CONDITIONS ON THE COEFFICIENTS OF A REDUCED CUBIC POLYNOMIAL SUCH THAT IT AND ITS DERIVATIVE ARE FACTORABLE OVER THE RATIONAL NUMBERS

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1. INTRODUCTION

The purpose of this article is to highlight the conditions on the coefficients of a reduced cubic of the form $P_{3R}(x) = Ax^3 + Bx^2 + Cx$ such that it and its derivative are factorable over the rational numbers. Such a cubic is useful to instructors of first semester Calculus courses when constructing a cubic which has rational zeroes (or, x-intercepts) and local extrema at rational values of x. Included is Matlab code to generate coefficients of the polynomial.

2. BACKGROUND

This project was a result of attempting to construct a factorable cubic polynomial from a known factorable quadratic polynomial. The intent was to use these polynomials in a Calculus 1 course where students would find the zeroes of the cubic to determine the x-intercepts, then take the derivative and find the critical numbers to examine local extrema of the polynomial.

This process began by considering a quadratic polynomial which has two rational zeroes, $x_1 = -\frac{c_1}{a_1}$ and $x_2 = -\frac{c_2}{a_2}$. With such zeroes, the quadratic is constructed as

$$P_2(x) = (x - x_1)(x - x_2) = \left(x + \frac{c_1}{a_1}\right)\left(x + \frac{c_2}{a_2}\right)$$

or more conveniently

$$P_2(x) = (a_1 x + c_1)(a_2 x + c_2).$$

Expanding,

$$P_2(x) = a_1a_2x^2 + (a_1c_2 + a_2c_1)x + c_2c_2$$

and integrating

$$\int P_2(x)dx = \frac{a_1a_2}{3}x^3 + \frac{a_1c_2 + a_2c_1}{2}x + c_1c_2x + D$$

for some arbitrary constant of integration D. By defining

$$A \equiv \frac{a_1a_2}{3}, B \equiv \frac{a_1c_2 + a_2c_1}{2}, C \equiv c_2c_2$$

then

$$P_3(x) = Ax^3 + Bx^2 + Cx + D.$$

What values could D have such that $P_3(x)$ is factorable over the rational numbers? An initial investigation presented in this paper focuses on establishing conditions on the coefficients under the assumption $D = 0$.

3. Methods

It is sufficient to consider a reduced cubic polynomial $P_{3R}(x) = Ax^3 + Bx^2 + Cx$ with integer coefficients. The reduced cubic polynomial can be rewritten in a factored form

$$P_{3R}(x) = Ax^3 + Bx^2 + Cx = x(Ax^2 + Bx + C).$$

Buddenhagen, *et al.*, provided solutions to cubic polynomials termed "nice" cubics of the form

$$y = (x + a)x(x - b)$$

for nonnegative integers a and b by using multiplication and translations. The approach taken in this paper is different, yet yields similar results.

Zeroes of $P_{3R}(x)$ are

$$x = 0$$

and all values of x where

$$Ax^2 + Bx + C = 0.$$

To solve the second equation, one may use the quadratic formula and thus require the discriminant to be a perfect square, or

$$B^2 - 4AC = s_1.$$

Thus, the first condition for $P_{3R}(x)$ to be factorable over $\mathbb{Q}$ has been established.

The derivative of the cubic is a quadratic

$$P'_{3R}(x) = 3Ax^2 + 2Bx + C.$$

Solutions to $P'_{3R}(x) = 0$ require the associated discriminant

$$(2B)^2 - 3(4AC) = 4B^2 - 12AC = 4(B^2 - 3AC)$$

to be a perfect square. Note that

$$\sqrt{4\left(B^2 - 3AC\right)} = 2\sqrt{B^2 - 3AC}$$

which gives us our second condition:

$$B^2 - 3AC = s_2.$$

The problem is to find conditions on A, B, and C such that a solution to the system

$$\begin{cases} B^2 - 4AC = s_1 \\ B^2 - 3AC = s_2, \end{cases} \qquad \textbf{(1)}$$

where each s_i is a perfect square, can be found.

Solving the system algebraically by subtracting the first equation of **(1)** from the second equation of **(1)** leads to

$$AC = s_2 - s_1.$$

Substituting the result for AC into **(1)**

$$\begin{cases} B^2 - 4(s_2 - s_1) = s_1 \\ B^2 - 3(s_2 - s_1) = s_2 \end{cases} \Rightarrow \begin{cases} B^2 = 4s_2 - 3s_1 \\ B^2 = 4s_2 - 3s_1 \end{cases}$$

leads to

$$B = \pm\sqrt{4s_2 - 3s_1}$$

In order for B to be real we require that

$$4s_2 - 3s_1 \geq 0$$

or

$$s_2 \geq \frac{3}{4}s_1.$$

For B to be an integer, we require $4s_2 - 3s_1$ to be a perfect square. A series of cases follows, leading to conditions on the coefficients of $P_{3R}(x)$ which give the desired results.

3.1 Case 1: $4s_2 - 3s_1 = 0$

If $4s_2 - 3s_1 = 0$, then $B = 0$ and **(1)** reduces to

$$\begin{cases} -4AC = s_1 \\ -3AC = s_2, \end{cases} \quad \textbf{(2)}$$

and therefore either $A < 0$ or $C < 0$, with not both A and C negative.

The cubic $P_{3R}(x)$ reduces to $P_{3R}(x) = x(Ax^2 + C)$ and the derivative is $P'_{3R}(x) = 3Ax^2 + C$.

3.1.1 Scenario 1: $|A|$ and $|C|$ are both perfect squares

With opposite signs, $Ax^2 + C$ is factorable as the difference of squares, but $P'_{3R}(x)$ is not factorable as the difference of squares. Assuming that the leading term is positive (if not, just factor -1),

$$3Ax^2 + C = (3ax + c)(ax - c)$$

where $a^2 = A$ and $-c^2 = C$. Distributing, we require $-2ac = 0$. This scenario is not possible unless either A or C is zero. In the event $A = 0$, then the polynomial is no longer a cubic as required. If $C = 0$, then the polynomial is further reduced to the cubic $P_{3R}(x) = Ax^3$ which is always factorable over the rational numbers and whose derivative is also factorable over the rational numbers but which does not yield polynomials useful to the calculus instructor since zero is the only "zero" of the polynomial.

3.1.2 Scenario 2: Either $|A|$ or $|C|$ is a perfect square, but not both

For discussion, assume A is a perfect square. From the first equation of the reduced system **(2)**, an additional requirement is

$$C = \frac{s_1}{-4A}.$$

From the conditions on case 1, we already established that A and C would have opposite signs. The additional requirement above states that C is the ratio of two perfect squares, which implies C is a negative of a perfect square. This contradicts the condition for this scenario, and therefore is not possible.

3.1.3 Scenario 3: Neither $|A|$ nor $|C|$ is a perfect square

In this scenario, the first equation of **(2)** becomes

$$C = -\left(\frac{s_1}{4}\right)/A.$$

or $-AC = \frac{s_1}{4}$. This gives us another condition: the product $-AC$ must be a perfect square. If either $|A|$ or $|C|$ is a prime, then the other is that prime multiple of a perfect square (e.g., $|C| = |-3| = 3$, $A = 12 = 3 \cdot 4$). In the event either $|A|$ or $|C|$ is a composite whose factors are not perfect squares, then the other is a perfect square multiple of that composite (e.g., If $|A| = |-15| = 15 = 3 \cdot 5$, then $|C| = k^2|A| = 15k^2$ for any rational number k, therefore $-AC = -(-15) \cdot 15k^2 = 225k^2$ which is a perfect square). Thus the product $-A \cdot C$ is not only a difference of squares ($-AC = s_1 - s_2$), but also is a perfect square.

An additional requirement exists from the second equation of (2),

$$C = \frac{s_2}{-3A}.$$

Therefore, the product $-3AC = s_2$ is required to be a perfect square to make the derivative factorable. However, from the first condition of this scenario, it was required the product $-A \cdot C$ must be a perfect square. This is a contradiction. It is trivial that if x is a perfect square, then $3x$ cannot be a perfect square.

In conclusion the only viable option in the case $4s_2 - 3s_1 = 0$ is the first scenario; however, the result is trivial.

3.2 Case 2: $4s_2 - 3s_1 \neq 0$

If $4s_2 - 3s_1$ is a non-zero perfect square, then B is an integer whose value is calculated by

$$B = \pm\sqrt{4s_2 - 3s_1}.$$

3.2.1 Scenario 1: $s_1 = 0$

If $s_1 = 0$, then

$$B = \pm 2\sqrt{s_2} = \pm 2\sqrt{B^2 - 3AC} \Rightarrow B^2 = 4(B^2 - 3AC) \Rightarrow B^2 = 4AC$$

which is identical to the result from the first equation of (1). As a result,

$$B = \pm 2\sqrt{AC}.$$

To be an integer value, it is then required the product AC is a perfect square and the signs of A and C must be identical.

3.2.2 Scenario 2: $s_1 = s_2 \neq 0$

Let $p = s_1 = s_2$. Then $B = \pm\sqrt{4s_2 - 3s_1} = \pm\sqrt{4p - 3p} = \pm\sqrt{p}$ for some perfect square p. The system defined by **(1)** becomes

$$\begin{cases} p - 4AC = p \\ p - 3AC = p, \end{cases}$$

which leads to $AC = 0$. If $A = 0$ then the polynomial is not a cubic. If $C = 0$, then the polynomial reduces further to $P_{3R}(x) = Ax^3 + Bx^2 = x^2(Ax + B)$ and the derivative is factorable as $P'_{3R}(x) = x(3Ax + 2B)$. Both the polynomial and derivative are factorable over the rational numbers. This is another instance of a simplified scenario and is typically uninteresting to the instructor.

3.2.3 Scenario 3: $s_1, s_2 \neq 0$ and $s_1 \neq s_2$

If both $s_1 \neq 0$ and $s_2 \neq 0$, then the number of combinations of s_1 and s_2 which lead to integer values of B is far fewer than the other scenarios; however, identifying such combinations is more difficult.

Let k and $k + n$ be positive integers ($n \in \mathbb{Z}^+$). Set $s_2 = k^2$ and $s_1 = (k + n)^2$. Under this assumption, $s_2 < s_1$ for all n. In order for B to be a non-zero real number, we require $4s_2 - 3s_1 > 0$, or

$$4k^2 - 3(k + n)^2 > 0.$$

Simplifying,

$$k^2 - 6kn - 3n^2 > 0.$$

The figure below displays $y = k^2 - 6kn - 3n^2$ for six different values of n.

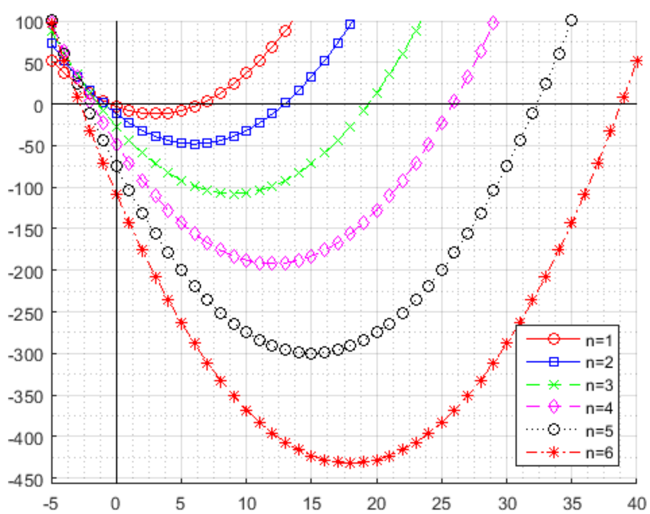

Figure 1: $y = k^2 - 6kn - 3n^2$ for six different values of n

As can be seen, there always exist values of $k < 0$ such that $k^2 - 6kn - 3n^2 > 0$; however, only positive values of k are considered so that $s_2 < s_1$. The k-intercepts identify where $k^2 - 6kn - 3n^2 = 0$. Consequently, values such that k is an integer greater than the intercept are to be considered. Table 1 lists minimum values of k for each value of n.

Table 1

n	0	1	2	3	4	5	6
k	0	7	13	20	26	33	39

Table 1: Minimum positive integer k for each number n, such that $4k^2 - 3(k+n)^2 > 0$

The minimum positive integer can be generated by the following recursive sequence for $n \geq 3$:

$$k_n = k_{n-1} + k_{n-2} - k_{n-3}$$

where $k_0 = 0, k_1 = 7$, and $k_3 = 13$.

As a result, if one selects consecutive squares (i.e., $n = 1$), then are no useful integers less than $k = 7$ to generate $s_2 = k^2$. Similarly, if one selects squares of integers separated by 2, then only integers greater than or equal to 13 are useful and so on.

If we set $s_2 = (k+n)^2, s_1 = k^2$, then $s_2 > s_1$ for all $k > 0, n > 0$, so $4s_2 - 3s_1 > 0$ for all $k > 0, n > 0$.

4. Generating Cubic Triples

Recall that for B to be an integer, we need $4s_2 - 3s_1 = q^2$ for some $q \in \mathbb{Z}$. Values of s_1, s_2, and q which satisfy that equation are coined cubic triples. Generation of the cubic triples is done by the first Matlab code found in the Appendix. For the first 20 perfect squares, 1, 4, 9, 16, …, 400, the code generates the eight unique sets of cubic triples shown in Table 2. Included in Table 2 are the values of the quadratic coefficient, B, as well as the product of the cubic and linear coefficients, $A \cdot C$.

Table 2

$(\sqrt{s_1}, \sqrt{s_2}, q)$	$(3,7,13)$	$(5,7,11)$	$(8,7,2)$	$(7,13,23)$
(B, AC)	$(13,40)$	$(11,24)$	$(2,-15)$	$(23,120)$
$(\sqrt{s_1}, \sqrt{s_2}, q)$	$(8,13,22)$	$(15,13,1)$	$(5,19,37)$	$(16,19,26)$
(B, AC)	$(22,105)$	$(1,-56)$	$(37,336)$	$(26,105)$

Table 2: Cubic triples and resulting value of the coefficient, B, and the product of the coefficients A and C. Note each value of B can be expressed as $\pm B$.

Geometrically, these triples represent sides of a triangle whose measurements are $\sqrt{3s_1}, q$, and $2\sqrt{s_2}$ (see Figure 2). Base pairs, defined as $(\sqrt{s_1}, \sqrt{s_2})$, are used calculate to the third term of the cubic triple, q. For any $c \in \mathbb{N}$, $4s_2 - 3s_1 = q^2 \Rightarrow 4s_2c^2 - 3s_1c^2 = (cq)^2$, so any base pair actually generates an infinite sequence of cubic triples.

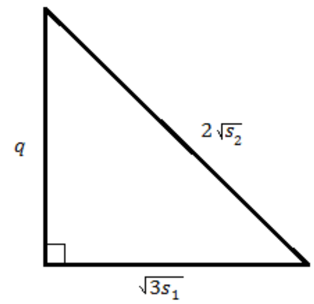

Figure 2: Physical rendition of a triangle with side lengths that satisfy the equation $4s_2 - 3s_1 = q^2$

For instance if $c = 2$ then

$$4(7)^2 - 3(3)^2 = (13)^2 \Rightarrow 4(14)^2 - 3(6)^2 = (26)^2$$

$$4(7)^2 - 3(8)^2 = (2)^2 \Rightarrow 4(14)^2 - 3(16)^2 = (4)^2$$

$$4(7)^2 - 3(5)^2 = (11)^2 \Rightarrow 4(14)^2 - 3(10)^2 = (22)^2$$

$$4(13)^2 - 3(7)^2 = (23)^2 \Rightarrow 4(26)^2 - 3(14)^2 = (46)^2$$

$$4(19)^2 - 3(5)^2 = (37)^2 \Rightarrow 4(38)^2 - 3(28)^2 = (74)^2$$

$$4(19)^2 - 3(16)^2 = (26)^2 \Rightarrow 4(38)^2 - 3(32)^2 = (13)^2$$

etc.

Under the conditions from **(1)**,

$$B = \pm\sqrt{4s_2 - 3s_1} \text{ and } AC = s_2 - s_1.$$

Using the first cubic triple (3,7,13), $s_2 = 49$ and $s_1 = 9$. Then $B = \pm q = \pm 13$ and $AC = 40$. The general cubic

$$P_{3R}(x) = x(a_1x + c_1)(a_2x + c_2)$$

requires $a_1a_2c_1c_2 = 40$ and $a_1c_2 + a_2c_1 = \pm 13$. Factors of 40: 1, 2, 4, 5, 8, 10, 20, and 40 can be used to find values of a_1, a_2, c_1, and c_2. However, only a few combinations lead to $a_1c_2 + a_2c_1 = \pm 13$.

Obvious factors of 40 which yield a sum of 13 are 8 and 5. Thus, one possible solution is

$$P_{3R}(x) = x(x+5)(x+8) = x^3 + 13x^2 + 40x.$$

Consequently, the derivative is factorable over the rational numbers:

$$P'_{3R}(x) = 3x^2 + 26x + 40 = (3x+20)(x+2).$$

However, simply satisfying the first criterion $a_1a_2c_1c_2 = 40$ is not sufficient to yield a viable solution to the problem. For instance, let $a_1 = 5, a_2 = 1, c_1 = 1$, and $c_2 = 8$. The cubic is factorable, but its derivative is not.

$$P_{3R}(x) = x(5x+1)(x+8) = 5x^3 + 41x^2 + 8x$$

$$P'_{3R}(x) = 15x^2 + 82x + 8$$

It is therefore necessary as stated to satisfy the condition $a_1c_2 + a_2c_1 = \pm 13$. By observation, one key point is that the quadratic coefficient in the cubic was not ± 13 and is a clue there may an error in the cubic polynomial. This does not imply that the only solutions using the cubic triple (3,7,13) are $P_{3R}(x) = 5x^3 \pm 13x^2 + 8x$. Other possible cubic polynomials that satisfy the criteria include

$$P_{3R}(x) = x(5x+1)(8x+1) = 40x^3 + 13x^2 + x,$$

$$P_{3R}(x) = x(2x+5)(x+4) = 2x^3 + 13x^2 + 20x,$$

$$P_{3R}(x) = x(5x+2)(4x+1) = 20x^3 + 13x^2 + 2x,$$

$$P_{3R}(x) = x(4x+5)(x+2) = 4x^3 + 13x^2 + 10x,$$

$$P_{3R}(x) = x(5x+4)(2x+1) = 10x^3 + 13x^2 + 4x,$$

$$P_{3R}(x) = x(x+1)(5x+8) = 5x^3 + 13x^2 + 8x, \text{ and}$$

$$P_{3R}(x) = x(x+1)(8x+5) = 8x^3 + 13x^2 + 5x.$$

Additional cubic polynomials can be developed using other cubic triples derived.

5. Additional Matlab Code

The Appendix contains an additional Matlab code which is used to generate the coefficients of a cubic polynomial that satisfy the criteria presented in this paper. The coefficients generated are assumed to be integers and the cubic coefficient is positive. In addition, situations where the cubic is a simplified cubic ($A = 0$ or $C = 0$) are ignored.

The storing of solutions eliminates both the repetition of identical results and those coefficients that are different only by a constant scaling factor. For instance, the cubic

$$P_{3R}(x) = x^3 + 11x^2 + 24x$$

is a valid result. Any scalar multiple of the cubic is also valid. Only the base coefficient triple $(A, B, C) = (1,11,24)$ is generated from the code. Results are formatted as a matrix whose rows are of the form $[A\ B\ C\ i\ j\ k\ m]$, where the first three columns are the coefficients of the cubic, in order, and the next four columns are the indices used to generate the factors of A $(a_1 = i, a_2 = j)$ and C $(c_1 = k, c_2 = m)$, respectively.

The code given uses values of a_1 and a_2 from 1 to 10 and values of c_1 and c_2 from -10 to 10, resulting in a total of 442 generated sets of coefficients. Simply doubling the values (1 to 20 and -20 to 20, respectively) the number of sets of coefficients increases nearly seven-fold to 2954 sets. The number of possible reducible cubic polynomials whose derivatives are factorable over the rational numbers is larger than once thought.

6. Future Work

The reduced cubic $P_{3R}(x) = Ax^3 + Bx^2 + Cx$ is addressed in this article. The authors have begun modifying the reported codes to work with a full cubic $P(x) = Ax^3 + Bx^2 + Cx + D$. Possible methods of tackling the full cubic include use of transformations to take the full cubic to a reduced cubic or a depressed cubic, and using Eisenstein's Criterion to find reducible cubic polynomials.

7. Appendix

The Matlab program for generating the *cubic triples* is given below.

```
%%cubic triple generator
clc; clear;
m=31;       %%m is a positive integer used to be the maximum number of
             %%integers investigated.
s=zeros(m);
for i=1:m
    s(i)=(i-1)^2;  %%s is a vector whose entries are perfect squares of
                     an %%integer
end

N=zeros(m,m);
results=[0 0 0 0];
for i=1:m
    for j=1:m
        N(i,j)=4*s(i)-3*s(j);   %%Matrix composed of all combinations
                                  of %%4(s_2)-3(s_1)
        Q=sqrt(N(i,j));         %%Square root of each entry of A
        if imag(Q)==0           %%Only considering real numbers
            temp=mod(Q,1);      %%Testing to see if value is an integer
            if temp~=0
                N(i,j)=0;       %%Toss out number if it is not an
integer
            end
        else                    %%If imaginary part exists
            N(i,j)=0;           %%toss out numbers with imaginary parts
        end
        if N(i,j)~=0 && s(j)~=0 && s(i)~=s(j)
            results=[results; sqrt(s(j)) sqrt(s(i)) Q s(i)-s(j)];
            %%results in order of [√s1 √s2 B AC]
        end
    end
end

%%Elimination of non-base triples (repeated by a common factor)
[m,n]=size(results);
results=results(2:m,1:n);    %%Eliminated 1st row of zeros
Soln=results(1,1:n);         %%Defines size of Solution matrix
for r=2:m-1
     g=gcd(results(r,1),gcd(results(r,2),results(r,3)));
        %%Identifies Greatest Common Devisor
     sqs1=(results(r,1)/g); %%Divides √s1 by the GCD
     sqs2=(results(r,2)/g); %%Divides √s2 by the GCD
     B=(results(r,3)/g);    %%Divides B by the GCD

     if sqs1~=results(1:r-1,1) | sqs2~=results(1:r-1,2) |
B~=results(1:r-1,3)
      %%Elimination of non-base triples (repeated by a common factor)
        oldresults=Soln;
        Soln=[oldresults; results(r,1:n)];
       end
end
```

The Matlab program for generating the coefficients of a cubic that is factorable over the rational numbers and whose derivative is also factorable over the rational numbers is given below.

```
%%Coefficient Generator, D=0
clc; clear;
soln=[0, 0, 0, 0, 0, 0, 0]; %%Initializing a solution

for i=1:10;
    for j=1:10;
        for k=-10:10;
            for m=-10:10;
              a1=i;
              a2=j;
              c1=k;
              c2=m;

              A=a1*a2;         %%Cubic coefficient
              B=a1*c2+a2*c1;  %%Quadratic coefficient
              C=c1*c2;        %%Linear coefficient
              if A~=0 && C~=0 %%Eliminates simplified cubic
                Discrim=(B^2-(4*A*C))^(1/2); %%Discriminant of reduced
cubic
                DDiscrim=((2*B)^2-(4*3*A*C))^(1/2); %%Discriminant of
                                                      derivative
               integerTestA=~mod(Discrim,1);
                    %%Tests discriminant of reduced cubic to make sure
it is
                      %% an integer
                integerTestB=~mod(DDiscrim,1);
                %%Tests discriminant of derivative to make sure it is
an
                        %% integer

                if integerTestA==1 && integerTestB==1 %%Only use
integer
                                                        results
                    oldSoln=soln;
                    soln=[oldSoln; A B C i j k m];
                    %%results store coefficients and index values a1,
a2, c1,
                        %%and c2 values
                end
              end
            end
        end
    end
end

[m n]=size(soln);   %%Sets size of solution matrix
soln=soln(2:m,1:n); %%Eliminate 1st row of zeros
Soln=soln(1,1:n);   %%Defines size of Solution matrix

for r=2:m-1
```

```
        g=gcd(soln(r,1),gcd(soln(r,2),soln(r,3)));
        %%Identifies GCD of the polynomial
        A=(soln(r,1)/g); %%Reduces A by the GCD
        B=(soln(r,2)/g); %%Reduces B by the GCD
        C=(soln(r,3)/g); %%Reduces C by the GCD

          if A~=soln(1:r-1,1) | B~=soln(1:r-1,2) | C~=soln(1:r-1,3)
           %%Throws out polynomials in solution that once divided by the
            GCD are %%the same as a previous solution
           oldsoln=Soln;
           Soln=[oldsoln; soln(r,1:n)]; %%Storage of unrepeated solution

          end
   end
```

8. References

[1] Jim Buddenhagen, Charles Ford, Mike May. "Nice Cubic Polynomials, Pythagorean Triples, and the Law of Cosines", *Mathematics Magazine*, Volume 65, Number 4, pp. 244-249, Online ISSN: 0025570X (1992).