Spring 99 **Math 211** Bogacki

Sample Test 3 - Answer Key

1. Find $\frac{dy}{dx}$

(a) $y = \arcsin(3x^2 + x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x^2+x)^2}}(6x+1)$$

(b) $y = x\,\text{arccot}\,x$

$$\frac{dy}{dx} = \text{arccot}\,x + x\frac{-1}{x^2+1}$$

(c) $y = \cosh\left(\frac{1}{x}\right) = \frac{e^{1/x}+e^{-1/x}}{2}$

$$\frac{dy}{dx} = \frac{e^{1/x}(-1/x^2)+e^{-1/x}(1/x^2)}{2} = \frac{-1}{x^2}\sinh(\tfrac{1}{x})$$

(d) $y = \text{sech}\,(\sec x) = \frac{2}{e^{\sec x}+e^{-\sec x}}$

$$\frac{dy}{dx} = 2(-1)(e^{\sec x}+e^{-\sec x})^{-2}(e^{\sec x}\sec x\tan x + e^{-\sec x}(-\sec x\tan x))$$

$$= -\text{sech}(\sec x)\ \tanh(\sec x)(\sec x\tan x)$$

2. Use the first derivative test to find the local extrema of $f(x) = -x^3 + 3x^2 - 2$. Determine the intervals where the function is increasing and the intervals where it is decreasing.

$f'(x) = -3x^2 + 6x = -3x(x-2)$. Critical points: x=0, x=2. No singular points.

x		0		2	
$f'(x)$	-	0	+	0	-
$f(x)$	↘	min	↗	max	↘

Local minimum $f(0) = -2$.
Local maximum: $f(2) = 2$.
The function is increasing on the interval (0, 2).
The function is decreasing on the intervals $(-\infty, 0)$ and $(2, \infty)$.

3. Use the second derivative test to find the local extrema of $f(x) = 8x^3 - 2x^4$. Determine the intervals where the function is concave up and intervals where it is concave down. Find the points of inflection.

$f'(x) = 24x^2 - 8x^3 = 8x^2(3-x)$. Critical points: x=0, x=3.
$f''(x) = 48x - 24x^2 = 24x(2-x)$.
Apply the second derivative test at each critical point:
$f''(0) = 0$ - test fails
$f''(3) = -72 < 0$ - local maximum

x		0		2	
$f''(x)$	-	0	+	0	-
$f(x)$	$\frown$	point of inflection	$\smile$	point of inflection	$\frown$

f is concave upwards on the interval (0,2)
f is concave downwards on the intervals $(-\infty, 0)$ and $(2, \infty)$
Points of inflection: (0, 0) and (2, 32).

4. Find all the vertical, horizontal, and/or slant asymptotes of $y = \dfrac{x^3 + 2x^2}{x^2 - 1}$.

<u>Vertical asymptotes</u> can occur at the x values outside the domain, i.e., x= -1 and x=1.

$\lim_{x\to -1^-} \dfrac{x^3+2x^2}{x^2-1} = \infty$ $\quad$ $\lim_{x\to -1^+} \dfrac{x^3+2x^2}{x^2-1} = -\infty$ $\quad \Rightarrow$ x = - 1 is a vertical asymptote

$\lim_{x\to 1^-} \dfrac{x^3+2x^2}{x^2-1} = -\infty$ $\quad$ $\lim_{x\to 1^+} \dfrac{x^3+2x^2}{x^2-1} = \infty$ $\quad \Rightarrow$ x = 1 is a vertical asymptote

<u>Horizontal asymptote</u>

$\lim_{x\to -\infty} \dfrac{x^3+2x^2}{x^2-1} = \lim_{x\to -\infty} \dfrac{x^3/x^2 + 2x^2/x^2}{x^2/x^2 - 1/x^2} = \lim_{x\to -\infty} \dfrac{x}{1} = -\infty$ $\quad \Rightarrow$ no horiz. asymptote on the left

$\lim_{x\to \infty} \dfrac{x^3+2x^2}{x^2-1} = \lim_{x\to \infty} \dfrac{x^3/x^2 + 2x^2/x^2}{x^2/x^2 - 1/x^2} = \lim_{x\to \infty} \dfrac{x}{1} = \infty$ $\quad \Rightarrow$ no horiz. asymptote on the right

<u>Slant asymptote</u>
Rational function where degree in the numerator exceeds the degree in the denominator by **one** $\Rightarrow$ has a slant asymptote
Polynomial division: $\dfrac{x^3+2x^2}{x^2-1} = x + 2 + \dfrac{x+2}{x^2-1} \Rightarrow$ slant asymptote: y = x + 2

5. Make use of domain, asymptotes, intercepts, relative extrema and points of inflection to obtain an accurate graph of

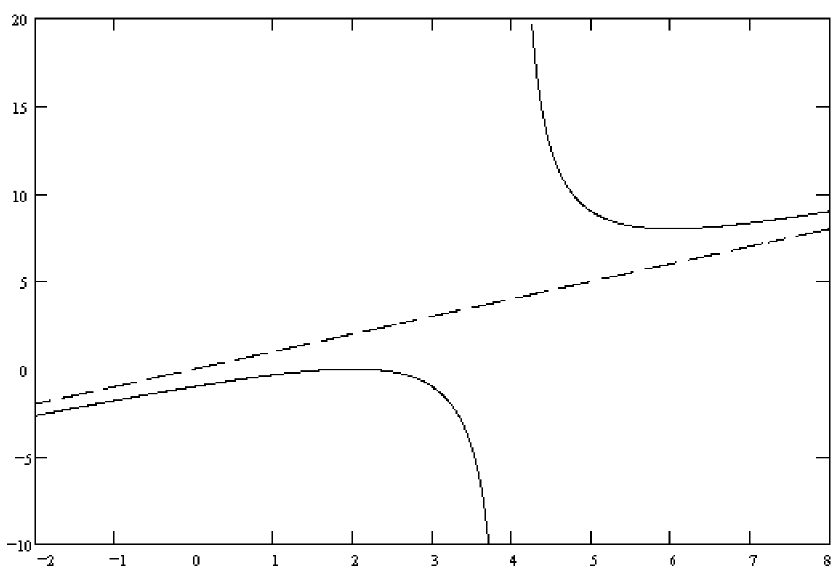
$$f(x) = \frac{x^2 - 4x + 4}{x - 4}$$

- Domain: all real numbers except $x = 4$.
- Vertical asymptotes: $x = 4$: $\lim_{x \to 4^-} f(x) = -\infty \quad \lim_{x \to 4^+} f(x) = \infty$.
- Horizontal asymptotes: none - $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{\frac{x^2}{x} - \frac{4x}{x} + \frac{4}{x}}{\frac{x}{x} - \frac{4}{x}} = \infty \quad \lim_{x \to -\infty} f(x) = -\infty$.
- Slant (oblique) asymptote: $y = x$, since $\dfrac{x^2 - 4x + 4}{x - 4} = x + \dfrac{4}{x - 4}$.
- x-intercepts: Solving $\dfrac{x^2 - 4x + 4}{x - 4} = 0$ we have $x^2 - 4x + 4 = (x-2)^2$ obtaining $x = 2$.
- y-intercepts: $f(0) = -1 \quad \Rightarrow \quad y = -1$.
- $f'(x) = \dfrac{(2x-4)(x-4) - (x^2 - 4x + 4)(1)}{(x-4)^2} = \dfrac{2x^2 - 8x - 4x + 16 - x^2 + 4x - 4}{(x-4)^2} = \dfrac{x^2 - 8x + 12}{(x-4)^2}$

  $f'(x) = \dfrac{(x-6)(x-2)}{(x-4)^2}$ Critical values: $x = 2, x = 6$
- $f''(x) = \dfrac{(2x-8)(x-4)^2 - (x^2 - 8x + 12)(2)(x-4)(1)}{(x-4)^4}$

  $= \dfrac{(x-4)(2x^2 - 8x - 8x + 32 - 2x^2 + 16x - 24)}{(x-4)^4} = \dfrac{8}{(x-4)^3}$ No inflection points.

x		2		4		6	
$f'(x)$	+	0	-	undefined	-	0	+
$f''(x)$	-	-	-	undefined	+	+	+
$f(x)$		max 0		undefined		min 8	



6. Evaluate and simplify

(a) $$\lim_{x\to 0^-}\frac{e^x-x-1}{\sin x - x} \underset{\text{l'Hopital}}{\overset{0/0}{=}} \lim_{x\to 0^-}\frac{e^x-1}{\cos x-1} \underset{\text{l'Hopital}}{\overset{0/0}{=}} \lim_{x\to 0^-}\frac{\overset{\overset{1}{\uparrow}}{e^x}}{\underbrace{-\sin x}_{\substack{\downarrow\\0^+}}} = \infty$$

(b) $$\lim_{x\to 0^+}(\cos x)^{1/x^2} = L$$

$$\ln L = \lim_{x\to 0^+}\ln(\cos x)^{1/x^2} = \lim_{x\to 0^+}\frac{1}{x^2}\ln(\cos x) = \lim_{x\to 0^+}\frac{\ln(\cos x)}{x^2} \underset{\text{l'Hopital}}{\overset{0/0}{=}} \lim_{x\to 0^+}\frac{\frac{-\sin x}{\cos x}}{2x}$$

$$= \lim_{x\to 0^+}\frac{-\tan x}{2x} \underset{\text{l'Hopital}}{\overset{0/0}{=}} \lim_{x\to 0^+}\frac{-\sec^2 x}{2} = \frac{-1}{2} \quad\Rightarrow\quad L = e^{-1/2} = \frac{1}{\sqrt{e}}$$

(c) $$\lim_{x\to\infty}\frac{\ln(x^2)}{x^3} = \lim_{x\to\infty}\frac{2\ln x}{x^3} \underset{\text{l'Hopital}}{\overset{\infty/\infty}{=}} \lim_{x\to\infty}\frac{2/x}{3x^2} = \lim_{x\to\infty}\frac{2}{3x^3} = 0$$

(d) $$\lim_{x\to 0^+}(1+x)^{1/x} = L$$

$$\ln L = \lim_{x\to 0^+}\ln(1+x)^{1/x} = \lim_{x\to 0^+}\frac{1}{x}\ln(1+x) = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \underset{\text{l'Hopital}}{\overset{0/0}{=}} \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1} = 1$$

$$\Rightarrow\quad L = e^1 = e$$