10.1 Linear Tranformations - Definition

DEF ($\rightarrow$ p. 434)

If V and W are vector spaces then a function $L : V \rightarrow W$ is called a **linear transformation** of V into W if it satisfies the following conditions:

(a) $L(\overrightarrow{u} + \overrightarrow{v}) = L(\overrightarrow{u}) + L(\overrightarrow{v})$, for every $\overrightarrow{u}$ and $\overrightarrow{v}$ in V.

(b) $L(c\overrightarrow{u}) = cL(\overrightarrow{u})$, for every vector $\overrightarrow{u}$ in V and every scalar c.

$L(\overrightarrow{u})$ is the **image** of $\overrightarrow{u}$

EXAMPLES from Section 4.3:

- Projection onto the x-axis: $L(x, y) = (x, 0)$.
- Projection onto the xy-plane:
  $L(x, y, z) = (x, y, 0)$
  (or $L(x, y, z) = (x, y)$)
- Dilation and contraction: $L(\overrightarrow{u}) = r\overrightarrow{u}$
- Reflection with respect to the x-axis:
  $L(x, y) = (x, -y)$.
- Rotation with respect to the origin by the angle ϕ:

$$L(\overrightarrow{u}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \overrightarrow{u}$$

EXAMPLE 1

Consider $L : P_2 \to P_3$ defined by $L(at^2 + bt + c) = ct^3 + (a + b)t$. Check:

(a) $LHS = L(\overrightarrow{u} + \overrightarrow{v})$

$= L((at^2 + bt + c) + (a't^2 + b't + c'))$

$= L((a + a')t^2 + (b + b')t + (c + c'))$

$= (c + c')t^3 + (a + a' + b + b')t$

$RHS = L(\overrightarrow{u}) + L(\overrightarrow{v})$

$= L(at^2 + bt + c) + L(a't^2 + b't + c')$

$= (ct^3 + (a + b)t) + (c't^3 + (a' + b')t)$

Therefore, $LHS = RHS$ for every $\overrightarrow{u}$ and $\overrightarrow{v}$ in P_2.

(b) $LHS = L(k\overrightarrow{u}) = L(kat^2 + kbt + kc)$

$= kct^3 + (ka + kb)t$

$RHS = kL(\overrightarrow{u}) = k(ct^3 + (a + b)t)$

Therefore, $LHS = RHS$ for every vector $\overrightarrow{u}$ in P_2 and every scalar k.

Conclude: L <u>is</u> a linear transformation.

EXAMPLE 2 → Example 2 p. 435

Consider $L : P_1 \to P_2$ defined by $L(at+b) = t(at+b)+t^2$. Check:

(a) $LHS = L(\overrightarrow{u} + \overrightarrow{v})$

$= L(at+b+a't+b')$

$= L((a+a')t+(b+b'))$

$= t((a+a')t+(b+b')) \ + t^2$

$RHS = L(\overrightarrow{u}) + L(\overrightarrow{v}) =$

$= L(at+b) + L(a't+b')$

$= t(at+b)+t^2+t(a't+b')+t^2$

$= t(at+b)+t(a't+b')+2t^2$

$LHS \neq RHS$ therefore L is not a linear transformation.

$\rightarrow$ Example 3 p. 435

TH 10.1 $\rightarrow$ p. 436

If L is a linear transformation then

$$L(c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}) = c_1L(\overrightarrow{v_1}) + \cdots + c_kL(\overrightarrow{v_k})$$

TH 10.2 $\rightarrow$ p. 436

If $L : V \rightarrow W$ is a linear transformation then

(a) $L(\overrightarrow{0_V}) = \overrightarrow{0_W}$

(b) $L(\overrightarrow{u} - \overrightarrow{v}) = L(\overrightarrow{u}) - L(\overrightarrow{v})$

TH 10.3 $\rightarrow$ p. 437

If $L : V \rightarrow W$ is a linear transformation and $S = \{\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}\}$ is a basis for V then L is completely determined by $L(\overrightarrow{v_1}), \ldots, L(\overrightarrow{v_k})$.