ -3 & -2 \end{pmatrix}$,
- the unknown vector $\overrightarrow{x} = \begin{pmatrix} x \\ y \end{pmatrix}$,
- the right-hand-side vector $\overrightarrow{b} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$,
- and the augmented matrix
  $[A|\overrightarrow{b}] = \begin{pmatrix} 2 & -1 & -4 \\ -3 & -2 & -1 \end{pmatrix}$.