
8 Infinite Series

The main mathematical topic in this chapter is to determine the behavior of a given sequence (a function defined on n the natural numbers) as $n \rightarrow \infty$. At first, one might think this question could be easily answered by simply using a computer to calculate more and more terms. That this is not the case will be demonstrated in many of the assignments in this chapter.

This chapter includes two laboratory assignments. The first assignment deals exclusively with sequences, and explores the behavior of sequences defined recursively, in particular the logistic equation. In the second laboratory assignment, infinite series are the main topic. It is shown how to consider an infinite series in terms of a sequence of partial sums.

8.1 Activity: Sequences

Prerequisite: Read Section 8.1 LHE.

This activity is intended to enhance your understanding of what it means for a sequence to converge. In particular, you will explore the convergence of recursive sequences.

Instructions

After reading the comments and studying the worked example, open a blank Mathcad document and create your report there. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

Comments

1. Earlier in this manual, we introduced Mathcad's *literal* subscripts, which are used for cosmetic reasons, e.g., to introduce descriptive names like y_{\tan} (see Comment 1 in Section 2.1). In this activity, we introduce Mathcad's *numeric* subscripts, which can be used to define sequences of numbers. To produce such a subscript, use the $\boxed{x_i}$ button on the first palette strip, or type the "]" (left bracket) key.

Example 1

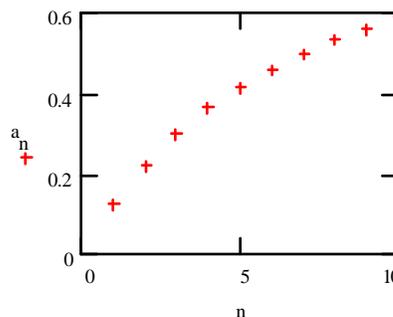
Consider the sequence whose general term is $a_n = \frac{n}{7+n}$ where $n = 1, 2, \dots$

- (a) Tabulate the first ten terms of the sequence and then plot the points (n, a_n) . Select Symbol: *+'s* and Line: *none* from the Graph Format dialog box for the plot.

Solution

$$n := 1..10 \quad a_n := \frac{n}{7+n}$$

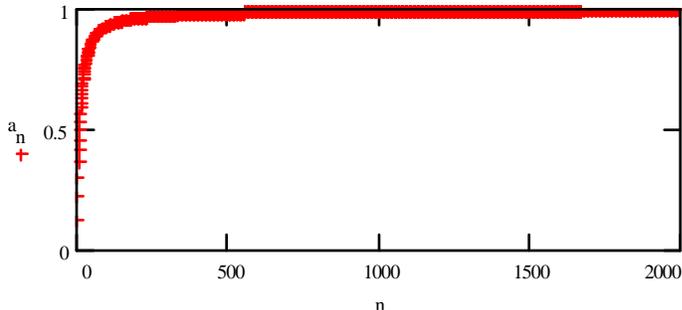
n	a_n
1	0.125
2	0.222
3	0.3
4	0.364
5	0.417
6	0.462
7	0.5
8	0.533
9	0.563
10	0.588



- (b) Plot the first 2000 terms of the sequence. Evaluate a_{100} , a_{500} , a_{1000} and a_{2000} . What does the limit of the sequence appear to be?

Solution

$$n := 1..2000 \quad a_n := \frac{n}{7+n}$$



$$a_{100} = 0.935$$

$$a_{500} = 0.986$$

$$a_{1000} = 0.993$$

$$a_{2000} = 0.997$$

The limit appears to be 1.

Example 2

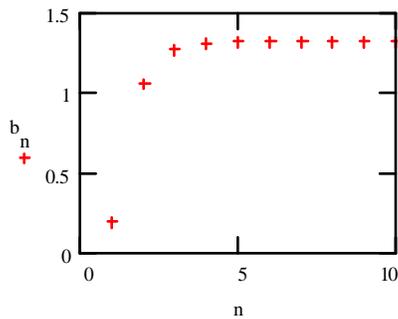
Consider the sequence defined by the recursion formula $b_{n+1} = (b_n + 1)^{1/3}$ where $n = 1, 2, \dots$ and the starting input value is $b_1 = 0.2$. This formula gives the iterated sequence of terms b_1, b_2, b_3, \dots where the output value b_{n+1} of the n -th iteration is the input for the $(n + 1)$ -st iteration.

- (a) Tabulate the first 10 terms of the sequence and plot these terms against the index. What does the limit of the sequence appear to be?

Solution

$$n := 1..10 \quad b_1 := 0.2 \quad b_{n+1} := (b_n + 1)^{\frac{1}{3}}$$

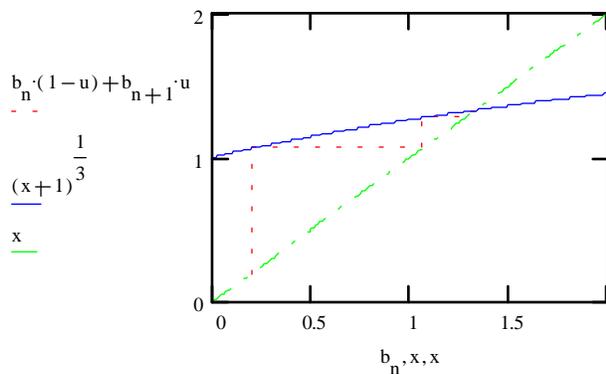
n	b_n
1	0.2
2	1.06266
3	1.27294
4	1.31481
5	1.32283
6	1.32436
7	1.32465
8	1.32471
9	1.32472
10	1.32472



Limit appears to be approximately 1.32472

- (b) On the plot given here you will find three graphs. The graph of $y = (x+1)^{1/3}$ is rendered with a solid line, while the graph of $y = x$ uses "dash-dot". The dotted "stair-step" path begins at the point (b_1, b_1) , then goes to the point (b_1, b_2) , then (b_2, b_2) , then $(b_2, b_3), \dots$, and then finally to the point (b_{10}, b_{11}) , since $n = 1, 2, \dots, 10$.

$$u := 0..1 \quad x := 0, 0.01..2$$



Explain the relationship between this plot and the sequence in part (a).

Solution

The recursion formula for the sequence has the form $b_{n+1} = f(b_n)$ where $f(x) = (x + 1)^{1/3}$. (Keep in mind that the output value b_{n+1} of the n -th iteration becomes the input value for the $(n + 1)$ -st iteration, and that the starting input value is b_1 .)

The first vertical line segment from (b_1, b_1) to (b_1, b_2) corresponds to calculating $b_2 = f(b_1)$. Moving along the horizontal segment from (b_1, b_2) to the line $y = x$ yields the point (b_2, b_2) . This locates the next input value b_2 . Then moving along the vertical segment from (b_2, b_2) to (b_2, b_3) corresponds to calculating $b_3 = f(b_2)$. This provides us with the next input value b_3 . Continuing this procedure will generate the iterated sequence of terms discussed in part (a).

The "stair-step" path approaches the intersection of the line $y = x$ and the function $f(x) = (x + 1)^{1/3}$. Therefore, the limit of the sequence must be a solution of $x = f(x)$. This also follows from the general observation that if a sequence converges, then successive terms b_n and b_{n+1} must get closer and closer in value as n gets large.

Problems

1. (a) Use Mathcad to define sequences a_n, b_n, c_n and d_n from p.554 LHE for $n = 1, \dots, 100$.
 - (b) Define another range variable, say m , to take the values $1, 2, \dots, 10$ and then tabulate the first 10 terms of each sequence side by side. Observe that the first three terms of these sequences are identical. Plot the four sequences against the index m on a single graph. In the Graph Format dialog change Line to none, and set a different symbol for each sequence.
 - (c) Plot the first 100 terms of each sequence on a single graph (you may first want to copy and paste the plot from part (b)). Based only on this graph, what can you infer about the convergence or divergence of each sequence, if anything?
 - (d) Copy and paste the graph from part (c), and zoom in to help distinguish between the sequences which seem to converge. Based only on this graph, what can you infer about the convergence of each sequence? For each sequence, evaluate some terms to estimate the limit.
 - (e) By hand, determine the limit of each sequence. In each case, briefly explain the reasoning behind how you arrived at the answer. Check that your answers are consistent with your conclusions in parts (c) and (d).
2. (a) In Example 2 (b) we observed that if a recursive sequence $b_{n+1} = f(b_n)$ converges to a limit x , then x must be a solution of the equation $x = f(x)$. Hence, the limit of the convergent sequence discussed in Example 2 must be a solution of the equation $x = (x + 1)^{1/3}$. Let's verify this. Find an approximation to the solution to this problem by plotting both sides of the equation and "zooming" in on the point of intersection. From an initial guess based on your graph, use the root function to find a more accurate solution. (Remember: Mathcad's root function finds values where $F(x) = 0$. So, you must create an appropriate $F(x)$.) Have Mathcad display the solution value to ten decimal places, and select **Built-In Variables...** from the **Math** menu to change the value of TOL to 0.00001 (this will make the root function more accurate - see the footnote on page 13). Comment on how this solution value compares with b_{10} in Example 2.
 - (b) Suppose we "reverse" the recursion formula of Example 2 to obtain the new recursion formula $b_n = f(b_{n+1}) = (b_{n+1} + 1)^{1/3}$. Notice that if this sequence converges to a limit p , then p must be a solution of the same equation as in part (a), namely $p = (p + 1)^{1/3}$. By hand, solve the equation $b_n = (b_{n+1} + 1)^{1/3}$ for b_{n+1} as an expression $g(b_n)$, and

use the result to set up a recursion formula for the new sequence b_1, b_2, b_3, \dots . Tabulate the first five terms of the sequence, starting with $b_1 = 1.5$. Does the sequence appear to converge?

- (c) Now create a "stair-step" plot corresponding to the sequence in part (b) by rendering the plot of Example 2 (b), replacing $(x+1)^{1/3}$ with the expression $g(x)$ obtained in part (b) defined as a function of x . (Don't forget to define $u = 0..1$.) Change the value of x to cover the interval $[0, 6]$, and restrict both the x and y axes to $[0, 6]$.
- (d) Observe that in part (c), the "stair-step" path moves away from the intersection of the line $y = x$ and the new function, whereas in Example 2(b), the path approaches the intersection of $y = x$ and the function $f(x) = (x+1)^{1/3}$. These opposite behaviors are related to the magnitude of the slope of the functions at the point of intersection with $y = x$. We won't go into the details here. Rather, evaluate the slope of the function $f(x)$ and compare to the slope of $y = x$ at their point of intersection. Repeat for the function $g(x)$. How do you think the steepness of the function at the point of convergence is related to the convergence of the iteration?

3. The "logistic model" of population growth is defined by the recursion formula

$$P_{n+1} = rP_n(1 - P_n)$$

where the value P_n is the population of the n -th generation, expressed as a fraction of the maximum population. Different values of the parameter r (between 0 and 4) represent different environments.

- (a) By hand, find the non-zero limit of the logistic model by solving the equation $x = rx(1-x)$ for x (i.e find x expressed in terms of r). In Mathcad, define a function $xsol(r)$ to be equal to this expression. Set $r = 1.5$ and define the sequence P_n for $n = 1, 2, \dots, 100$ with $P_1 = 0.5$. Tabulate the 91st through the 100th terms of the sequence and make sure the sequence is converging to $xsol(r)$. Copy the plot you created in problem 2(c): changing the sequence name and function as necessary, and define the range variable x on $[0,1]$. Restrict both axes to $[0,1]$, and if necessary, redefine the index so that the plot reflects the first 100 terms of the sequence. You should be able to see the "stair-step" path approach the intersection of the parabola and $y = x$.
- (b) Copy your ENTIRE answer in part (a) onto the clipboard, and paste it three times below. In the first copy, begin increasing the value of r by increments of 0.2. Each time after the document recalculates, observe that the sequence P_n converges to the new value given by $xsol(r)$ and that this convergence is exhibited by the plot. Continue increasing r by 0.2 until you reach the value for which the sequence P_n no longer converges. STOP. Notice that the sequence "oscillates" between two values.
- (c) In the second copy you made, begin increasing the value of r starting at a value 0.1 bigger than the last value you used in part (b). Continue incrementing r by 0.1 until the sequence begins to oscillate between four values. STOP.
- (d) Finally, in the third copy you made, increase the value of r by 0.1 beyond the value reached in part (c). You shouldn't have to wait very long to see a dramatic change in

the behavior of the sequence: it becomes chaotic, with no indication of the sequence approaching any values as n gets large.

- (e) By hand, find the derivative of $rx(1-x)$ with respect to x . Evaluate the derivative at the point of convergence (In other words, evaluate the derivative when $x = x_{sol}(r)$. This should give you an answer in terms of r only.). For which values of r is the slope of the derivative, at the point of convergence, bigger than 1 in magnitude? How is this related to the behavior of the sequence P_n observed above for various values of r ? (Note: Problem 2(d) is related to this one.)

8.2 Activity: Infinite Series

Prerequisites: Read Sections 8.2 - 8.10 LHE.

In this activity you will use Mathcad to numerically and graphically investigate the convergence of infinite series. You will also learn how to use Mathcad to obtain Taylor polynomials. In addition, you will investigate how the Taylor polynomial changes with the order, and with the center of expansion.

Instructions

After reading the comments and studying the worked examples, open a blank Mathcad document and create your report there. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

Comments

1. In this activity we focus on the concept of convergence of a partial sum sequence. Given the sequence a_i ($i = 1, 2, \dots$), the partial sum sequence S_n is defined by the sum:

$$S_n = \sum_{i=1}^n a_i .$$

While we could use this definition to compute S_n in Mathcad, in most cases we opt for a more efficient way of computing partial sums, based on the following *recursive* formula

$$S_n = S_{n-1} + a_n \text{ for all } n \geq 1 \text{ and } S_0 = 0.$$

By iterating this formula m times (and performing m additions in the process), we generate the first m partial sums. On the other hand, using the definition would require $n-1$ additions to obtain S_n , thus, obtaining the first m partial sums S_1, \dots, S_m would require

$$0 + 1 + 2 + \dots + (m-2) + (m-1) = \frac{(m-1)m}{2}$$

additions (we used the summation formula 2. of Theorem 4.2, p.260 LHE). For example, if $m = 1000$, then the recursive formula requires 1000 additions, whereas the definition requires 495,000 additions! (For most series, computation of a_i values takes more time than does their addition. With respect to this criterion, again, the recursive formula is far more efficient.)

Example 1

- (a) For $x = 1$, use a recursive assignment to define the first 50 partial sums of the infinite series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Tabulate the first 10 terms of the series along with the corresponding partial sums. Plot these values on the same graph against the index.

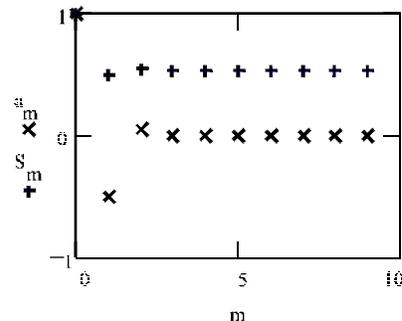
Estimate the sum of the infinite series to 4 decimal place accuracy.

Solution

$$x := 1 \quad n := 1..49 \quad a_n := \frac{(-1)^n \cdot x^{2n}}{(2 \cdot n)!} \quad a_0 := 1 \quad S_0 := a_0 \quad S_n := S_{n-1} + a_n$$

m := 0..9

m	a_m	S_m
0	1	1
1	-0.5	0.5
2	0.0416666666666667	0.5416666666666667
3	-0.0013888888888889	0.5402777777777778
4	0.000024801587302	0.540302579365079
5	-0.000000275573192	0.540302303791887
6	0.000000002087676	0.540302305879563
7	-0.0000000000011471	0.540302305868092
8	0.000000000000048	0.54030230586814
9	0	0.54030230586814



Note: We changed the global settings in the Numerical Format box (selected from the **Math** menu):

- * Displayed Precision: 15
- * Exponential Threshold: 15
- * Zero Tolerance: 15

The sum of the infinite series appears to be approximately 0.5403

- (b) In Problem 3(b), you will prove that the infinite series of part (a) converges (i.e. has a

”sum”) for any number x . Thus, the infinite series defines a function of x whose range value is the sum of the infinite series corresponding to a given domain value x . Therefore, the n -th partial sum should approximate this function with increasing accuracy as n increases. What does the graph of this function look like for x on the interval $[-10, 10]$? What familiar function do you think it is?

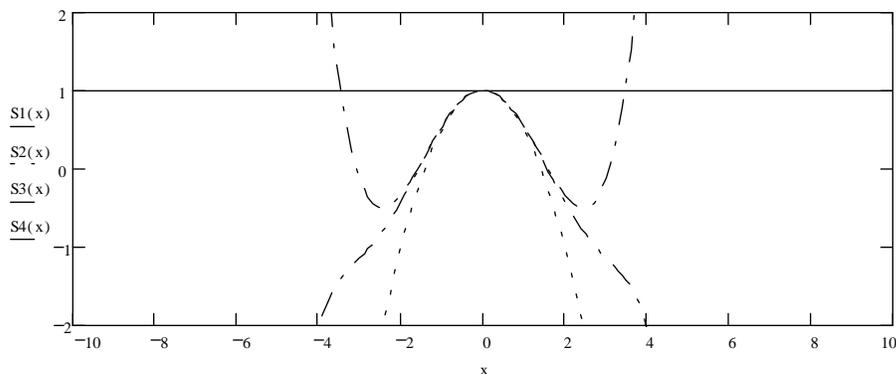
Partial Solution

We plot the first four partial sums as functions of x and restrict the y-range to obtain a useful picture.

$$S1(x) := \sum_{n=0}^0 \frac{(-1)^n \cdot x^{2n}}{(2n)!} \qquad S2(x) := \sum_{n=0}^1 \frac{(-1)^n \cdot x^{2n}}{(2n)!}$$

$$S3(x) := \sum_{n=0}^2 \frac{(-1)^n \cdot x^{2n}}{(2n)!} \qquad S4(x) := \sum_{n=0}^3 \frac{(-1)^n \cdot x^{2n}}{(2n)!}$$

`x:=-10,-9.9..10`



Note: You are asked to complete the solution to this example in Problem 3.

Example 2

Obtain the 3rd degree Taylor polynomial $P_3(x)$ of $\sin(x)$ about the center $c = 2$. Graph $P_3(x)$ and $\sin(x)$ on the same plot for x on the interval $[-1, 5]$.

Solution

First obtain the 3rd degree Taylor polynomial of $\sin(x + 2)$ about $c = 0$. To do this, select the variable x , then choose **Expand to Series...** and specify 4 as the Order of Approximation:

$$\sin(x+2)$$

converts to the series

$$\sin(2) + \cos(2) \cdot x + \left(\frac{-1}{2} \cdot \sin(2)\right) \cdot x^2 + \left(\frac{-1}{6} \cdot \cos(2)\right) \cdot x^3 + O(x^4)$$

Mathcad generates a Taylor series (discussed in Section 8.10 LHE), rather than a Taylor polynomial. Thus, $O(x^4)$ is a symbol for all the terms of degree 4 and higher that were not included in the series. Since we only want the 3rd degree Taylor polynomial, we should delete $O(x^4)$. When we do this and use **Substitute for Variable** to replace x with $x - 2$, we obtain $P_3(x)$:

$$\sin(2) + \cos(2) \cdot x + \left(\frac{-1}{2} \cdot \sin(2)\right) \cdot x^2 + \left(\frac{-1}{6} \cdot \cos(2)\right) \cdot x^3$$

by substitution, yields

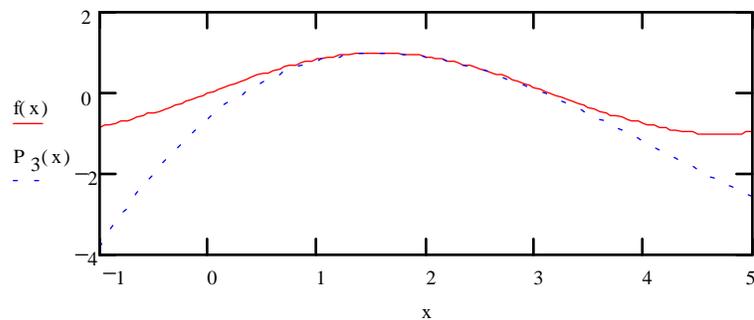
$$\sin(2) + \cos(2) \cdot (x-2) - \frac{1}{2} \cdot \sin(2) \cdot (x-2)^2 - \frac{1}{6} \cdot \cos(2) \cdot (x-2)^3$$

Therefore,

$$f(x) := \sin(x)$$

$$P_3(x) := \sin(2) + \cos(2) \cdot (x-2) - \frac{1}{2} \cdot \sin(2) \cdot (x-2)^2 - \frac{1}{6} \cdot \cos(2) \cdot (x-2)^3$$

$$x := -1, -.95..5$$



Problems

- (a) For $r = 0.5$, use a recursive assignment to define the first 101 partial sums of the geometric series

$$\sum_{n=0}^{\infty} r^n$$

(Note that $S_0 = 1 \neq 0$.) Tabulate and plot the terms of the series and the corresponding

partial sums for $n = 1, 2, \dots, 10$. Follow the template of part (a) of Example 1, including the global **Numerical Format** settings.

- (b) Copy-and paste five copies of your entire answer to part (a) into your document. In the first copy, change the value of r to 0.9, in the second copy to 1, in the third to 1.5, in the fourth to -0.5 , and in the fifth to -1 .
 - (c) Of the six geometric series in parts (a) and (b), identify the series for which divergence is predicted solely on the basis of the n -th *Term Test*. For these series, can their divergence also be inferred solely on the basis of the computed values of their partial sums obtained in part (b)? Briefly explain why.
 - (d) By hand or Mathcad, use the sum formula for a geometric series to obtain the exact sum of each convergent geometric series in parts (a) and (b). For these series, compute additional partial sums (say, $S_{25}, S_{50}, S_{75}, S_{100}$) to verify that they approach the exact sum for large n .
2. (a) Position the pointer (the red cross) below your answer to the Problem 1. From the **File** menu select **Insert ...**; then, in the dialog box, select the file **SERIES1.MCD** (your instructor will indicate which drive and directory should be selected). You will see the rendering of the first ten partial sums of the geometric series of Problem 1(a). Notice that on the graph on the right side, the arrows representing the terms of the series are "stacked-up", and the horizontal axis represents r .
- 
- Change the value of r to each value specified in Problem 1(b), and watch how both graphs are affected. Describe what you see.
- (b) Copy the graph on the right side from the solution to part (a), and paste it below. Above the copy you've just created, define: $r := -1.5, -1 .. 1.5$. You should understand that your graph now depicts terms of the geometric series corresponding to seven different values of r . Now, redefine r to be the range: $-2, -1.9 .. 2$. Resize the graph - make it much wider. This time, forty-one geometric series are represented, each with different r . Click on the graph - you will see that six functions are plotted to generate the graph (do not worry now about the technicalities involved in creating this graph - ask your instructor later if you are interested). At the bottom of the list of functions plotted, add the expression:

$$\frac{1}{1-r}$$

When you click the left mouse button outside the graph, this expression will be plotted in black. Explain how this new graph is related to the rest of the picture.

3. This problem refers to the series in Example 1:
- (a) Repeat part (a) of Example 1 for $x = 10$. In addition, tabulate the terms of the series and the corresponding partial sums with $n = 5, 10, \dots, 50$ and then plot these values on the same graph for $n = 1, 2, \dots, 50$. Estimate the sum of the infinite series to 4 decimal place accuracy.

- (b) Using pencil and paper, apply an appropriate test for convergence in order to conclude that the infinite series in part (a) is convergent for *any* value of x . Are the numerical results obtained in part (a) of this problem and Example 1 consistent with this conclusion?
- (c) Plot the 12th, 13th, 14th and 15th partial sums against x .
- (d) What familiar function do you think we are approximating, i.e., values of which function do these partial sums appear to be converging towards? Verify that your guess agrees with the numerical results obtained in part (a) of this problem and Example 1.
- (e) The infinite series of Example 1(a), taken as a function of x , is actually a Taylor series. Evaluate *symbolically* its fourth partial sum, written as:

$$\sum_{n=0}^3 \frac{(-1)^n x^{2n}}{(2n)!}$$

Now, type the function of x which you gave as the answer in part (d), have the blue box contain x in this function, then from the **Symbolic** menu select **Expand to Series...** Choose the appropriate value of "Order of Approximation" to obtain an exact match with the polynomial that resulted from symbolically evaluating the fourth partial sum above.

4. (a) Have Mathcad find the 8th Maclaurin polynomial for $f(x) = \frac{1}{1-x}$ (see Example 2) and define it as a function $M8(x)$. Also, define $M3(x)$ to be the third Maclaurin polynomial for the same function (you may just copy a few terms from $M8$). Plot $f(x)$, $M3(x)$ and $M8(x)$ on the same graph, with $x = -2, -1.9, \dots, 6$, making sure you select "trace type" "draw" for $f(x)$ so that the vertical asymptote is not rendered. Restrict the vertical range to obtain a useful graph.

How does this graph compare to the final graph you obtained in Problem 2(b)?

Do our Maclaurin polynomials of order n appear to get closer and closer to $f(x)$ as n increases?

If your answer was yes, then do they get closer and closer to $f(x)$ for *all* x , or only *some* x values? (which ones?)

- (b) Follow the procedure outlined in Example 2, to find the 8th Taylor polynomial for $f(x) = \frac{1}{1-x}$ centered at $c = 3$. (Remember, you must first type in the *expression* for $f(x+3)$, expand it to a series with desired number of terms, and substitute $x-3$ for x into the answer.) Define this polynomial as a function $T8(x)$. Define $T3(x)$ to be the 3rd Taylor polynomial for $f(x)$ centered at the same c (again, you may just copy a few terms from $T8$). Copy the graph from your answer to part (a), and paste it. Add the functions $T3(x)$ and $T8(x)$ to the list of functions being plotted.

Do our Taylor polynomials centered at $c = 3$, appear to get closer and closer to $f(x)$ as n increases?

If your answer was yes, then do they get closer and closer to $f(x)$ for *all* x , or only *some* x values? (which ones?)

5. (a) For $n = 1, 2, \dots, 2000$, create the partial sum sequences S_n , T_n , and U_n for the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.995}} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1.005}}$$

Use a recursive assignment for the partial sums (see Comment 1 and Example 1(a)).

Create a range variable $m=1000, 1100..2000$ and tabulate m , S_m , T_m and U_m . From the data, are you able to determine the behavior of any of the sequences?

- (b) From your knowledge of p -series which, if any, of the series converge?

- (c) Recall that (page 570 LHE) $\int_1^n f(x)dx \leq \sum_{k=1}^{n-1} f(k) \leq \sum_{k=1}^n f(k)$ (i.e. the n -th partial sum is greater than the integral value).

For each convergent series in part (a), use the symbolic processor to evaluate $\int_1^n f(x) dx$.

Determine the limit (as $n \rightarrow \infty$) of this integral. After 2000 terms, is the convergent series close to this limit?

6. (a) Use Mathcad to show that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ satisfies the Alternating Series Test (page 582 LHE) and, therefore, is convergent.
- (b) Use Theorem 8.15 (p. 584 LHE) to determine how many terms we need to take before our partial sum approximates the exact sum of the series to 2 decimal places. (Use Mathcad to do the arithmetic for this problem.)
- (c) Suppose Mathcad can create 1000 terms in our partial sum sequence every second. How long would it take Mathcad to calculate the number of terms needed in part (b)? Use an appropriate unit of time measurement, e.g. seconds, hours, days etc.

8.3 Homework Help

- When solving exercises related to the convergence of sequences, you may want to use Mathcad to illustrate your answers (using a table of values and/or a graph)
 - Exercises 35-54 p.557 LHE
 - Exercises 3-10 p.631 LHE
- You may try plotting and/or tabulating partial sums of the series, whose convergence/divergence is to be discussed in these exercises:
 - Exercises 7-22, 41-52 p.566 LHE

- Exercises 1-32 p.573 LHE
 - Exercises 1-26 p.579 LHE
 - Exercises 1-36 p.587 LHE
 - Exercises 5-24, 27-52 p.594 LHE
 - Exercises 27-36 p.632 LHE
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- In exercises involving Taylor polynomials, try plotting the polynomial you obtained and the original function on the same graph.
 - Exercises 1-20 p.604 LHE
 - Exercises 1-10 p.630 LHE

9 Conic Sections

9.1 Homework Help

You can use the Maple template `IMPLICIT.MS` to sketch graphs of the various conic sections discussed in this chapter. Adjust the x and y ranges to view the region of interest to you.

