

An Introduction to the Robustness of GeoGebra

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In this minicourse, I presented a set of tasks that I use in a course for pre-service secondary mathematics teachers to help them develop proficiency in using GeoGebra to explore geometry, algebra, statistics, trigonometry, and calculus. I first became aware of GeoGebra as a free alternative to commercial dynamic geometry software, but as I have used it and incorporated it into my classes, I have become amazed by the versatility that the software has for creating dynamic environments for exploring all branches of undergraduate mathematics, not just **geometry** and **algebra** which give GeoGebra its name. In this paper, I will highlight the tasks that I have used with my students and presented in the minicourse. Complete GeoGebra files for all of the tasks in this paper can be found on my website: <http://faculty.ung.edu/tecooper/GGB/ICTCM/ICTCM-2016-files.zip>

Geometry

The session started, as do all of my technology courses, with an introduction to geometric constructions, including the four classic triangle centers. As these are common dynamic geometry constructions, I would refer interested readers to the following videos that I created to introduce my students to the basic features of GeoGebra. In addition to the constructions, these videos cover such basics as saving files, changing styles, and creating tools.

https://www.youtube.com/watch?v=muDeGh_B6og (Centroid Construction)

<https://www.youtube.com/watch?v=QeajZPizHE8> (Orthocenter Construction)

<https://www.youtube.com/watch?v=miRCfnU9jBE> (Circumcenter Construction)

<https://www.youtube.com/watch?v=-mq4ZPe21mY> (Incenter Construction)

Mathematicians have actually defined thousands of different triangle centers (Kimberling, 2016), and I also have my students discover the Gergonne Point (Weisstein, 2016a) and the Symmedian Point (Weisstein, 2016b) because this gives a chance for exploration and involves the use of many of the geometry tools, including reflections.

Dynamic Geometry software such as GeoGebra is great for allowing students to discover results and properties that they may not have encountered. For example, I presented the following task:

Create a dynamic quadrilateral $ABCD$. Find the midpoints M , N , O , and P of sides AB , BC , CD , and AD respectively. Construct quadrilateral $MNOP$. What type of figure is this?

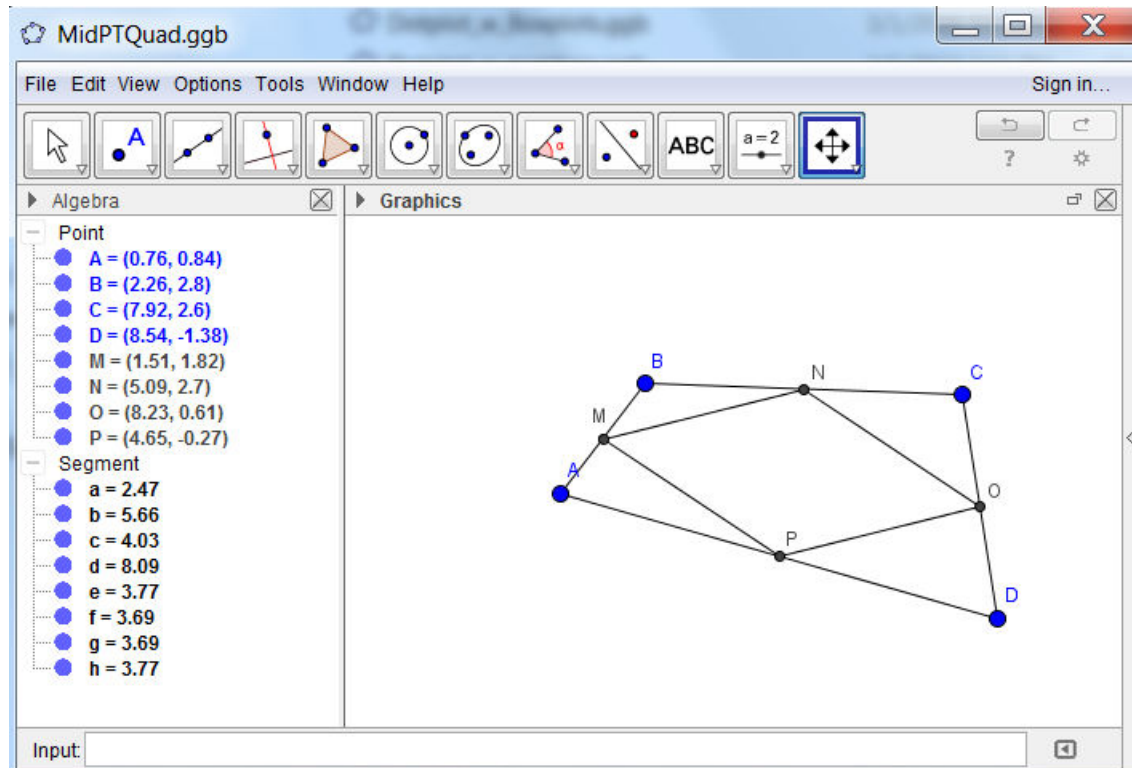


Figure 1. Midpoint quadrilaterals

Students may be surprised to learn that the midpoint quadrilateral is always a parallelogram regardless of the type of quadrilateral used. They can also explore the types of quadrilaterals that will produce rectangles and squares.

Algebra

Once of the most powerful features of GeoGebra for teaching algebra is the use of sliders to control parameters. With the following task, I showed participants in detail how to construct a slider and explore a family of quadratic functions.

Algebra Task 1: Create a file to explore $y = a(x - h)^2 + k$

1. Make a new GGB file with three sliders a , h , and k . Let each run from -5 to 5 by 0.1.

Creating a slider:

- Select the slider tool from the toolbar.



Figure 2. Slider tool

- Click where you want to place the slider in the graphics view, and a dialog box will open.

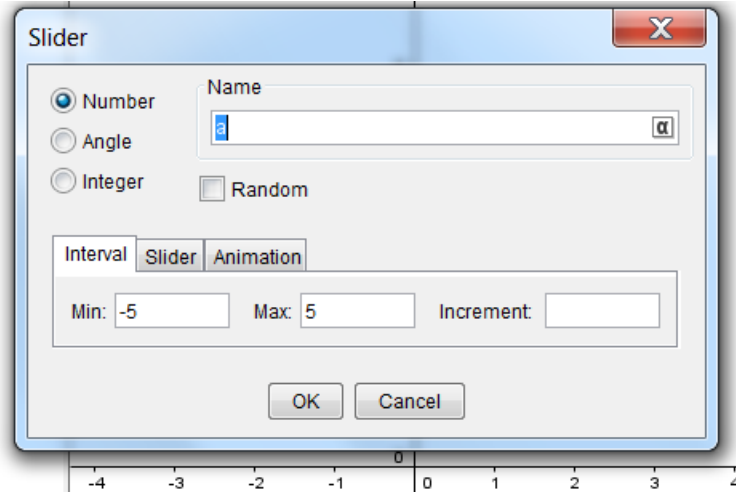


Figure 3. Slider menu

- Type the desired name; set the minimum, maximum, and increment; and click “OK”.
2. Plot the function $f(x) = a(x-h)^2 + k$.
Simply type the following in the input bar: $f(x) = a*(x-h)^2+k$

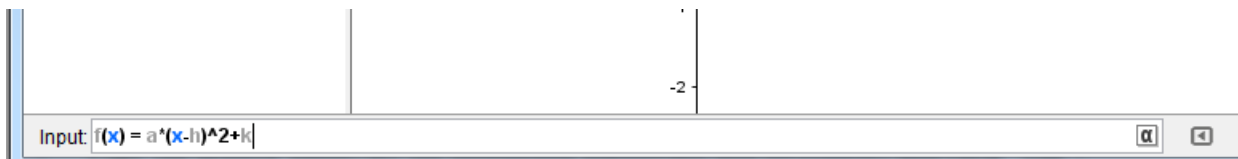


Figure 4. Entering the function

3. Adjust the value of a to determine the effect of a on the graph. What happens when a changes signs? What happens as $|a|$ increases?
4. Adjust the values of h to determine the effect of h on the graph. Describe this.
5. Adjust the values of k to determine the effect of k on the graph. Describe this.

The following tasks show further uses of sliders and algebra explorations that can be done with GeoGebra. In Task 2, most students are surprised to discover the effect of b on the graph of $y = ax^2 + bx + c$. Task 3 allows me to introduce the “Roots” and “Extremum” commands, and Task 4 has been an excellent problem solving task connecting several branches of mathematics.

Algebra Task 2: Create a file to explore $y = ax^2 + bx + c$

1. Make a GGB file with sliders a , b , and c that all go from -10 to 10 by 0.1. Then plot the function $f(x) = a*x^2 + b*x + c$.
2. Vary the value of a with the slider. What effect does a have on the graph?
3. Vary the value of c with the slider. What effect does c have on the graph?
4. What effect does b have on the graph? Add the minimum to the graph (use Extremum[f]). Then right click on the minimum and select “trace on.” Now vary the value of b . What shape is traced out by the vertex? Find a formula for the function that is traced out. Add that function to the graph. Once the function is added remove the traces of the vertex.
5. Make a check box to turn the function added in part d off and on.

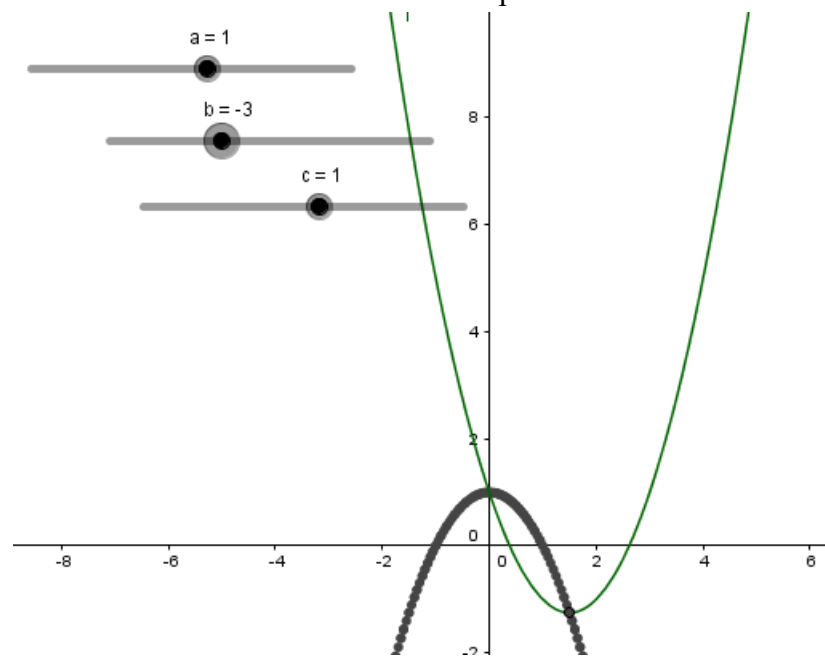


Figure 4. $f(x) = ax^2 + bx + c$ moving along $y = -ax^2 + c$ as b changes

Algebra Task 3: Fitting a polynomial to intercepts

Find an equation for a polynomial with x -intercepts $x = -6, -1, \text{ and } 4$ and y -intercept at $y = 8$. Make a sketch with this function. Plot the x -intercepts as red points, the y -intercept as a blue point, and all extrema as green points.

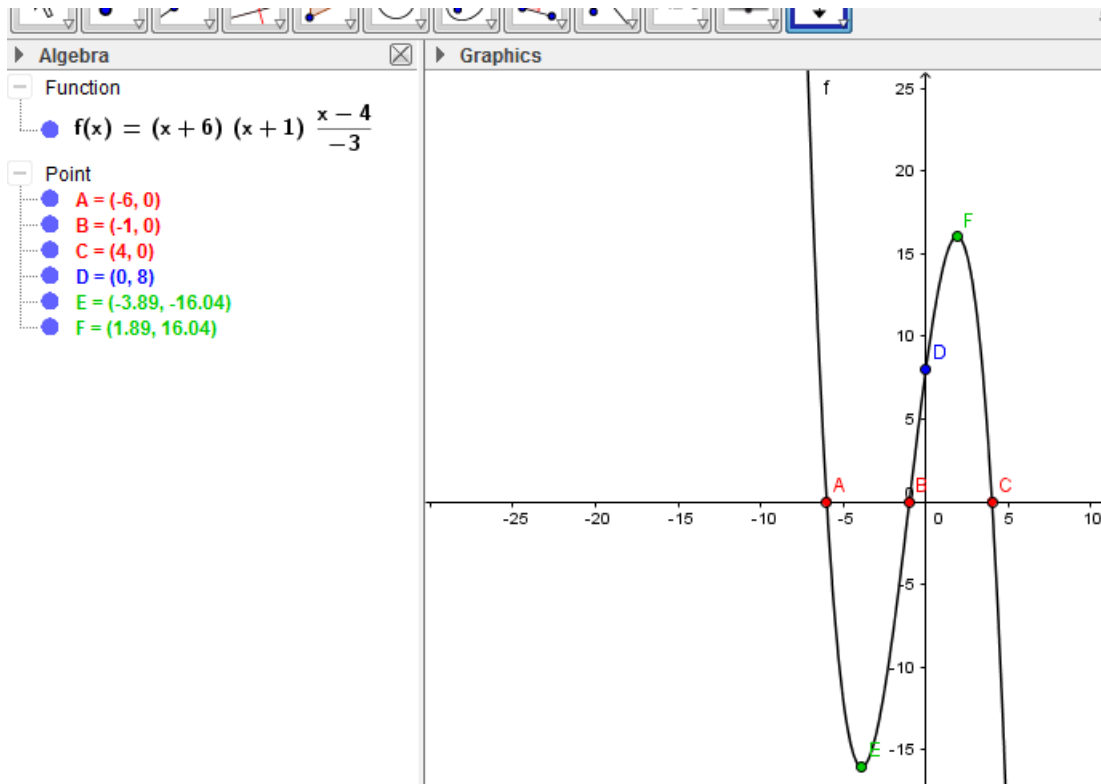


Figure 5. Algebra Task 3 screenshot

Algebra Task 4: Maximizing an area with fixed perimeter

1. Make a sketch with a slider that adjusts to show all given rectangles of perimeter 9 units along with the area. Note that the area is automatically added when you create a polygon.
2. Use the sketch to find the dimensions that give the maximum area.
3. Use calculus to show that the rectangle with maximum area given a perimeter of 9 units is a square.

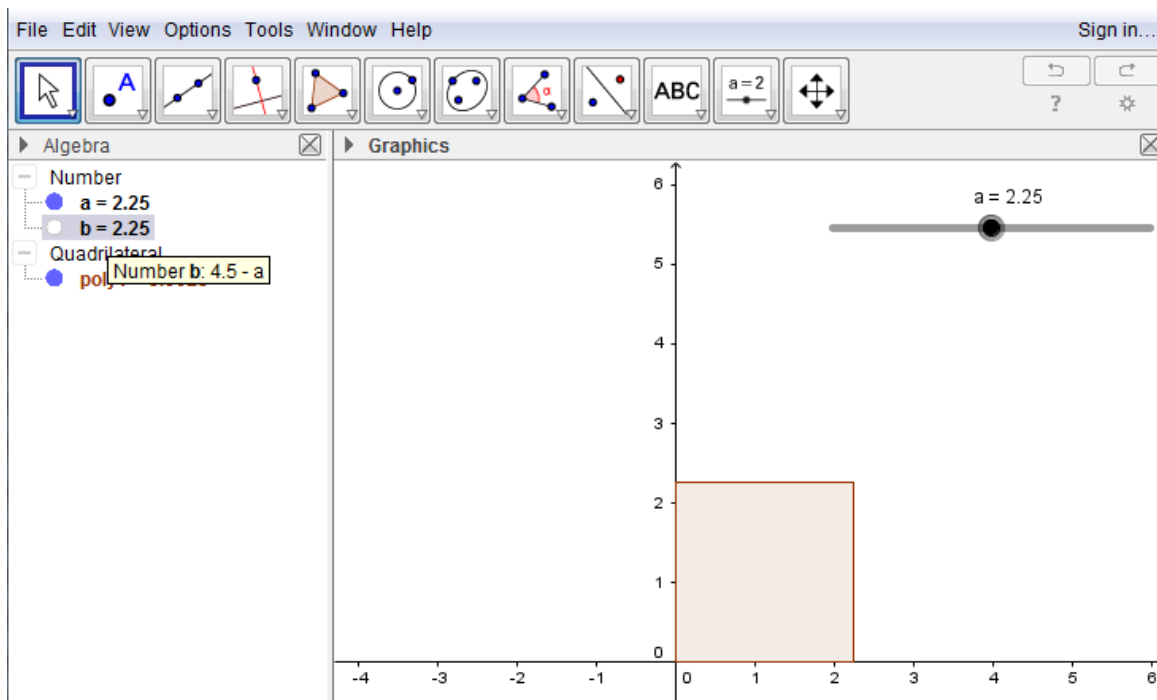


Figure 6. Optimizing rectangular area with a fixed perimeter

Trigonometry

For trigonometry, I provided four tasks which demonstrate using GeoGebra to create visualizations of concepts, verifying families of solutions to equations, and using sliders to fit an equation to temperature data.

Trigonometry Task 1: The Sine Graph

1. Plot the unit circle by typing the input: $x^2 + y^2 = 1$
2. Make a slider named a that runs from 0 to 4π with $\pi/12$ increments. We want radians instead of degrees so make it a “number” slider, not an “angle.”
3. Plot the point $(\cos(a), \sin(a))$. You should be able to rotate the point around the circle with the slider.
4. Plot the point $(a, \sin(a))$.
5. Make a segment connecting the two points.
6. Turn the trace on for point B . Move the slider around the circle and see the sine graph mapped out.

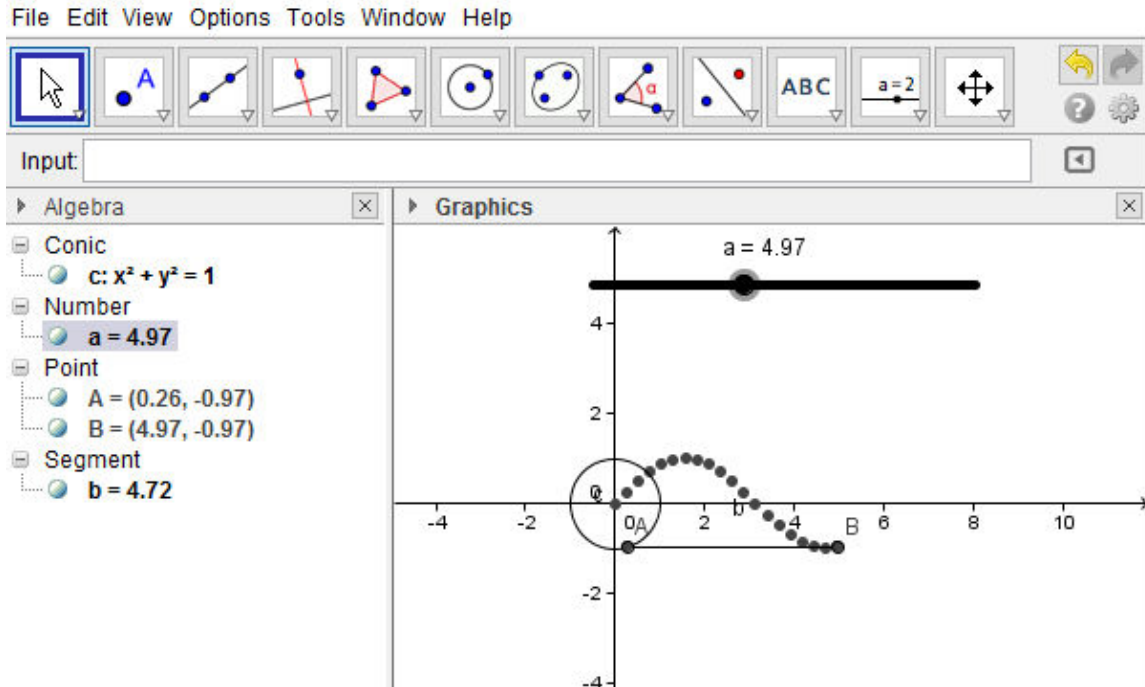


Figure 7. Mapping out a sign graph from the unit circle

Trigonometry Task 2: The Right Triangle Trigonometric Ratios

1. Plot the point $A=(0,0)$
2. Plot a point B somewhere in quadrant I
3. Plot the point $C=(x(B),0)$
4. Use the polygon tool to construct triangle A, B, C .
5. Place a point D on the positive x -axis (not on C).
6. Construct angle DAB . Then hide point D . You should now be able to move B to any quadrant and see a triangle with one side on the x -axis, a vertex at zero, and an angle between 0° and 360° .
7. Type each of the following in the text boxes to get some measurements:
 $\text{opp} = y(B)$ $\text{adj} = x(B)$ $\text{hyp} = \text{Distance}[A,B]$
8. Now add the following text to the graphic view by typing each in the input bar:
 “opposite =”+opp “adjacent =”+adj “hypotenuse =”+hyp
9. Add text to show the computation of $\sin(a)$ by typing:
 “ $\sin(a) = \text{opposite/hypotenuse} = \text{opp}/\text{hyp} = (\text{opp}/\text{hyp})$ ”
10. Add similar text to show the computations of $\cos(a)$ and $\tan(a)$.

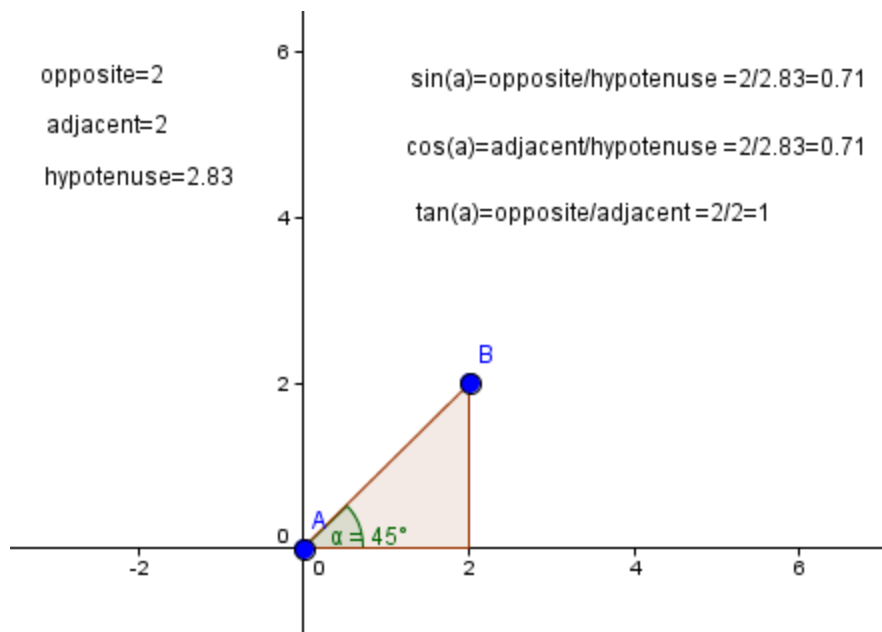


Figure 8. Demonstrating trigonometric ratios

Trigonometry Task 3: Modeling solutions to a trigonometric equation

Solve the trigonometric equation $(2\sin(x)-1)(2\cos(x)-1) = 0$. Graph $f(x) = (2\sin(x)-1)(2\cos(x)-1)$. Make a slider k running from -10 to 10 by increments of 1. Then add the solutions as points controlled by the slider.

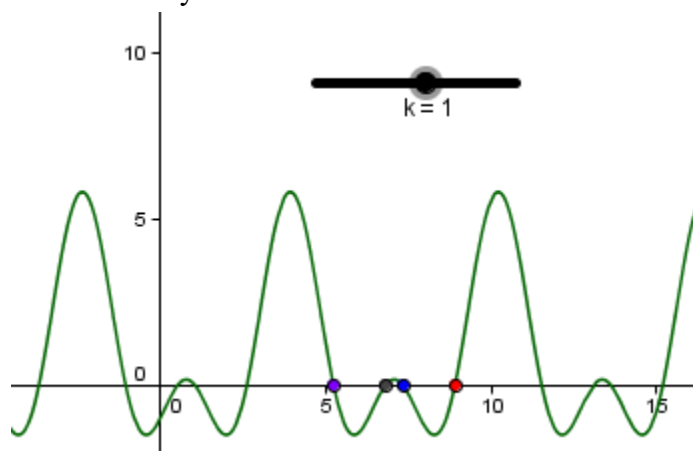


Figure 9. Solutions to a trigonometric equation

Trigonometry Task 4: Finding a Temperature Function

The following table gives the monthly average high temperatures in Atlanta.

Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
52	57	65	73	80	87	89	88	82	73	63	55

1. Create a new Geogebra file.
2. Open the “spreadsheet view” under the view tab. Adjust the window sizes as necessary to see the algebra, graph, and spreadsheet views at the same time.
3. Enter 1 through 12 in the column *A* for the 12 months.
4. Enter the corresponding average high temperatures in column *B*.
5. Highlight the data (left click, hold, drag, release), then right click and select “Create List of Point.” You should see the list in the algebra view. The points are also plotted, but you need to adjust the window.
6. Right click on an axis (make sure the points are deselected) and select “Graphics view.” Adjust the min and max of the x-axis and y-axis so that all of the points are visible.
7. Create sliders *a* (min = -20, max = 20), *b* (min = -5, max = 5), *k* (min = -5, max = 5), and *d* (min = -100, max = 100).
8. Plot the function $y = a \cdot \cos(k \cdot (x - b)) + d$.
9. Adjust the slider values until the function fits the points as nicely as possible (it won't fit exactly).
10. Record your best fit function with the numerical values of the sliders plugged into it.
11. Use your model to predict the average high temperature in August. How close is the estimate to the actual value? (You can evaluate the function with Geogebra.)

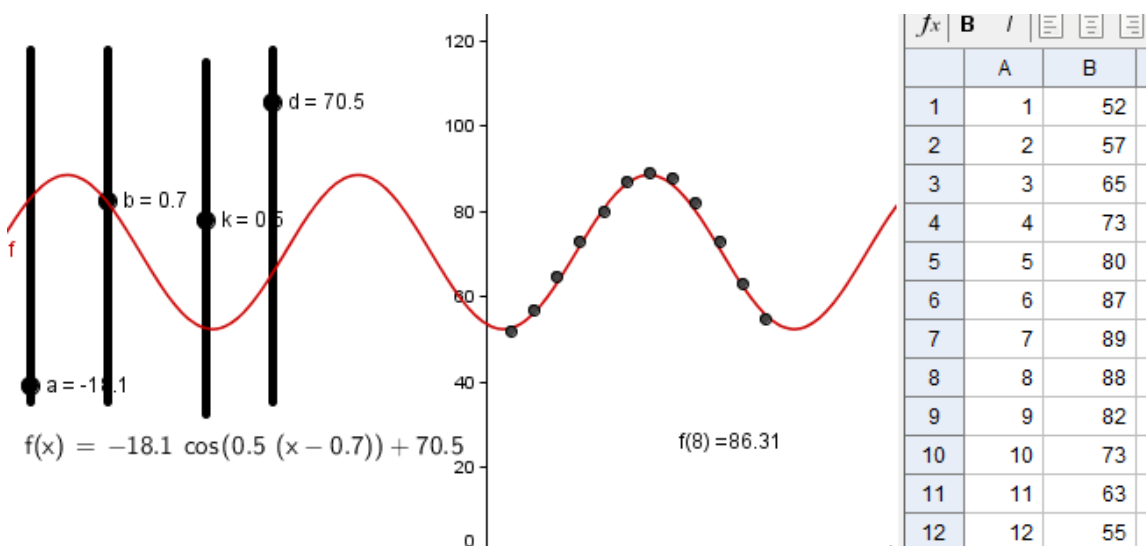


Figure 10. Fitting a temperature function

Calculus

Geogebra is an excellent tool for building visual explorations of calculus concepts. Due to time constraints, I selected two calculus tasks for the minicourse. The first leads to a demonstration of secant and tangent lines, and the second involves creating a visualization of left hand Riemann sums.

Calculus Task 1: Plotting secant and tangent lines

Create a file with sliders for a , b , and c . Then add the following:

1. $f(x) = a*x^2 + b*x + c$
2. a point P on the parabola
3. a slider h running from -1 to 1 by .01
4. a point $Q = (x(P)+h, f(x(P)+h))$ (This is a point “close to P ”.)
5. the secant line between P and Q .
6. the tangent line at P

You should see that the secant line approaches the tangent as $h \rightarrow 0$.

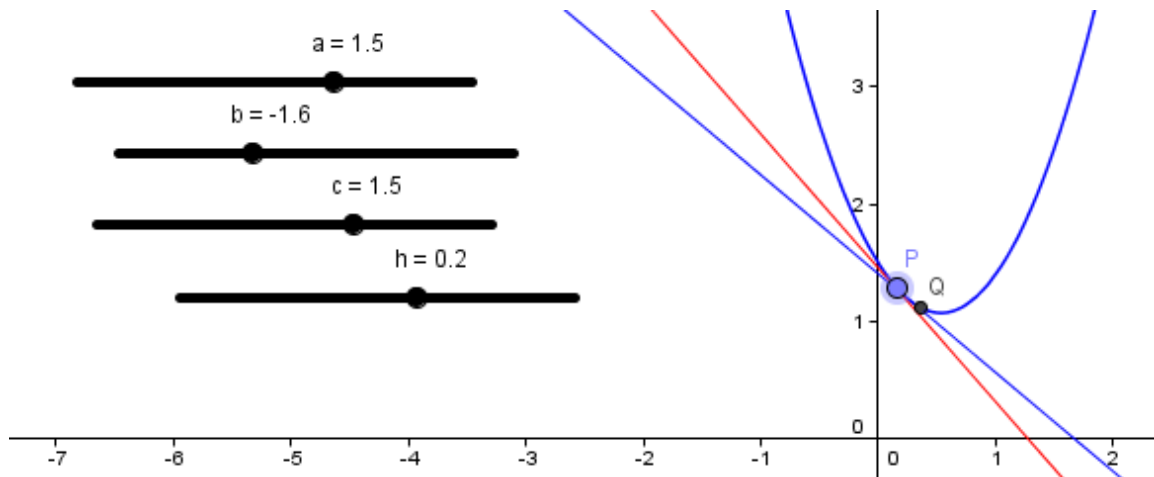


Figure 11. Secant and tangent lines

Calculus Task 2: Left hand sums

Construct the left hand sums and integral.

1. Enter the function: $f(x) = x^2 + 1$
2. Create sliders a (running from 0 to 5 by .1), b (running from a to 5 by 0.1), and n (running from 1 to 50 by 1)
3. Enter: $dx = (b-a)/n$
4. Enter: $L = \text{Sequence}[\text{Polygon}[(i, f(i)), (i+dx, f(i)), (i+dx, 0), (i, 0)], i, a, b-dx, dx]$
5. Enter: $\text{LHS} = \text{Sum}[\text{Sequence}[dx * f(a+i * dx), i, 0, n-1]]$
6. Enter: $\text{integral}[f, a, b]$

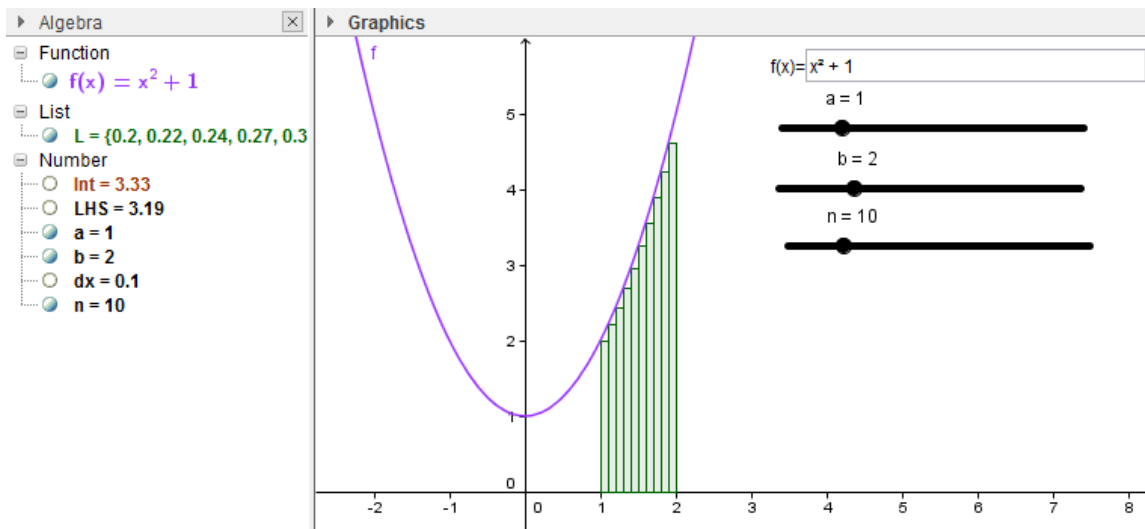


Figure 12. Left hand sums

Statistics

Geogebra has many built in inferential statistics procedures as well as tools that allow for creation of probability simulation that I encourage readers to explore. In the minicourse, I focused on the following tasks which help students learn to use GeoGebra for descriptive statistics.

Statistics Task 1: Making a dot plot

- Open the spreadsheet view and enter the following data in column *A*:
75, 74, 80, 84, 83, 76, 76, 70, 89, 75, 75, 64, 56, 55, 57, 81, 55, 79, 96, 67, 62, 86, 61, 70, 90, 81, 69, 75, 70, 72, 81, 86, 67, 80, 47, 45, 32, 75
- Select all of the data by clicking on the column label.
- Right click, go to “create,” and create a list.
- Right click on *list1* in the algebra view and rename it *L*.
- In the input bar type: $D = \text{DotPlot}[L]$
This will create a new list of points. They are plotted as a dot plot, but you can't see them unless you adjust the axes. Since they are exam scores, none are below zero and none are above 100.
- Right click somewhere on the axes and select “graphics” at the bottom of the options.
- Set the *x Min* to -10 and the *x Max* to 110. You should then be able to see the dot plot.
A dot plot simply plots a point above any score that is made. If the same score occurs more than once, you see a stack of points at that *x* value.
- Is the distribution roughly symmetric, skewed left, or skewed right?

Statistics Task 2: Finding some numerical values

- To find the “average” (sum of the scores divided by the number of scores), type the following in the input bar: $\text{meanL} = \text{mean}[L]$
- To find the sample standard deviation, type: $\text{sdL} = \text{sampleSD}[L]$

3. The mean and the standard deviation are good descriptors of a data set when the distribution is roughly symmetric and bell shaped. If not, the five number summary (min , $q1$, $median$, $q3$, max) is more informative.
4. Find the minimum test score by typing: $minL = \min[L]$
5. Find the maximum test score by typing: $maxL = \max[L]$
6. The first or lower quartile, $q1$, is the value at which 25% of the scores lie at or below.
To find this type: $q1L = Q1[L]$
7. The median is the value at which 50% of the scores lie above and 50% lie below. To find this type: $medL = \text{median}[L]$
8. The third or upper quartile, $q3$, is the value at which 75% of the scores lie at or below.
To find this type: $q3L = Q3[L]$
9. Is the five number summary or the mean and standard deviation more appropriate for this data?

Statistics Task 3: Making a box plot

Geogebra has several different types of possible inputs to make a box plot. We will look at two of them. The first will use the actual raw data and the second will use the 5 number summary.

One way to construct a box plot is: $\text{BoxPlot}[\langle y\text{Offset} \rangle, \langle y\text{Scale} \rangle, \langle \text{list of raw data} \rangle]$
The $y\text{Offset}$ argument controls where the center of the boxplot will be in the y -direction.
The $y\text{Scale}$ argument controls how thick the box and whiskers will be. It will be the distance from the center of the boxplot to the bottom or top of the box.

1. Change the y -axis so that it will be tall enough to place two box plots above the dot plot.
2. Make a box plot above the dot plot using the raw data.
Another way is to use the 5 number summary instead of the data:
 $\text{BoxPlot}[\langle y\text{Offset} \rangle, \langle y\text{Scale} \rangle, \text{min}, Q1, \text{median}, Q3, \text{max}]$
3. Make another box plot above the first one using the 5 number summary. Note the two box plots will be the same (except for the $y\text{Offset}$). You are just showing that you could make one either way.

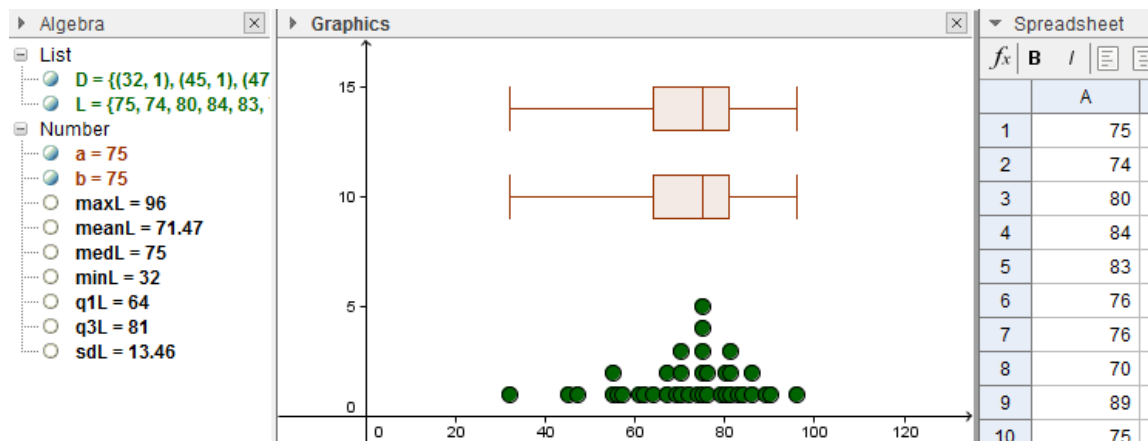


Figure 13. Box plots, dot plots, and numerical summaries

Statistics Task 4: Making a frequency table and histogram

1. In a new Geogebra file enter the data from Statistics Task 1.
2. Use the data to create a list, and name it L as before.
3. Create a list of boundary values: $B = \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$
4. To create a frequency table type: `FrequencyTable[B, L]`
The result is a table that tells how many of the data points fall in each interval. Note that each interval is of the form $[a, b)$ except the last one which is $[a, b]$. So for example, a 60 is counted in the interval $60 - 70$.
5. Move the table to the lower right portion of the graph screen. Right click on the table and select "Absolute Position on Screen." This will keep the table in place when you change the scale.
6. Before we make a histogram, we need to adjust the scale. As with the dot and box plots, set the x Min to -10 and the x Max to 110.
7. To make a histogram, type: `Histogram[B, L, false]`
The last command tells it to use the frequencies instead of densities.
8. Adjust the graph as necessary to see the entire histogram and the frequency table.

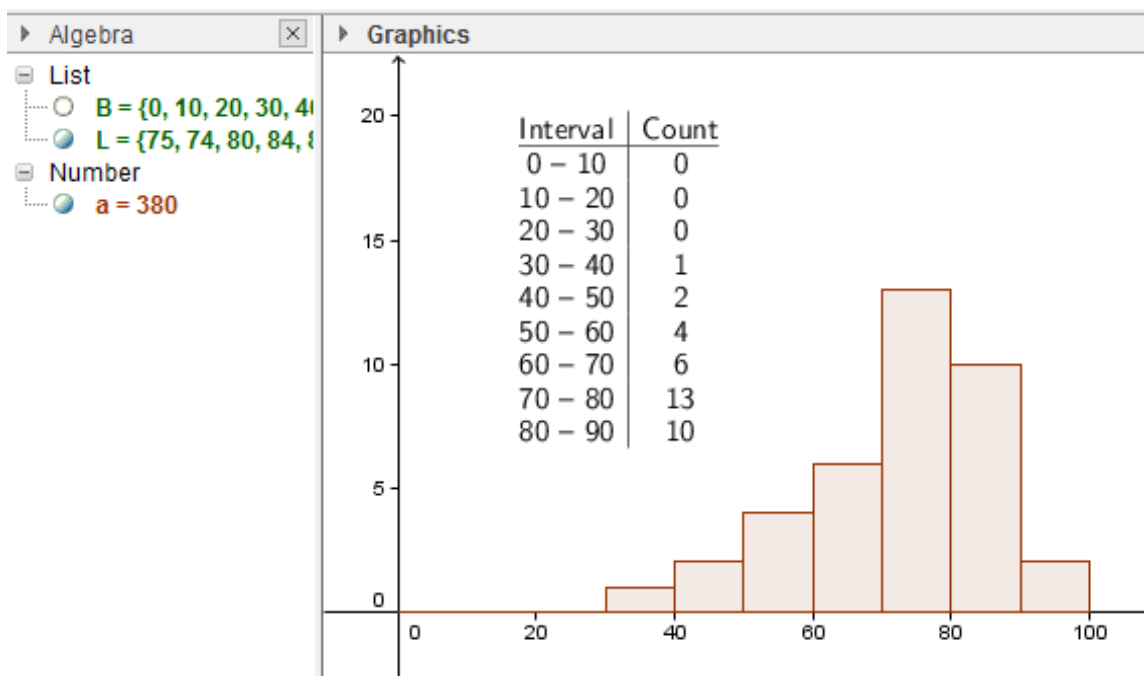


Figure 14. Histogram and frequency table

References

Kimberling, C. (2016). Clark Kimberling's encyclopedia of triangle centers. Retrieved from <http://faculty.evansville.edu/ck6/encyclopedia/ETC.html>

Weisstein, E. W. (2016a). Gergonne point. *MathWorld – A Wolfram Web Resource*. Retrieved from <http://mathworld.wolfram.com/GergonnePoint.html>

Weisstein, E. W. (2016b). Symmedian point. *MathWorld – A Wolfram Web Resource*. Retrieved from <http://mathworld.wolfram.com/SymmedianPoint.html>