VIRTUAL SPIROGRAPH: A LIBERAL ARTS MATH PROJECT

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Introduction

There is a variety of virtual simulations of the popular Spirograph® toy, both online and as mobile apps. Because the simulations aren't bound by the physical realities of interlocking gears, they can create designs that the actual toy cannot. For example, in a physical Spirograph, the pen position must be within the moving disk, but in a virtual simulation, the pen may be outside the boundary of the moving disk. The project that I describe here is a multi-part activity that is suitable for liberal arts students and future teachers. It can be assigned as homework or serve as an in-class activity. Successful completion of this project requires that students collect data, form conjectures, prove those conjectures, and communicate clearly about their results.

Getting Started

Some spirograph simulators go so far as to show geared disks that move according to the physics of the actual toy. The more versatile simulators use two circles; one is fixed and the other rolls around the boundary of the fixed circle. The design is drawn by a "pen" that is located a fixed distance from the center of the moving circle.

While it is relatively easy to find web sites or apps that draw spirograph designs, you'll need to choose one that allows you and your students to gather data. Specifically, choose a simulator that has these features:

- Fixed and moving circles that are visible.
- Option to situate the moving circle inside or outside the fixed circle.
- Changeable, numerical parameters for the radii of the circles and the pen location.
- Appropriate drawing speed – slow enough so that students can count revolutions, but not so slow that the process drags.
- A way to visually orient the pen location with respect to the moving circle – typically a line segment from the pen to the center or boundary of the moving circle. This is helpful, as it shows how the moving circle rotates about its center, but is not crucial for most of the project.

In order to foster communication about the project, everyone in the class should use the same simulator.

Part 1 – Learning how to use the simulator

Each parameter in the simulator measures something. Either tell the students what these measurements represent, or ask them to use their exploration to discover what each parameter measures. For simulators that don't use words to identify the parameters, help the students settle on common terminology for each parameter.

Ask the students to play with the simulator by changing the settings and noticing the results. Direct the students to write at least three math-related questions about the simulation, indicating that they do not need to know the answers to their questions.

Students will write questions that range from trivial to insightful. Many students will have difficulty communicating their questions clearly. For example, some students will write questions that mention "triangles," "squares," "lines," "circles," and "points." This is an opportunity to discuss the difference between a "line" and a "curve," and the difference between a "triangle" and a shape that is approximately a triangle. Students who mention "points" are often referring to a spirograph design that resembles a star. Help them develop wording that is unambiguous; for example "When the pen is on the boundary of the moving ring, is there a formula for determining the number of times the pen touches the fixed ring?" as opposed to "Is there a formula for determining the number of points in the design?"

Before proceeding with the rest of the project, formalize the language and notation with the students. In this paper, we use the following conventions:

- R is the radius of the fixed circle.
- r is the radius of the moving circle.
- p is the pen location, measured as a distance from the center of the moving circle.
- k is the number of times the moving circle travels around the fixed circle in order to complete the design. It's easiest to determine this value by counting the number of times the point of tangency returns to the starting position while the design is being drawn.
- n is the number of times the design touches the fixed circle in the completed design, when $p = r$.

Additionally, the moving circle can be located inside or outside the fixed circle. When the moving circle is smaller than the fixed circle, this designation is unambiguous. When the moving circle is larger than the fixed circle but "inside" the fixed circle, though, the description is counterintuitive. Formally, the moving circle is *outside* the fixed circle when the interiors of the disks are disjoint; otherwise, the moving circle is *inside* the fixed circle. Informally, your students may wish to think of the boundary of the fixed circle as having measurable thickness, and use the terms *inside* and *outside* to indicate whether the moving circle is tangent on the inside of that boundary or tangent on the outside of that boundary. In Figures 1(a) and 1(b), the moving circle is indicated by a dashed border and is located inside the fixed circle. In (a), the moving circle is smaller

than the fixed circle, while in (b) the moving circle is larger than the fixed circle. In both cases, the pen is located on the boundary of the moving circle.
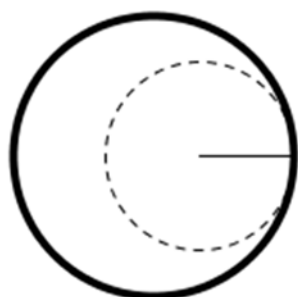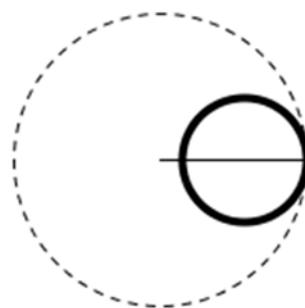

Figure 1(a).   Figure 1(b).

Part 2 – How the location of the pen and the location of the moving circle affect the finished design

In this part of the project, students explore different pen positions and the difference between situating the moving circle inside or outside the fixed circle. At this stage of the project, students may be looking for numerical answers, rather than relationships between various quantities. Remind them to vary the values of R and r as they experiment, since many of the properties of the design are determined by the pen's position relative to the centers of the two circles.   If necessary, explain that most of the answers will be equations or inequalities, rather than numbers.

Ask the students these questions. Suggest that they try to answer Questions 3-7 without using the simulator, and then use the simulator to test their conjectures.

1. Does the value of p affect the value of k?
2. Does the location of the moving circle (inside or outside the fixed circle) affect the value of k? Does the location of the moving circle affect the value of n?
3. What does the finished design look like if $p = 0$?
4. What happens if the moving circle is inside the fixed circle, and $R = r$?
5. Set the values of R and r so that the moving circle is smaller than the fixed circle, and set the location of the moving circle to be inside the fixed circle. What must be true about the value of p in order for the finished design to lie completely inside the fixed circle? What must be true about the value of p in order for the finished design to go through the point at the center of the fixed circle?
6. Set the values of R and r so that the moving circle is smaller than the fixed circle, and set the location of the moving circle to be outside the fixed circle. What must be true about the value p in order for the finished design to lie completely outside the fixed circle? What must be true about the value of p in order for the finished design to go through the point at the center of the fixed circle?
7. How do your answers to the last two questions change if R is smaller than r?

It should be clear that the values of p and k are independent, and that the location of the moving circle affects neither k nor n. Since the center of the moving circle traces out a circle, the finished design is a circle if $p = 0$. If the circles are the same size, no movement is possible when the moving circle is inside the fixed circle.

When the moving circle is inside the fixed circle, the design remains inside the fixed circle provided that the pen remains inside the fixed circle. So if $r < R$, this requires that $p < r$. When $r > R$, we need $p < 2R - r$. Since the centers of both circles and the point of tangency are always collinear, we must have $p = |R - r|$ in order for the design to go through the center of the fixed circle.

Regardless of which circle is larger, if the moving circle is outside the fixed circle, then $p < r$ or $p > r + 2R$ will yield a design that is completely outside the fixed circle, while the design goes through the center of the fixed circle when $p = r + R$.

Part 3 – How the circle sizes affect the finished design

In this part of the project, students explore the relationship between the circle sizes and the values of k and n. Students will almost certainly need some guidance in collecting data in order to form the conjecture in Question 1 below. It helps to give them a table with values of R and r already included. Most students also need a lot of data, so a table with at least 12 different values of r for each of at least 2 different values of R is a good start.

Ask students to answer these questions. Warn them to double check the settings before drawing the designs. In each trial, they should set the pen location to equal the value of r.

1. Make a conjecture for formulas for k and n in terms of R and r. Test your conjecture using different values of R and r.
2. Give an equation for the circumference of the fixed circle.
3. Give an equation for the circumference of the moving circle.
4. Imagine a dot that moves along the fixed circle while the design is being drawn, in such a way that the dot is always at the point where the two circles touch. Give an equation for the distance that the dot travels along the boundary of the fixed circle during the time it takes to draw the design, in terms of R and k. Does this equation depend on whether the point of tangency is inside or outside the fixed circle?
5. In the scenario from Question 4, the dot is also traveling along the boundary of the moving circle. If the dot starts at the pen location, it will make one circuit of the moving circle before returning to the pen location. Give an equation for the distance that the dot travels along the boundary of the moving circle during the

time it takes to draw the design, in terms of r and n. Does this equation depend on whether the point of tangency is inside or outside the fixed circle?

6. The distances in Questions 4 and 5 are equal. Use this fact to verify the conjectures you made in Question 1.

Many students will need help formulating the conjecture in Question 1. The conjectured formulas will take several forms:

$$k = \frac{r}{\gcd(R,r)} \text{ or } k = \frac{\text{lcm}(R,r)}{R};$$

$$n = \frac{R}{\gcd(R,r)} \text{ or } n = \frac{\text{lcm}(R,r)}{r}.$$

The design touches the fixed circle every time the pen location is at the point of tangency. So the distance that the imaginary dot moves is $2\pi Rk = 2\pi rn$. The design is complete when k and n are as small as possible in this equation. So $Rk = rn = \text{lcm}(R,r)$. Thus,

$$k = \frac{\text{lcm}(R,r)}{R} \text{ and } n = \frac{\text{lcm}(R,r)}{r}.$$

The fact that $\text{lcm}(R,r) = \dfrac{Rr}{\gcd(R,r)}$ yields the other formulas for k and n.

Part 4 – Create specific designs

Once students know how the radii of the two circles affect the values of k and n, and how the pen location affects other aspects of the design, you can ask them questions like these.

1. Create a design in which the moving circle travels around the fixed circle 8 times and the pen touches the fixed circle 15 times.
2. Is it possible to create a design where the moving circle travels around the fixed circle 8 times and the pen touches the fixed circle 20 times?
3. Draw two designs that are geometrically similar – i.e., that are scaled versions of one another.
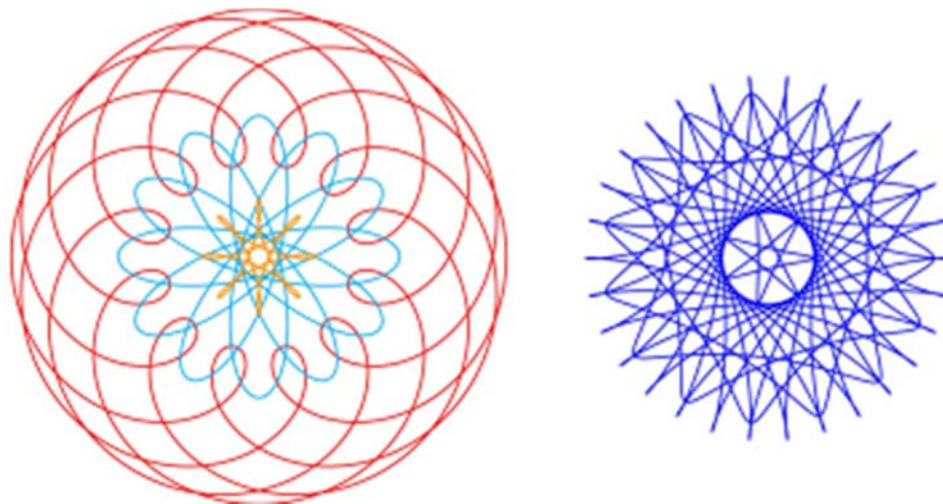4. Create designs that look like those in Figure 2. (Each involves more than one drawing.)

Figure 2: Spirograph designs

Part 5 – Extra credit

If your simulator uses a line segment from the center or boundary of the moving circle to the pen location, students can watch the design being drawn and count the number of times the segment returns to the same orientation relative to the center of the moving circle. The result is the number of times the moving circle rotates about its center while the design is being drawn. Call this number m, and ask students these questions.

1. Does the location of the moving circle (inside or outside the fixed circle) affect the value of m?
2. Find a formula for m in terms of k and n.

In light of their previous investigations, students may be surprised to find that the value of m does depend on the location of the moving circle. If the moving circle is inside the fixed circle, then the rotation of the moving circle around its center is in the opposite direction as the motion of the moving circle around the center of the fixed circle. That is, if the moving circle rolls clockwise around the fixed circle, it will simultaneously be rotating counterclockwise around its own center. On the other hand, if the moving circle is outside the fixed circle, then the rotation of the moving circle around its center is in the same direction as its motion around the center of the fixed circle.

Recall that if the pen starts at the point of tangency, n measures the number of times the pen returns to the point of tangency while the design is being drawn. Using this interpretation, if the moving circle were to roll along a line for a finite distance, starting and ending with the pen at the point of tangency, then $m = n$. In a spirograph design, the moving circle rolls along a circle rather than a line; each full revolution around the fixed circle either adds or subtracts a revolution of the moving circle about its center,

depending on whether the motion around its center is in the same direction as the motion around the fixed circle. So if the moving circle is outside the fixed circle, $m = n + k$. If the moving circle is inside the fixed circle, $m = |n - k|$.

Part 6 – Extra extra credit

Every spirograph design has reflectional symmetry through at least one line, since the pen traces out the same design when the moving circle travels clockwise as when the moving circle travels counterclockwise. The vast majority of spirograph designs have rotational symmetry as well: if $p = r$ and $n > 1$, you can retrace the design starting at any point where the design touches the fixed circle. Challenge your students to create a spirograph design that does not have rotational symmetry.

Figure 3: Spirograph design with no rotational symmetry

Conclusion

This project requires that students collect and analyze data and form conjectures, and highlights the need for standardized language and notation in order to facilitate good communication. Because there are four distinct parameters in the simulator, there are many combinations for students to explore, but the number of possibilities is not overwhelming. The mathematics in this project is not particularly deep, but students are forced to think beyond the basic operations of addition, subtraction, multiplication, and division. Furthermore, understanding the interplay among the parameters allows students to create intentional, aesthetically pleasing designs, which may help students to appreciate the activity as more than just a set of tasks that they must complete.