THE BRACHISTOCHRONE ON A ROTATING EARTH

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References:

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In 1696 Johann Bernoulli challenged the mathematicians of his day to find the curve of quickest descent for a body to fall through a frictionless homogeneous Earth between two given points in a vertical plane [Calinger, 1995]. The problem languished for months until Newton chanced upon the problem, quickly solved it, and published the solution anonymously. But Bernoulli recognized the "lion by its claw" – the path is a cycloid. We have shown the solution of this problem by various authors in previous communcications. We will instead consider an extension of the problem, that is, to fall through the rotating Earth with a variable gravity. A complete solution was first presented by Simoson in 2010, and the path is a trochoid [Simoson, 2010, 2007].

We will trace the development of the solution to this problem, including a historical background, and displaying the equations of motion. Our free falling pebble may be realized by a black hole with the mass of a mountain and an event horizon of an atomic nucleus, from a recent suggestion from Hawking [1998].

We will briefly review the properties of the cycloid and the trochoid as they relate to the free fall through a rotating Earth. Examples will be provided for the case of motions in an equatorial plane, and those initiating in the northern hemisphere. The path for the former is a hypocycloid. For the latter, it turns out that the motion is not planar, but rather is three-dimensional whose projection on the *xy*-plane is either a hypotrochoid or an epitrochoid depending on the rate of rotation.

The cycloid is the curve traced out by a point on the circumference of a circle rolling without slipping on a straight line. Its parametric equations are:

$$x = r(\theta - sin\theta), \qquad y = r(1 - cos\theta),$$

where *r* = the radius of the circle, and θ is the rotation angle ($\theta = 0$ when the tracing point *P* is at the origin) (Figure 1).

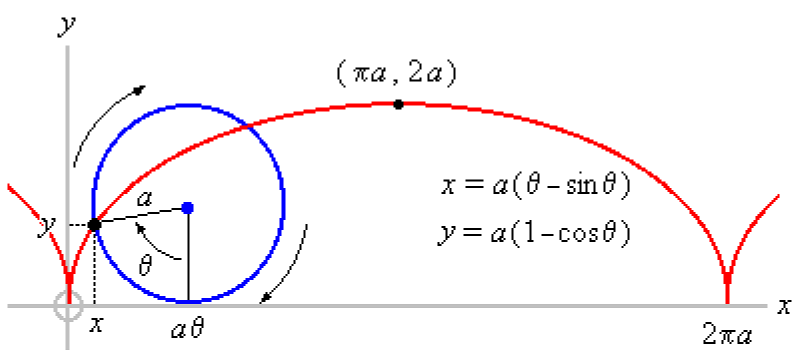


Figure 1. The cycloidal path

Two noteworthy points about the cycloidal track are that: (1) Of all curves joining the two points *A* and *B* in a vertical plane, the cycloid gives the shortest time of descent, a brachistochrone; and (2) The cycloid is also a tautochrone, that is, no matter where the particle *P* is placed on an inverted cycloid, it takes the same time to slide to the bottom.

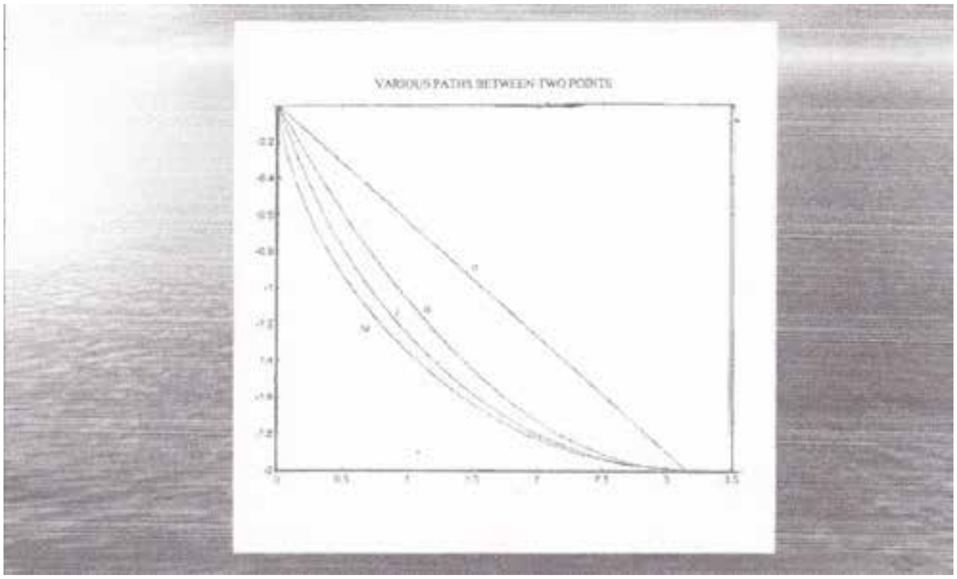


Figure 2. Shortest descent time

I. Cycloid, $a = 15.0\ ft; T_0 = \pi\sqrt{\frac{a}{g}} = 2.144\ s$

II. Straight line; $T = 1.185T_0 = 2.541\ s$
III. Parabola; $T = 1.160T_0 = 2.487\ s$
IV. Circle; $T = 1.149T_0 = 2.463\ s$

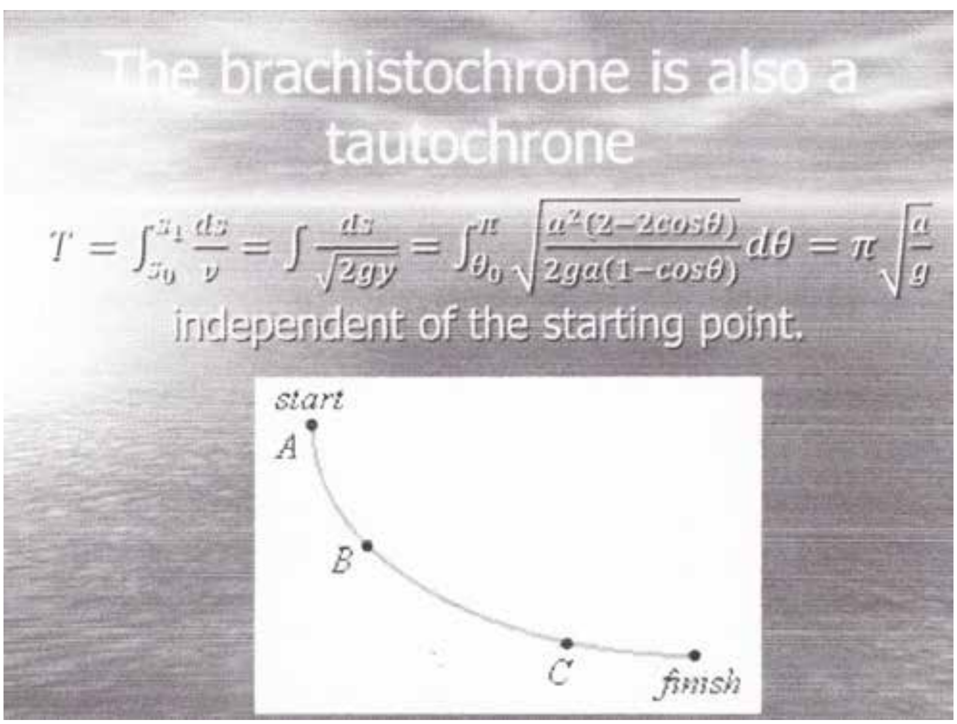


Figure 3. Same time from *A, B,* or *C* to finish.

An extension of the problem is to consider falling through the whole Earth, first, when the Earth is taken stationary, and, next, to allow it to rotate. We have always been fascinated by a fall through a hole in the Earth.

- In 700 BC, the Greek poet Hesiod claimed that an anvil would fall from the heavens to the Earth in 9 days, and through the Earth again in 9 days before reaching hell.
- Alice, in 1865 (through Lewis Carroll), fell down a rabbit hole for ever so long that she grew sleepy and thought she must have fallen through the Earth's center.
- Henrik van Etten, in 1864, argued that a millstone dropped from a hole would reach the Earth's center in 2.5 days and "hang" in the air.
- Plutarch, around 100 AD, pointed out that in a spherical Earth boulders would pass through the Earth's center and beyond and again oscillating in a perpetual motion.
- Galileo, in 1632, analyzed the simple harmonic motion of a cannonball dropped from a hole in the Earth.
- Hawking, in 1988, drops a black hole, with mass of a mountain and radius that of a nucleus of an atom, from the Earth's surface. (This entry is important to us because it is the closest thing we have for now of a particle falling through the Earth by gravity alone, unimpeded by friction.)

Through a stationary homogeneous Earth, the motion is shown to be simple harmonic, with a known period of oscillation (Figure 4).

$$F = ma$$

$$m\ddot{r} = -G\frac{mM\prime}{r^2}$$

$$= -\frac{Gm}{r^2}\cdot\frac{4}{3}\pi r^3\rho$$

$$= -kr, \;\; k = \frac{4}{3}\pi Gm\rho$$

i.e., $$\ddot{r} = -\frac{k}{m}r \Rightarrow SHM \text{ with period } T = 2\pi\sqrt{\frac{m}{k}} = 84.4\,min.$$

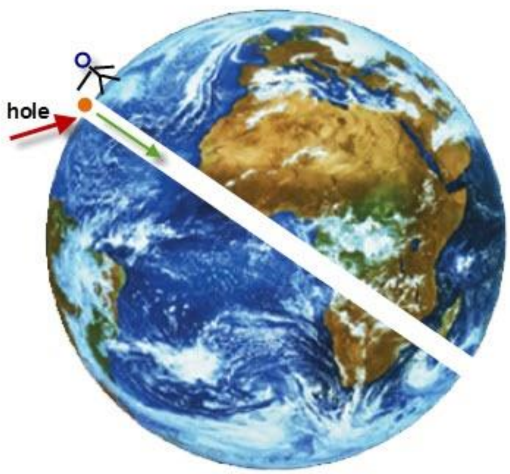
We note that this transit time of 42.2 minutes, one-way, is the same value between any two antipodes on the Earth. It turns out that the hole through the Earth need not pass through the center; any chordal tunnel in the Earth gives rise to a simple harmonic motion, with the same period as the diametrical path (Figure 5).

$$m\ddot{x} = -G\frac{mM'}{r^2}sin\theta, \;\; sin\theta = \frac{x}{r}$$

$$= -\frac{Gm}{r^2}\cdot\frac{4}{3}\pi r^3\rho\cdot\frac{x}{r}$$

$$= -kx, \quad k = \frac{4}{3}\pi G m \rho$$

i.e., $$\ddot{x} = -\frac{k}{m}x \Rightarrow SHM \text{ with period } T = 2\pi\sqrt{\frac{m}{k}} = 84.4\, min.$$



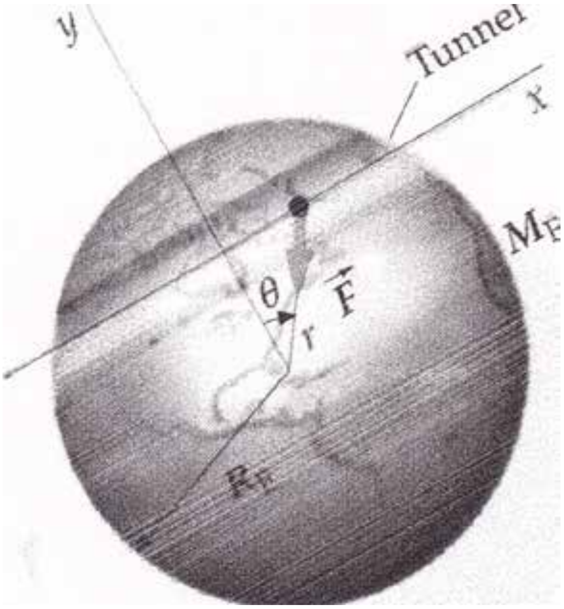


Figure 4. Hole along a diameter

Figurer 5. Hole along a chord

Allowing for a rotating Earth, the motion now adds a new dimension. This was first completely described by Simoson in 2010. From a frame of reference on a fixed star, the simple harmonic motion through a tunnel in the Earth appears as an ellipse that is precessing in a clockwise sense. From the point of view of an observer on the Earth, the to-and-fro motion along the tunnel appears as a trochoidal motion. This is reminiscent of the motion of a Foucault pendulum, first demonstrated by Foucault in 1851. Hanging a 62-lb bob by a string of length 220 ft at the Pantheon in Paris, latitude $48^0 51'$, the pendulum appeared to precess around 11^0 every hour clockwise, for a total period of 32.7 hours.

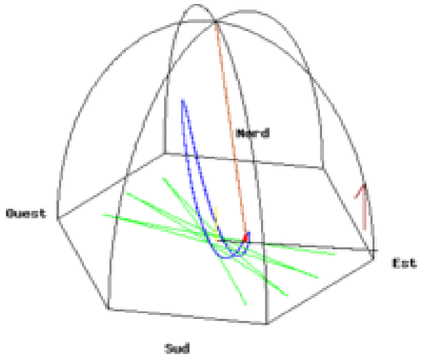


Figure 6. Foucault pendulum (Pantheon)

Figure 7. Foucault pendulum, model

The equations of motion for motion on a plane (the equatorial plane) is given by Newton's laws:

$$\frac{d^2r}{dt^2} + kr = \frac{h^2}{r^3}, \qquad \frac{d^2z}{dt^2} + z = \frac{k}{h^2z^3}$$

with r as the radial distance from the Earth's center to the particle, θ the radial angle of the position vector, $z = 1/r$, h = the angular momentum, and k is a proportionality constant to the gravitation force $F = -kmr$. The solutions are (details from [Simoson, 2010]):

$$r(t) = R\left[cos^2(\sqrt{k}t) + \frac{\omega^2}{k}sin^2(\sqrt{k}t)\right]^{1/2} \quad \text{and}$$

$$r(\theta) = R\left[cos^2\theta + \frac{k}{\omega^2}sin^2\theta\right]^{-1/2}, \quad \text{with } \omega = \frac{2\pi}{Q},$$

Q = Earth's period, and ω = Earth's angular frequency. The solution can be written as a matrix:

$$P(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = R\begin{bmatrix} cos(\sqrt{k}t) \\ \frac{\omega}{\sqrt{k}}\sin(\sqrt{k}t) \end{bmatrix},$$

which is an elliptical path, for an observer fixed on the stars. For an Earth observer, this path is made to precess by introducing a dynamic rotation matrix:

$$Q(\alpha, t) = \begin{bmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{bmatrix}.$$

The resulting path, $Q(-\omega, t)P(t)$, is a hypocycloid along the equatorial plane of a rotating homogeneous spherical Earth.

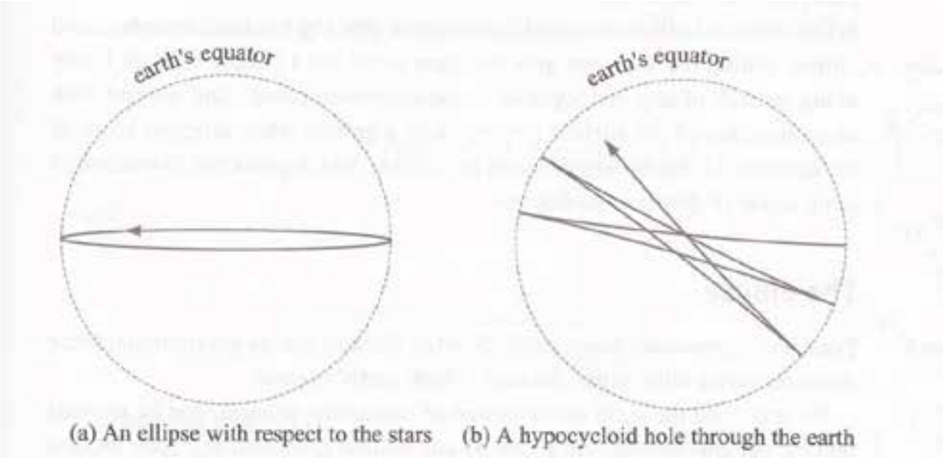


Figure 8. Pebble motion relative to the stars and the Earth.

A trochoid is generated by a point on a circle rolling, without slipping, on another circle. If the rolling circle is outside the other circle, the curve generated is called an epitrochoid, inside the curve is a hypotrochoid. When the cusps of the curves are on the stationary circle, the curves are called epicycloids and hypocycloids (Figure 9). Generating a trochoid is shown in Figure 10. The exact shape of the curve depends on the ratio of the radii of the two circles, r'/r, $r' =$ the ring, *r* = the wheel. If this ratio is a fraction m/n in lowest terms, the curve will have *m* cusps, and it will be completely traced after moving the wheel *n* times around the inner rim. If this ratio is irrational, the curve will not close, although going many times around the rim will nearly close it.

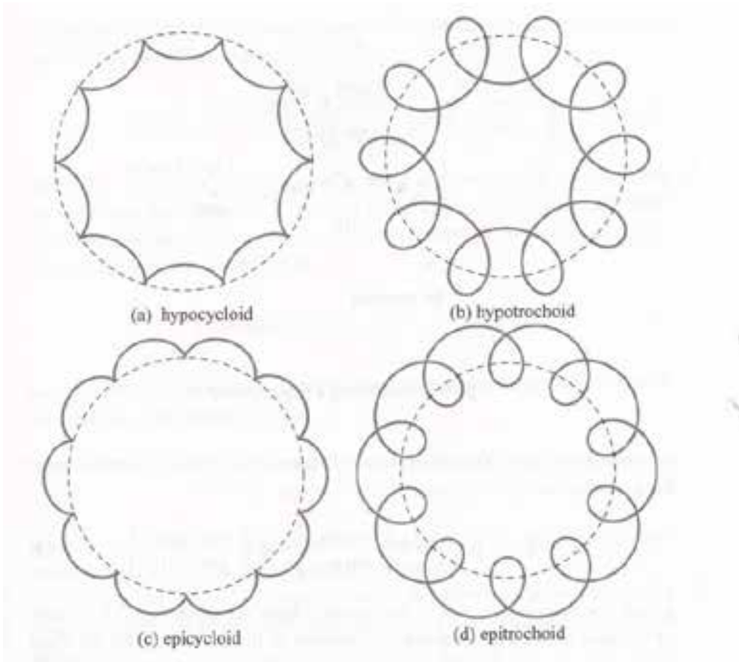


Figure 9. Trochoid curves

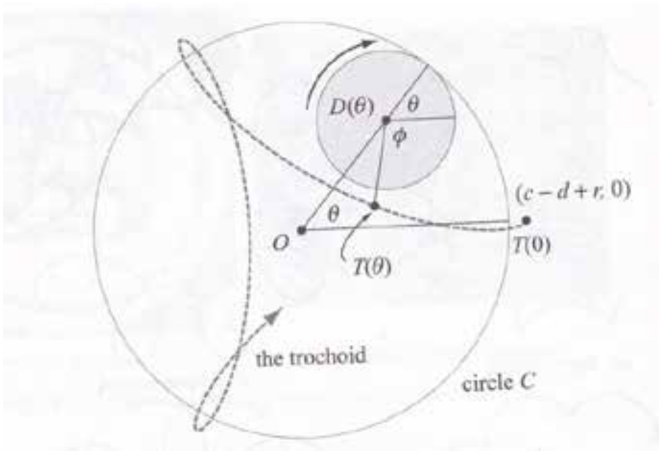


Figure 10. Generating a trochoid

As an application of the equations of motion, we can calculate the transit times between any two cities along the equator. One further feature of the equations of the path is that it allows for the variation of the rate of rotation of the Earth. The solutions, for the equatorial plane, may be written:

$$r(t)[cos(\theta(t)-\omega t),\ sin(\theta(t)-\omega t)],$$

with $\omega =$ the Earth's angular frequency, and

$$\theta(t)=\sqrt{k}t+tan^{-1}\left[\frac{\left(\frac{\omega}{\sqrt{k}}-1\right)sin\sqrt{k}t\,cos\sqrt{k}t}{1+\left(\frac{\omega}{\sqrt{k}}-1\right)sin^2\sqrt{k}t}\right],\ t>0.$$

This last equation allows us to calculate the transit time between launch point *A* and resurface point *B* for various hypothetical rates of rotation of the Earth (Figure 11), with *t* = the number, in hours, as argument in the parentheses of *B*. For example, $B(\infty)$ corresponds to a stationary Earth; $B(24)$ is for normal rotating Earth at 24 hours/day; and for $B(1.5)$, the pebble immediately veers off into space because the centripetal acceleration exceeds gravitational acceleration. The hypocycloid between *A* and *B*(24) is the same hypocycloid shown in Figure 8b. In particular, for the hypocycloid between launch point *A*: Quito, Ecuador (78^0W), resurface point *B*: Entebbe, Uganda (32.5^0E), the transit time is 42.31 minutes. Compared to the transit time for a stationary Earth, 42.24 minutes, this is slower by 4.2 seconds.

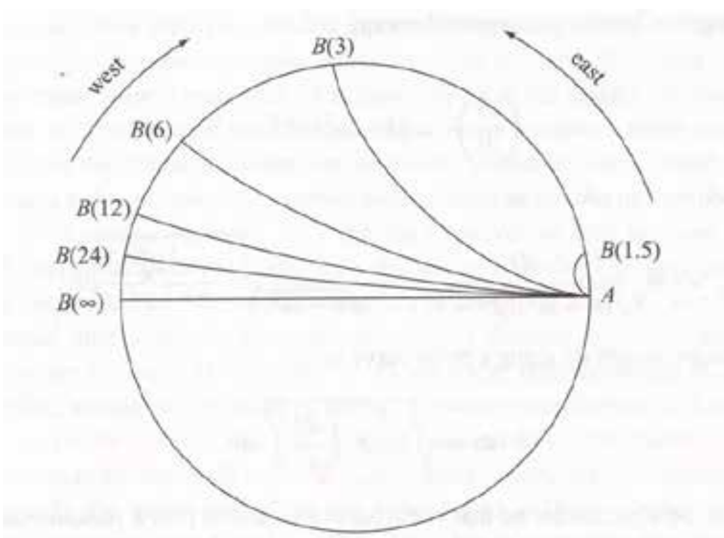


Figure 11. Hypocycloid arches with respect to Earth period

For release points *A* in higher latitudes, say the Northern Hemisphere (Figure 12), the paths through the tunnel get a new shape. From an observer in the stars, the path is still elliptical, but for an Earth observer, a new factor comes in. Since the pebble's initial tangential speed at latitude ψ is less by a factor of $cos\psi$ compared to that of a particle at the equator, the semi-minor axial length for the ellipse originating at ψ will be smaller by the same factor to that of the ellipse originating at the equator. Besides, the tunnel curve between *A* and *B* will no longer be planar, but a 3-*D* curve between $\pm\psi$, resembling the curves on a wicker basket, Figure 13. Their projection onto the equatorial plane is still a hypocycloid. Simoson [2010] gives an example of a drop at Torino, Italy $(45^0 N, 7.5^0 E)$ and a resurface point at $(45^0 S, 177^0 E)$, which is 300 miles southeast off the coast from Wellington, New Zealand. (No transit time was given, however.)

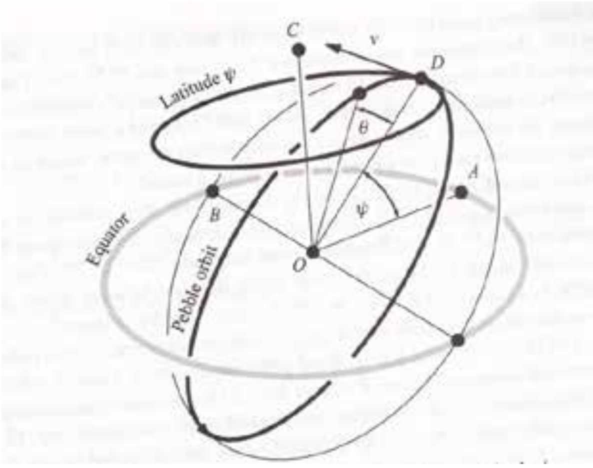


Figure 12. Ellipse of motion for latitude ψ

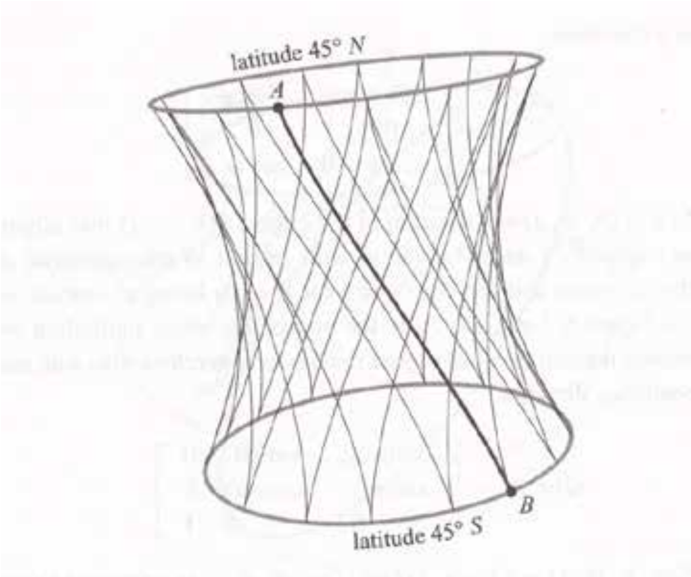
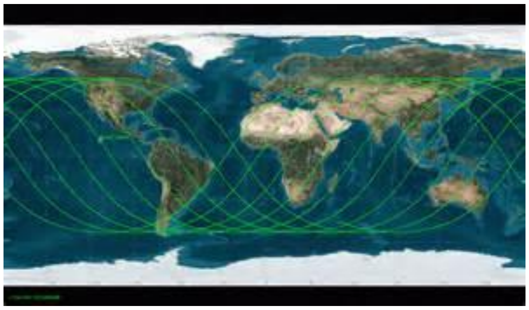


Figure 13. Path of fall for a pebble dropped at latitude ψ

Such paths as shown for a free fall at higher latitudes are not unfamiliar. They are reminiscent of the orbits of low-altitude polar satellites around the Earth, as shown in Figure 14a, b.



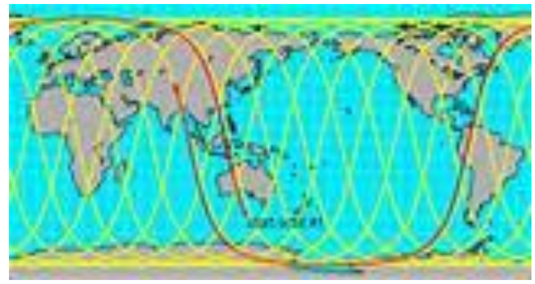


Figure 14a. Polar low-altitude satellite orbits 14b.

Summary of Results:

- The path of quickest descent in a free fall through a constant gravity field is a cycloid.
- An extended fall through a stationary Earth is a simple harmonic motion, with the same period between any two points on the Earth.
- Even when the tunnel is a chord not through the center, the motion is still simple harmonic, with the same period as before (an isochrone).
- When rotation is allowed, the path of the fall becomes a trochoid along the equator, which is still the least time path. Along the equator, the trochoidal motion is planar.
- In the North (or South) Hemisphere, the motion is nonplanar, but its projection on the equator is still trochoidal. The motion between release point and resurface point is accomplished in the least time.