10.3 The Matrix of a Linear Transformation

Let $S = \{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ be a basis for a vector space V. If

$$[\overrightarrow{v}]_S = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \text{ and } [\overrightarrow{w}]_S = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

i.e.

$$\overrightarrow{v} = c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} \text{ and } \overrightarrow{w} = d_1\overrightarrow{u_1} + d_2\overrightarrow{u_2}$$

then

$$\begin{aligned} a\overrightarrow{v} + b\overrightarrow{w} &= a(c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2}) + b(d_1\overrightarrow{u_1} + d_2\overrightarrow{u_2}) \\ &= (ac_1 + bd_1)\overrightarrow{u_1} + (ac_2 + bd_2)\overrightarrow{u_2} \end{aligned}$$

so that

$$\begin{aligned} [a\overrightarrow{v} + b\overrightarrow{w}]_S &= \begin{pmatrix} ac_1 + bd_1 \\ ac_2 + bd_2 \end{pmatrix} \\ &= a\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + b\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \\ &= a[\overrightarrow{v}]_S + b[\overrightarrow{w}]_S \end{aligned}$$

We generally have (→ p.296)

$$[c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}]_S = c_1[\overrightarrow{v_1}]_S + \cdots + c_k[\overrightarrow{v_k}]_S$$

TH 10.8 (→ p. 452)

Let $L : V \to W$ be a linear transformation and let the set $S = \{\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}\}$ be a basis for V, and $T = \{\overrightarrow{w_1}, \ldots, \overrightarrow{w_m}\}$ be a basis for W. Then for every vector $\overrightarrow{x}$ in V

$$[L(\overrightarrow{x})]_T = A[\overrightarrow{x}]_S$$

where A is the $m \times n$ matrix whose jth column is $[L(\overrightarrow{v_j})]_T$. A is the only matrix with this property - it is called the matrix of L with respect to S and T.

Proof

Let $[\overrightarrow{x}]_S = (c_1, \ldots, c_n)$, i.e.,

$\overrightarrow{x} = c_1\overrightarrow{v_1} + \cdots + c_n\overrightarrow{v_n}$

$$
\begin{aligned}
RHS &= A[\overrightarrow{x}]_S \\
&= \left([L(\overrightarrow{v_1})]_T | \cdots | [L(\overrightarrow{v_n})]_T\right) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= c_1[L(\overrightarrow{v_1})]_T + \cdots + c_n[L(\overrightarrow{v_n})]_T \\
&= [c_1 L(\overrightarrow{v_1}) + \cdots + c_n L(\overrightarrow{v_n})]_T \\
&= [L(c_1\overrightarrow{v_1} + \cdots + c_n\overrightarrow{v_n})]_T \\
&= [L(\overrightarrow{x})]_T \\
&= LHS
\end{aligned}
$$

EXAMPLE 1 → **EXAMPLE 2** from the previous lecture

Consider $L : R^2 \to R^3$ defined by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}.$$

(a) Find the matrix of L with respect to the bases $\{(1,1),(1,-1)\}$ and $\{(1,0,0),(1,1,0),(1,1,1)\}$.

(b) Use the matrix obtained in (a) to evaluate $L(\begin{pmatrix} -2 \\ 3 \end{pmatrix})$. Evaluate the same expression directly.

(a) $L(\overrightarrow{u_1}) = L(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$L(\overrightarrow{u_2}) = L(\begin{pmatrix} 1 \\ -1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right)$$

has r.r.e.f. $\left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & -1 \end{array}\right)$

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 3 & -1 \end{pmatrix}$$

(b)

$$\left(\begin{array}{cc|c} 1 & 1 & -2 \\ 1 & -1 & 3 \end{array}\right) \text{ has r.r.e.f. } \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{-5}{2} \end{array}\right)$$

Therefore $\left[\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right]_S = \begin{pmatrix} \frac{1}{2} \\ \frac{-5}{2} \end{pmatrix}$. Use the matrix A:

$$\left[L\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right)\right]_T = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{-5}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix}$$

$$L\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right)$$

$$= -3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

Evaluating directly:

$$L\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ -2+3 \\ -2+2(3) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$