ANSWER KEY - SAMPLE TEST 1

Spring 1999 - MATH 211 - Dr. Bogacki

1. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)+1}{x+h} - \frac{x+1}{x}}{h} = \lim_{h\to 0} \frac{\frac{(x+h+1)x-(x+1)(x+h)}{(x+h)x}}{h} = \lim_{h\to 0} \frac{x^2+hx+x-x^2-xh-x-h}{h(x+h)x}$

$= \lim_{h\to 0} \frac{-h}{h(x+h)x} = \lim_{h\to 0} \frac{-1}{(x+h)x} = \boxed{\frac{-1}{x^2}}$ (Shortcut: rewrite $f(x) = 1 + \frac{1}{x}$)

2. (a) $\lim_{x\to 2^-} \frac{x^2+4}{x^2-4} = \boxed{-\infty}$

(b) $\lim_{x\to 0} \frac{-2}{x\csc x} = \lim_{x\to 0} \frac{-2\sin x}{x} = -2\lim_{x\to 0} \frac{\sin x}{x} = -2(1) = \boxed{-2}$

(c) $\lim_{x\to -2} \frac{x^2+4x+4}{x^2-4} = \lim_{x\to -2} \frac{(x+2)^2}{(x+2)(x-2)} = \lim_{x\to -2} \frac{x+2}{x-2} = \boxed{0}$

(d) $\lim_{x\to 4} \frac{\sqrt{x^2+9}}{-x-4} = \frac{5}{-8} = \boxed{-\frac{5}{8}}$

(e) $\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{x+1} = \lim_{x\to -\infty} \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{\frac{x+1}{-x}} = \lim_{x\to -\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{x}{-x}+\frac{1}{-x}} = \frac{\sqrt{2+0}}{-1+0} = \boxed{-\sqrt{2}}$

(f) $\lim_{x\to \infty} \frac{5x^4-4}{2-x-x^3} = \lim_{x\to \infty} \frac{\frac{5x^4}{x^3}-\frac{4}{x^3}}{\frac{2}{x^3}-\frac{x}{x^3}-\frac{x^3}{x^3}} = \lim_{x\to \infty} \frac{5x-0}{0-0-1} = \boxed{-\infty}$

3. $f'(x) = 12x^3 - 12x^2$; Tangent line to the graph of $f(x)$ is horizontal whenever $f'(x) = 0$.

We set $f'(x) = 0 \Leftrightarrow 12x^3 - 12x^2 = 0 \Leftrightarrow 12x^2(x-1) = 0 \Leftrightarrow \boxed{x = 0 \text{ or } x = 1}$

4. (a) $y = 3x^4 - \sqrt{x} + \pi - \frac{1}{x} = 3x^4 - x^{1/2} + \pi - x^{-1}$

$\frac{dy}{dx} = 12x^3 - \frac{1}{2}x^{-1/2} + 0 - (-1)x^{-2} = \boxed{12x^3 - \frac{1}{2\sqrt{x}} + \frac{1}{x^2}}$

(b) $y = \cos^3(2x^4+x) = (\cos(2x^4+x))^3$

$\frac{dy}{dx} = 3(\cos(2x^4+x))^2(-\sin(2x^4+x))(8x^3+1) = \boxed{-3(8x^3+1)\cos^2(2x^4+1)\sin(2x^4+1)}$

(c) $y = \frac{\sec x + x}{\tan x - x}$; $\frac{dy}{dx} = \boxed{\frac{(\sec x\tan x+1)(\tan x-x)-(\sec x+x)(\sec^2 x-1)}{(\tan x-x)^2}}$

(d) $y = (3x^2 - \sin x + 1)(\csc x + \sqrt[3]{x})$

$\frac{dy}{dx} = \boxed{(6x-\cos x)(\csc x + \sqrt[3]{x}) + (3x^2 - \sin x + 1)(-\csc x\cot x + \frac{1}{3}x^{-2/3})}$

(e) $xy + x^3 + y^3 = \cos x \sin y$ - Implicit differentiation

$1y + xy' + 3x^2 + 3y^2y' = -\sin x\sin y + (\cos x)(\cos y)y'$

$xy' + 3y^2y' - (\cos x)(\cos y)y' = -y - 3x^2 - \sin x\sin y$

$y'(x + 3y^2 - \cos x\cos y) = -y - 3x^2 - \sin x\sin y$

$y' = \boxed{\frac{-y-3x^2-\sin x\sin y}{x+3y^2-\cos x\cos y}}$

(f) $y = \sqrt[3]{4x^2 - \cot x} = (4x^2 - \cot x)^{1/3}$

$\frac{dy}{dx} = \boxed{\frac{1}{3}(4x^2-\cot x)^{-2/3}(8x+\csc^2 x)}$

5. $f(x) = \frac{1}{x} + \sin x = x^{-1} + \sin x$

$f'(x) = -x^{-2} + \cos x$

$f''(x) = \boxed{2x^{-3} - \sin x}$

6. $f(x) = \frac{-2}{2x+1} = -2(2x+1)^{-1}$; Check that the point (-1,2) is on the graph: $2 \overset{\checkmark}{=} \frac{-2}{2(-1)+1}$

$f'(x) = 2(2x+1)^{-2}(2) = \frac{4}{(2x+1)^2}$; $f'(-1) = \frac{4}{(2(-1)+1)^2} = 4$.

Point-slope form: $y - y_0 = m(x - x_0)$; $y - 2 = 4(x+1)$; $\boxed{y = 4x + 6}$

7. Check whether the points are on the curve:

$4(-1)^2 + 4(-1) \overset{\checkmark}{=} 0 \quad 2(2^2) + 4(2) \neq 0$!!!

Therefore, there is no tangent or normal line at the point (2,2).

$xy^2 + 4y = 0$ Implicit differentiation:

$1y^2 + x(2y)(y') + 4y' = 0$

$y'(2xy + 4) = -y^2$

$y' = \frac{-y^2}{2xy+4}$

Slope of the tangent line: $y'|_{(4,-1)} = \frac{-(-1)^2}{2(4)(-1)+4} = \frac{-1}{-4} = \frac{1}{4}$; equation: $\boxed{y + 1 = \frac{1}{4}(x - 4)}$

Slope of the normal line: $-\frac{1}{1/4} = -4$; equation: $\boxed{y + 1 = -4(x - 4)}$

8. (a) $\int \frac{x+x^2}{\sqrt{x^3}} dx = \int \frac{x+x^2}{x^{3/2}} dx = \int (x^{-1/2} + x^{1/2}) dx = \frac{x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} + c = \boxed{2\sqrt{x} + \frac{2}{3}x\sqrt{x} + c}$

(b) $\int (\sin x - x + 1) dx = \boxed{-\cos x - \frac{x^2}{2} + x + c}$

9. $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 - 5x + 7) = 1$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \cos(x - 2) = \cos 0 = 1$

Therefore $\lim\limits_{x \to 2} f(x) = 1$.

Also, $f(2) = \cos(2 - 2) = \cos 0 = 1$.

Since $\lim\limits_{x \to 2} f(x) = f(2)$, $\boxed{\text{the function } f \text{ is continuous at } x = 2}$.