MATH 211 - Sample Test 2 Answer Key
Spring 1999, Dr. Bogacki

1. $y = x\cos y$
   $y' = \cos y - x(\sin y)\, y'$
   $y'(1 + x\sin y) = \cos y$
   $y' = \frac{\cos y}{1+x\sin y}$
   $y'' = \frac{-\sin y\, y'(1+x\sin y)-\cos y(\sin y+x\cos y\, y')}{(1+x\sin y)^2} = \frac{-\sin y\left(\frac{\cos y}{1+x\sin y}\right)(1+x\sin y)-\cos y(\sin y+x\cos y\left(\frac{\cos y}{1+x\sin y}\right))}{(1+x\sin y)^2}$

2. $\frac{dx}{dt} = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$
   $\frac{dx}{dt} = 0$ at $t = 1$ and $t = 3$.
   $\left.\frac{dx}{dt}\right|_{t=0} = 9 > 0$ therefore for $t < 1$, x is moving to the right;
   $\left.\frac{dx}{dt}\right|_{t=2} = -3 < 0$ therefore for $1 < t < 3$, x is moving to the left;
   $\left.\frac{dx}{dt}\right|_{t=4} = 9 > 0$ therefore for $t > 3$, x is moving to the right;
   $\frac{d^2x}{dt^2} = 6t - 12 = 6(t-2)$
   $\left.\frac{d^2x}{dt^2}\right|_{t=0} = -12 < 0$ therefore for $t < 2$, x is accelerating to the left;
   $\left.\frac{d^2x}{dt^2}\right|_{t=3} = 6 > 0$ therefore for $t > 2$, x is accelerating to the right;

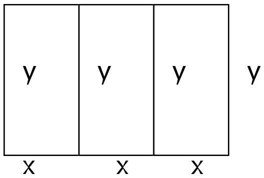
   (a) the point is moving to the right for $t < 1$ and for $t > 3$;
   (b) the point is accelerating to the right for $t > 2$;
   (c) the point is speeding up whenever the signs of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ match, i.e. for $1 < t < 2$ and for $t > 3$.

3. $V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$.
   Since $\frac{dV}{dt} = -20\ \text{ft}^3/\text{min}$; $r = 2$ ft, we have
   $-20 = 4\pi\,(2)^2\,\frac{dr}{dt}$ therefore $\frac{dr}{dt} = \frac{-20}{16\pi} = \frac{-5}{4\pi}$.
   Answer: the radius is decreasing at the rate of $1.25/\pi$ feet per minute.

4. $f(x) = 2x^3 - 9x^2 - 3$; $f'(x) = 6x^2 - 18x = 6x(x-3)$

   - Critical points: Set $f'(x) = 0$; thus $\underline{x = 0,\ f(0) = -3}$ and $\underline{x = 3,\ f(3) = -30}$ are the two critical points. However only the first point is inside the interval $[-1, 1]$.
   - Singular points (f' is undefined while f is defined)
     Since f' is a polynomial, it is defined for all real x, therefore there are no simgular points.
   - Endpoints: $\underline{f(-1) = -14}$; $\underline{f(1) = -10}$.

   The absolute maximum is $f(0) = -3$. The absolute minimum is $f(-1) = -14$.

5. 

Length of the fence: $2000 = 6x + 4y$.

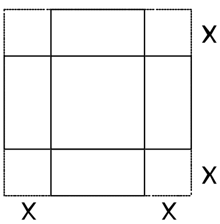
Maximize the area: $A = 3xy$.

Solve the fence length equation for y : $y = 500 - \frac{3}{2}x$ and substitute into the area formula: $A(x) = 3x(500 - \frac{3}{2}x) = 1500x - \frac{9}{2}x^2$ defined on $[0, \frac{2000}{6}]$.

- Critical points: $A'(x) = 1500 - 9x = 0 \Rightarrow x = \frac{1500}{9} = \frac{500}{3}$ is inside the interval; $A(\frac{500}{3}) = 500(500 - 250) = (500)(250) = 125,000$.
- No singular points (derivative defined for all x)
- Endpoints $A(0) = 0$; $A(\frac{2000}{6}) = 0$.

Absolute maximum at $x = \frac{500}{3}$.

Answer: the largest rectangular area that can be enclosed is $125,000$ sq ft.

6. 

Volume of the box: $V(x) = (12 - 2x)^2(x)$ for x in $[0, 6]$.

- Critical points: $V'(x) = 2(12-2x)(-2)(x)+(12-2x)^2 = (12-2x)(-4x+12-2x)$
  $= (12 - 2x)(-6x + 12) = 0$
  $x = 6 \Rightarrow V(6) = 0$
  $x = 2 \Rightarrow V(2) = 64(2) = 128$
- Singular points: none
- Endpoints: $V(0) = 0$, $V(6) = 0$ (already listed as a critical point)

Absolute maximum at $x = 2$.

Answer: The dimensions of the box with maximum volume are 8 in $\times$ 8 in $\times$ 2 in.

7. (a) $\frac{dy}{dx} = e^{2x}(2) + \frac{1}{x^2}(2x) = 2e^{2x} + \frac{2}{x}$
   (b) $\frac{dy}{dx} = e^{\sin^2 x}(2 \sin x)(\cos x)$
   (c) $\frac{dy}{dx} = (\sec^2(\ln x))\,\frac{1}{x}$
   (d) $\frac{dy}{dx} = \left(\frac{x^2-x+1}{5x^2+2x+1}\right)\left(\frac{(10x+2)(x^2-x+1)-(5x^2+2x+1)(2x-1)}{(x^2-x+1)^2}\right)$
   (e) $y = \frac{\ln(3x+1)}{\ln 2}$; $\frac{dy}{dx} = \left(\frac{1}{\ln 2}\right)\left(\frac{3}{3x+1}\right)$
   (f) $y = e^{(\ln 4)(x-\cos x)}$; $\frac{dy}{dx} = e^{(\ln 4)(x-\cos x)}(\ln 4)(1 + \sin x) = 4^{x-\cos x}(\ln 4)(1 + \sin x)$
   (g) $y = e^{(\ln x)x}$; $\frac{dy}{dx} = e^{(\ln x)x}(\frac{1}{x}x + \ln x) = x^x(1 + \ln x)$

(h) $y = e^{(\ln(\sin x))\cos x}$; $\frac{dy}{dx} = e^{(\ln(\sin x))\cos x}(\frac{\cos x}{\sin x}\cos x + \ln(\sin x)(-\sin x))$
$= (\sin x)^{\cos x}(\frac{\cos^2 x}{\sin x} - \sin x(\ln(\sin x)))$

8. (a) $\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x$

(b) $\frac{dy}{dx} = \frac{1}{\csc x + \cot x}(-\csc x\cot x - \csc^2 x) = \frac{-\csc x(\cot x + \csc x)}{\csc x + \cot x} = -\csc x$