13 Functions of Several Variables

13.1 Activity: Partial Derivatives and Chain Rules

Prerequisites: Read Sections 13.1-13.5 LHE.

The objective of this activity is to numerically and symbolically evaluate partial derivatives in Mathcad. We will then use this feature to verify chain rules and perform implicit differentiation.

Instructions

After reading the comments and studying the worked examples, open a blank Mathcad document and create your report there. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

Comments

1. Mathcad is capable of numerically and symbolically dealing with partial derivatives. In particular, the Mathcad symbol $\frac{d}{dx}$ assumes the role of a partial differentiation operator whenever applied to a function of many variables. For example, here we symbolically obtain the expression for f_x for $f(x,y) = x^2y + \sin x \cos y$:

$$\frac{d}{dx} \left(x^2 \cdot y + \sin(x) \cdot \cos(y) \right) \quad \text{ yields } \quad 2 \cdot x \cdot y + \cos(x) \cdot \cos(y)$$

2. It is acceptable mathematical notation to use the same letter to denote a variable (dependent or intermediate) and the corresponding function simultaneously, e.g., y = y(x). Unfortunately, this cannot be done in Mathcad, where once the name is used to define a function, it cannot be used as a variable (without redefining), and vice versa.

To circumvent this problem, in this section, whenever necessary, we adopt the convention where lowercase letters will be variable names, and uppercase letters denote functions (using this convention, the equality above would be rewritten as y = Y(x)).

Examples

- 1. Consider the function $F(x, y) = x \sin(2y^2) + \cos(x+y)$.
 - (a) Define the functions Fx(x, y), Fy(x, y) representing the first partials of F, as well as the second-partial functions Fxx(x, y), Fxy(x, y), Fyx(x, y), Fyy(x, y). Evaluate all the functions at (x, y) = (1, 2). Verify Theorem 13.3 p. 857 LHE at that point.

(b) Symbolically obtain the expressions for F_x , F_y , F_{xy} and F_{yx} . Verify Theorem 13.3 p. 857 LHE for general x and y values.

Solution

(a)
$$F(x,y) := x \sin(2 \cdot y^2) + \cos(x+y)$$

First partials:
$$Fx(x,y) := \frac{d}{dx}F(x,y)$$
 $Fx(1,2) = 0.848$
 $Fy(x,y) := \frac{d}{dy}F(x,y)$ $Fy(1,2) = -1.305$

Second partials:
$$Fxx(x,y) := \frac{d}{dx} \frac{d}{dx} F(x,y)$$
 $Fxx(1,2) = 0.99$

Fxy(x,y) :=
$$\frac{d}{dy}\frac{d}{dx}F(x,y)$$
 Fxy(1,2) = -0.174

$$Fyx(x,y) := \frac{d}{dxdy} F(x,y) \qquad Fyx(1,2) = -0.174$$
$$Fyy(x,y) := \frac{d}{dydy} F(x,y) \qquad Fyy(1,2) = -62.911$$

Fxy(1,2) = -0.174 equals Fyx(1,2) = -0.174

- verifies Theorem 13.3 p. 857 LHE

- (b) First partials:
- $$\begin{split} & \mathsf{F}_{x}: \quad \frac{d}{dx} \Big(x \cdot \sin \Big(2 \cdot y^{2} \Big) + \cos (x + y) \Big) \qquad \text{yields} \qquad \sin \Big(2 \cdot y^{2} \Big) \sin (x + y) \\ & \mathsf{F}_{y}: \quad \frac{d}{dy} \Big(x \cdot \sin \Big(2 \cdot y^{2} \Big) + \cos (x + y) \Big) \qquad \text{yields} \qquad 4 \cdot x \cdot \cos \Big(2 \cdot y^{2} \Big) \cdot y \sin (x + y) \end{split}$$

Mixed second partials

$$\begin{aligned} \mathsf{F}_{xy} &: \quad \frac{\mathrm{d}}{\mathrm{d}\,y} \Big(\sin\Big(2 \cdot y^2\Big) - \sin(x + y) \Big) \quad \text{yields} \quad 4 \cdot \cos\Big(2 \cdot y^2\Big) \cdot y - \cos(x + y) \\ \mathsf{F}_{yx} &: \quad \frac{\mathrm{d}}{\mathrm{d}\,x} \Big(4 \cdot x \cdot \cos\Big(2 \cdot y^2\Big) \cdot y - \sin(x + y) \Big) \quad \text{yields} \quad 4 \cdot \cos\Big(2 \cdot y^2\Big) \cdot y - \cos(x + y) \end{aligned}$$
(equal)

An equivalent way to evaluate the above:

$$\begin{aligned} \mathsf{F}_{xy} &: \quad \frac{d}{dy} \frac{d}{dx} \Big(x \cdot \sin \Big(2 \cdot y^2 \Big) + \cos (x + y) \Big) & \text{ yields } & 4 \cdot \cos \Big(2 \cdot y^2 \Big) \cdot y - \cos (x + y) \end{aligned} \\ \mathsf{F}_{yx} &: \quad \frac{d}{dx} \frac{d}{dy} \Big(x \cdot \sin \Big(2 \cdot y^2 \Big) + \cos (x + y) \Big) & \text{ yields } & 4 \cdot \cos \Big(2 \cdot y^2 \Big) \cdot y - \cos (x + y) \end{aligned}$$

2. Given $w = (\sin x)e^y$, $x = s^3t + s$ and y = s - t numerically verify that at s = 1 and t = -1, the partial derivative w_s obtained using the chain rule equals to that calculated directly. Solution

$$\begin{split} W(x,y) &:= \sin(x) \cdot e^{y} \\ X(s,t) &:= s^{3} \cdot t + s \qquad Y(s,t) := s - t \qquad F(s,t) := W(X(s,t),Y(s,t)) \\ s &:= 1 \qquad t := -1 \\ x := X(s,t) \qquad x = 0 \\ y &:= Y(s,t) \qquad y = 2 \end{split}$$

$$\begin{aligned} \text{Chain Rule:} \qquad \left(\frac{d}{dx}W(x,y)\right) \cdot \left(\frac{d}{ds}X(s,t)\right) + \left(\frac{d}{dy}W(x,y)\right) \cdot \left(\frac{d}{ds}Y(s,t)\right) = -14.778 \\ \text{Directly:} \qquad \frac{d}{ds}F(s,t) = -14.778 \end{aligned}$$

3. Consider the equation $x^3y - zy^2 + 3xz^2 - 2x + 3z = 0$. Symbolically evaluate the expressions for z_x , z_y and z_{xy} .

Solution

$$\frac{x^{3} \cdot y - z \cdot y^{2} + 3 \cdot x \cdot z^{2} - 2 \cdot x + 3 \cdot z}{\left[\frac{d}{dx}(x^{3} \cdot y - z \cdot y^{2} + 3 \cdot x \cdot z^{2} - 2 \cdot x + 3 \cdot z)\right]}{\frac{d}{dz}(x^{3} \cdot y - z \cdot y^{2} + 3 \cdot x \cdot z^{2} - 2 \cdot x + 3 \cdot z)}$$
 yields $\frac{(-3 \cdot x^{2} \cdot y - 3 \cdot z^{2} + 2)}{(-y^{2} + 6 \cdot x \cdot z + 3)}$
$$-\frac{\left[\frac{d}{dy}(x^{3} \cdot y - z \cdot y^{2} + 3 \cdot x \cdot z^{2} - 2 \cdot x + 3 \cdot z)\right]}{\frac{d}{dz}(x^{3} \cdot y - z \cdot y^{2} + 3 \cdot x \cdot z^{2} - 2 \cdot x + 3 \cdot z)}$$
 yields $\frac{(-x^{3} + 2 \cdot z \cdot y)}{(-y^{2} + 6 \cdot x \cdot z + 3)}$

Another way to obtain the above expressions:

Use **Substitute for Variable** to substitute Z(x,y) for z

$$x^{3} \cdot y - (Z(x,y)) \cdot y^{2} + 3 \cdot x (Z(x,y))^{2} - 2 \cdot x + 3 \cdot Z(x,y)$$

Highlight x (in a blue box) and use Differentiate on Variable

by differentiation, yields

$$3 \cdot x^2 \cdot y - \frac{d}{dx} Z(x,y) \cdot y^2 + 3 \cdot Z(x,y)^2 + 6 \cdot x \cdot Z(x,y) \cdot \frac{d}{dx} Z(x,y) - 2 + 3 \cdot \frac{d}{dx} Z(x,y)$$

 $\textit{Highlight} \quad \frac{d}{dx}Z(x,y) \quad \textit{ and use } \textit{Collect on Subexpression}$

by collecting terms, yields

$$(-y^{2}+6\cdot x \cdot Z(x,y)+3)\cdot \frac{d}{dx}Z(x,y)+3\cdot x^{2}\cdot y+3\cdot Z(x,y)^{2}-2$$

Now, we can solve easily by hand: $z_x = \frac{-(3 \cdot x^2 \cdot y + 3 \cdot z^2 - 2)}{(-y^2 + 6 \cdot x \cdot z + 3)}$

(This step could be done automatically if we first replace Z(x,y) and the derivative of Z(x,y) with a variable name, highlight the derivative name and use **Solve for Variable**)

We could repeat this to obtain z_v

To obtain the second partial, z_{xy} , only the second way remains available.

$$\left(-y^2+6\cdot x\cdot Z(x,y)+3\right)\cdot \frac{d}{dx}Z(x,y)+3\cdot x^2\cdot y+3\cdot Z(x,y)^2-2$$

Highlight y

by differentiation, yields

$$\left(-2\cdot y+6\cdot x\cdot \frac{d}{dy}Z(x,y)\right)\cdot \frac{d}{dx}Z(x,y)+\left(-y^2+6\cdot x\cdot Z(x,y)+3\right)\cdot \frac{d}{dy}\frac{d}{dx}Z(x,y)+3\cdot x^2+6\cdot Z(x,y)\cdot \frac{d}{dy}Z(x,y)$$

Replace Z(x,y) and all of its partials with variable names:

$$(-2 \cdot y + 6 \cdot x \cdot Zy) \cdot Zx + (-y^2 + 6 \cdot x \cdot z + 3) \cdot Zxy + 3 \cdot x^2 + 6 \cdot z \cdot Zy$$

Highlight Zxy and useSolve for Variable

has solution(s)
$$\frac{-(-2\cdot Zx \cdot y + 6\cdot Zx \cdot x \cdot Zy + 3\cdot x^{2} + 6\cdot z \cdot Zy)}{(-y^{2} + 6\cdot x \cdot z + 3)}$$

Now substitute the first partial expressions for Zx and Zy:

stitution, yields
$$\frac{\left[2 \cdot \frac{\left(-3 \cdot x^{2} \cdot y - 3 \cdot z^{2} + 2\right)}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)} \cdot y - 6 \cdot \frac{\left(-3 \cdot x^{2} \cdot y - 3 \cdot z^{2} + 2\right)}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)} \cdot x \cdot Zy - 3 \cdot x^{2} - 6 \cdot z \cdot Zy}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)}\right]}$$

by sub

by substitution, yields

$$\frac{\left[2 \cdot \frac{\left(-3 \cdot x^{2} \cdot y - 3 \cdot z^{2} + 2\right)}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)} \cdot y - 6 \cdot \frac{\left(-3 \cdot x^{2} \cdot y - 3 \cdot z^{2} + 2\right)}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)^{2}} \cdot x \cdot \left(-x^{3} + 2 \cdot z \cdot y\right) - 3 \cdot x^{2} - 6 \cdot z \cdot \frac{\left(-x^{3} + 2 \cdot z \cdot y\right)}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)}\right]}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)}$$

The final expression for z_{xy} .

simplifies to

$$\frac{-\left(-12 \cdot y + 27 \cdot x^{2} - 30 \cdot x^{3} \cdot y^{2} \cdot z + 72 \cdot y \cdot z^{3} \cdot x - 18 \cdot z^{2} \cdot y^{3} + 54 \cdot y \cdot z^{2} - 3 \cdot x^{2} \cdot y^{4} + 90 \cdot x^{3} \cdot z + 4 \cdot y^{3} - 12 \cdot x^{4} + 18 \cdot x^{6} \cdot y + 90 \cdot z^{2} \cdot x^{4}\right)}{\left(-y^{2} + 6 \cdot x \cdot z + 3\right)^{3}}$$

Problems

In Problems 1-3 find the second partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ by hand, then verify your answer using Mathcad (symbolically or numerically).

- 1. $z = x\cos y + y\sin x$
- 2. z defined in Exercise 40 p.859 LHE
- z defined in Exercise 42 p.859 LHE 3.

4. Solve Exercise 58 p.859 LHE by hand, then verify all third partial calculations using Mathcad's symbolics.

In Problems 5-7 use Mathcad to numerically verify that $\partial w/\partial s$ and $\partial w/\partial t$ obtained using the appropriate chain rule, agree with the values obtained directly. Test for a few different sets of values s and t.

- 5. $w = \sin x, \ x = e^t + \ln s$
- 6. $w = x^4 y \sqrt{y}, \ x = s + t, \ y = s t$
- 7. w = xyz + x + y + z, $x = s \cos t$, $y = s \sin t$, z = s
- 8. Solve Exercise 20 p.875 LHE by hand, then verify your answer using Mathcad's symbolics.
- 9. Solve Exercise 24 p.875 LHE by hand, then verify your answer using Mathcad's symbolics.

13.2 Activity: Normal Lines, Tangent Planes and Extrema

Prerequisites: Read Sections 13.7-13.8 LHE

During this activity you will use Maple to illustrate tangent planes, normal lines, and extrema for functions of several variables.

Instructions

Open a blank Mathcad document and create your report there. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

Problems

- 1. (a) Define the surface $f(x,y) = (\frac{1}{2} x^2 + y^2) e^{1-x^2-y^2}$ in Mathcad. Using Mathcad, define the partial derivatives of f(x,y) and evaluate them at P = (1.25, 0, f(1.25, 0)). Find the equation of the tangent plane to the surface f(x,y) at P and arrange it in the form z = g(x,y). Type this equation into your Mathcad document.
 - (b) Open the Maple file TANPLANE.MS. In the file, the surface and a plane are defined. The plane, however, is not the tangent plane. Redefine the plane in the Maple document to be the tangent plane, and execute the file. Copy and paste the plot into your Mathcad document. Don't exit from Maple (You should close ALL Maple plot windows.).



2. (a) By hand, determine the equation of the normal line to the surface $x^2 + y^2 = 2z^2$, at the point P = (-1, 1, 1). Type the parametric form of the equation of the normal line into your Mathcad document.

- (b) Open the Maple file NORMLINE.MS. The surface (in cylindrical form), and a line are defined. Change the parametric equations of the line L to generate the normal line determined in part (a). Execute the code, and then rotate the graph (using the mouse or the arrow keys) so that it is clear that this line is normal to the surface. Next, notice that the normal line appears to intersect the surface at another point Q. By hand, determine where the normal line and the surface intersect. Type your answer into your Mathcad document. Redefine the point Q in the Maple code to be the new point you just calculated. Execute the code, and then rotate the graph (using the mouse or the arrow keys) so that it is clear that the line does pass through P and Q. Copy and paste the plot into your Mathcad document. Don't exit from Maple (You should close ALL Maple plot windows.).
- 3. (a) By hand find the five critical points for $f(x, y) = xye^{-(x^2+y^2)/2}$ and type these into your Mathcad document. You may want to use the symbolics in Mathcad to help find the partial derivatives of f(x, y).
 - (b) Using theorem 13.17 on page 901 LHE, determine which of the critical points are relative maximum, relative, minimum, or saddle points.
 - (c) Open the Maple file MAXMIN3D.MS and execute the code. You will notice that the points MAX, MIN, and S (for saddle) are NOT in the correct place. Change the points in the Maple code so that they will be placed correctly. Rotate the surface to best view this and copy it to Mathcad.

13.3 Homework Help

- In the following exercises you can use Mathcad's numerical and/or symbolic partial differentiation to verify your answers (see Example 1 of Section 13.1):
 - Exercises 1-20, 37-44, 49-58 p.858 LHE
 - Exercises 11-26 p.922 LHE
- The Maple template file MAXMIN3D.MS can be used to visualize extrema of functions of two variables in
 - Exercises 5-24 p.904 LHE
 - Exercises 53-56 p.923 LHE

(note that you may want to use the Maple illustration even for those problems for which a graph is given in the text - unlike the plots in the book, Maple plots can be viewed from different



angles)

Appendix A Mathcad Summary

A.1 Menu items with button and keyboard shortcuts

File Menu Items

File menu item	button	key	comments
New	D	F7	open a new (empty) document
Open		F5	open an <i>existing</i> document
Save		F6	save document using the $current$ name
Save As			save document under a <i>different</i> name
Insert			insert a document into the current one
Close		Ctrl+F4	close the current document
Print	ð	Ctrl+O	print current document

Edit Menu Items

Edit menu item	button	key	comments
Undo Last Edit	×¥	Alt+Bksp	(note: to undo "Cut" use "Paste")
Cut		Ctrl+X	cut current selection to the clipboard
Сору	ee:	Ctrl+C	copy current selection to the clipboard
Paste	Ê	Ctrl+V	paste from the clipboard
View Regions			highlight the regions
F ind		Ctrl+F5	find a string in current document

Miscellaneous Items

menu	button	key	comments
\mathbf{T} ext Create Text \mathbf{R} egion			click outside the region to finish
$\mathbf{G} \text{raphics} \dashrightarrow \mathbf{C} \text{reate } \mathbf{X} \textbf{-} \mathbf{Y} \text{ Plot}$	1	0	see p.11
\mathbf{W} indow> \mathbf{R} efresh		Ctrl+R	redraw the screen
Help Index	7	$\mathbf{F1}$	

A.2 Mathematical Operators

button	key	operator
	+ - * /	addition, subtraction (or negation), multiplication, division
1 × ×	^	power
1 ,	\	square root
1 x		absolute value
1 5	Ctrl+Shift+4	summation
3 2	\$	range sum
1 🛃	?	derivative
3 4	Ctrl+?	<i>n</i> -th derivative $(n \le 5)$
1	&	definite integral
3. [Ctrl+I	indefinite integral (symbolic mode only)
1 8=7	Ctrl+=	
2 ≠	Ctrl+3	These six operators:
	> <	$=, \neq, >, <, \geq, \leq$
2 >	Ctrl+0	can be used in the "if" function
2 ≾	Ctrl+9	and in solve blocks

A.3 Mathematical Functions

Trigonometric functions

function	Mathcad form
$\sin x$	sin(x)
$\cos x$	cos(x)
$\tan x$	tan(x)
$\cot x$	cot(x)
$\sec x$	sec(x)
$\csc x$	csc(x)

Inverse trigonometric functions

function	Mathcad form
$\arcsin x$	asin(x)
$\arccos x$	acos(x)
$\arctan x$	atan(x)

Note that the remaining three inverse trigonometric functions are not built into Mathcad. One can define them using the identities

$$\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$$
 $\operatorname{arcsec} x = \operatorname{arccos} \frac{1}{x}$ $\operatorname{arccsc} x = \operatorname{arcsin} \frac{1}{x}$

Hyperbolic functions

function	Mathcad form
$\sinh x$	sinh(x)
$\cosh x$	cosh(x)
$\tanh x$	tanh(x)
$\coth x$	coth(x)
$\operatorname{sech} x$	sech(x)
$\operatorname{csch} x$	$\operatorname{csch}(x)$

Miscellaneous functions

function	Mathcad form
e^x	exp(x) or e ^x
$\ln x$	ln(x)
$\log_{10} x$	log(x)
$\lfloor x \rfloor$	<pre>floor(x)</pre>
$\lceil x \rceil$	ceil(x)

The functions $\lfloor x \rfloor$ and $\lceil x \rceil$ are defined as

 $\lfloor x \rfloor$ = greatest integer not exceeding x $\lceil x \rceil$ = least integer no smaller than x.

The function $\lfloor x \rfloor$ is discussed on p. 86 LHE (the notation used there is $\llbracket x \rrbracket$).

Also note that the only logarithmic functions defined in Mathcad are those to bases e and 10. To obtain a logarithm to different base, use the definition (p. 354 LHE)

$$\log_b x = \frac{1}{\ln b} \ln x$$

A.4 Built-in constants

constant	button	key	value
π	1 Л	Ctrl+P	3.14159 (15-digit approximation)
e		е	2.71828 (15-digit approximation)
∞	5 😋	Ctrl+Z	10^{307} (number larger than any number representable in Mathcad)

A.5 Additional Mathcad operations and functions

	operation	button	key	example	
	define	1 🚛	:	x:=3	
	evaluate	1 =	=	x+5=8	
	range	1. m.n	;	x:=1,1.1	2
fune	function		desc	ription	
if(condition,a,b)		See a	an example on p	page 14	
root(expression, variable)		See a	an example on p	page 11	
GivenFind solve block		See a	an example on p	bage 101	

Appendix B Seven Things to Avoid When Working in Mathcad

On the following list, you will find pitfalls that students often encounter. They range in severity from minor to major, but they all have one thing in common: they can be easily avoided. If you decide to ignore this list, then remember: you have been warned!

1. You are trying to evaluate $2\sin\frac{\pi}{5}$.



Explanation: Mathcad does not recognize 2x or even (2)(x) as products. Always use the multiplication operator (*) when multiplying.

2. Suppose you've begun entering an expression: 4+5 and then went on to do some other things in your document. Now, we want to finish typing the expression 4+5-1 and evaluate it.

WRONG	RIGHT
4+5-1=-1	4 + 5 - 1 = 8

What happened here? Here's a hint - a snapshot of the screen obtained with **View Regions** (from the **Edit** menu)switched ON:

WRONG	RIGHT
4 + 5 - 1 = -1	4 + 5 - 1 = 8

Explanation: When you are about to modify an existing expression, you must first click the left mouse button within it. If you do, you will see a vertical insertion bar, or a blue frame just where you clicked. If you see a small red cross instead, that means you clicked away from a region - if you start typing, you will be entering a new expression (that must have happened on the left above).

3. Define $x = 2\pi$ and evaluate x.

WRONG		RIGHT	
$\mathbf{x} := 2 \cdot \pi$	$\cos(x) = 0.54$	$\mathbf{x} := 2 \cdot \pi$	$\cos(x) =$

Explanation: Mathcad reads its documents from top to bottom, left to right. Bearing this in mind, you must define variables and function BEFORE they are used in any calculations, graphs, etc.

1

4. Graph $\cos x$ over the interval [-1, 1].



Explanation: The default step size for range variables is one. While this might be satisfactory in some cases, it frequently is not, thus, you should get into a habit of specifying all three numbers (first, second and last) in every range variable you define.

- 5. Symptom: "I clicked somewhere at the bottom of the screen, and everything disappeared!" *Explanation:* Look at the horizontal scroll bar at the bottom of Mathcad's screen. If the scroll button is at its right end, then click on the middle of the bar. This should restore the scroll button to its leftmost position, and will bring all your work "back". (This strange behavior results from the fact that Mathcad's documents are TWO PAGES wide.)
- 6. Symptom: "I swear I saved the document, but when I'm now trying to open it, it's not there". Explanation: You might have changed Mathcad's default extension "MCD" to a different one when saving your document. In order to see all the documents in your directory (rather than only ones with "MCD" extension), in the "Open Document" dialog's "File" line (top left corner) delete all of its contents, and replace with *.* (asterisk dot asterisk). If you still don't see your file, ask for assistance.
- 7. *Symptom:* "We worked on this report for three hours, and the system just crashed. We lost all our work!!!"

Explanation: You should save your work frequently, so that even if the system crashes, you don't lose a lot of it.