LINKAGES, ELLIPSES, AND ELLIPTICAL EXERCISE MACHINES: STUDENT EXPLORATIONS

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Exercise machines are splendid devices that help people in the modern age develop physical fitness, and their designs are excellent examples of mechanical linkages that students can study in mathematics class. There are numerous learning opportunities that arise when students create a model of such machines. Learning opportunities include exposing students to the basis of "how mechanical things work" and to the realization that curve-drawing technology can apply to actual machines. Finally, students answer and provide justification for the question, "Do elliptical exercise machines trace ellipses?" This presentation discusses project-based learning experienced by a class of accelerated precalculus students at Trinity Valley School in Fort Worth, TX.

The accelerated precalculus class consisted of one ninth grader, nine tenth graders, and one eleventh grader. Before receiving instructions for the assignment, students had been studying applications of trigonometry, specifically parametric equations. To develop the parametric equations for curves, students had been shown curves that had been created geometrically using dynamic geometry software. By identifying geometric properties of curves built with dynamic geometry software and representing these relationships symbolically, students were able to develop the parametric equations for curves such as the ellipse, hyperbola, parabola, involute, and witch of Agnesi. As well, students constructed ellipses on their own in the computer lab based on the following two well-known definitions of the conic:

1. the set of all points P in a plane for which the sum of the distances from point P to two fixed points (the foci) is constant,
2. the set of all points equally distant to a point (one focus) and a circle, when the point is inside the circle.

To produce the mechanical linkages using dynamic geometry software, students needed to be able to construct circles, congruent segments, parallel and/or perpendicular lines, and loci. The definitions of an ellipse were investigated and confirmed using dynamic geometry models. The students digested the definitions and discussed the major features of an ellipse, which include symmetry, identification of foci, the major and minor axes, and its equation in Cartesian and parametric form. Soon after students developed a significant understanding of parametric equations, the teacher wanted students to consider creating an actual curve-drawing device in the context of an elliptical exercise machine.

The project unofficially began with the teacher introducing two hands-on activities. The first activity instructed students to create an ellipse with pegboard, two nuts and bolts, and a piece of string that was tied in a loop. Since students had been studying various ways to create ellipses using dynamic geometry software, this activity progressed quickly, and students identified their ellipses as the set of all points P in a plane for which the sum of the distances from point P to two fixed points (the foci) was constant. Since many high school students are often not attuned to real-life engineering, understanding how an isosceles trapezoidal linkage is the base of a toy rocking horse that children ride in front of grocery stores or how the parallelogram linkage designs of sewing boxes exist to preserve parallel drawers helps students begin to make connections with mechanical understanding. To complete this activity, the teacher acquired hardware store pegboard and then cut it into strips with twelve holes, nuts, bolts and washers. Using Michael Serra's Discovering Geometry projects activity, students completed a lab that had them consider quadrilateral linkage designs. With the peg board strips, which served as geo-strips, students created mathematical linkages, moved their linkages based on situational prompts, and generalized features about these linkages. These two activities allowed students to get in a mindset that moved from computer designs of curves to mechanical linkage creations of curves.

At this point in the experience, the class was prepared to begin the project officially. After watching a web video of a functioning elliptical machine, students were assigned a project. With their newfound ellipse and parametric equation knowledge, they were ultimately charged with providing significant discussion about whether or not the locus of points traced by feet on an elliptical machine is an ellipse. This task allowed students to provide discussion and justification after construction and comparison.
The following sums up the parts of the project:

1. Develop the equation of an ellipse. *
2. Answer a set of problems involving ellipses.*
3. Share and model an interesting feature of an ellipse. *
4. Design an elliptical exercise machine and model it using a dynamic geometry program.
5. Create a scale model of the elliptical exercise machine.
6. Answer the question, "Do elliptical exercise machines trace ellipses?" As well, the question answer must be justified.

   Each * must be completed individually. Parts 4-6 can be created in groups of three or four provided no individual is left out of a group.

Students were given four days to complete part 1 through part 3. They completed the work away from class due to an ice storm that closed school from a Friday through Monday in early December of 2013. Because the students had been working with applications of trigonometry and continually beefing up their algebra skills, they did very well on these portions of the project. To complete part 4, students were given two days in the computer lab and any time that they wanted to spend on their own to create their elliptical machine models using a dynamic geometry program. This task required the students to make the computer create an animation that was similar to the typical movement of the pedals of an elliptical exercise machine.

Once students had acceptable models, the class discussed the linkage features of the constructions and how those linkage features created the locus that is the path the feet travel when using an elliptical machine. Then the following question was posed: Is this locus an ellipse? They definitely possessed enough supporting evidence to address the question at hand. This task allowed students to provide discussion and justification after construction and comparison.

There were three different groups working on designs, and students attacked the charge differently. All groups addressed part 6 of the project with their computer-developed representation. All students limited their creations to the bare minimum construction of the device needed to evaluate whether or not an ellipse was traced. Each group's creation and discussion of part 6's question are summarized below:

Group 1 Demonstration/Discussion

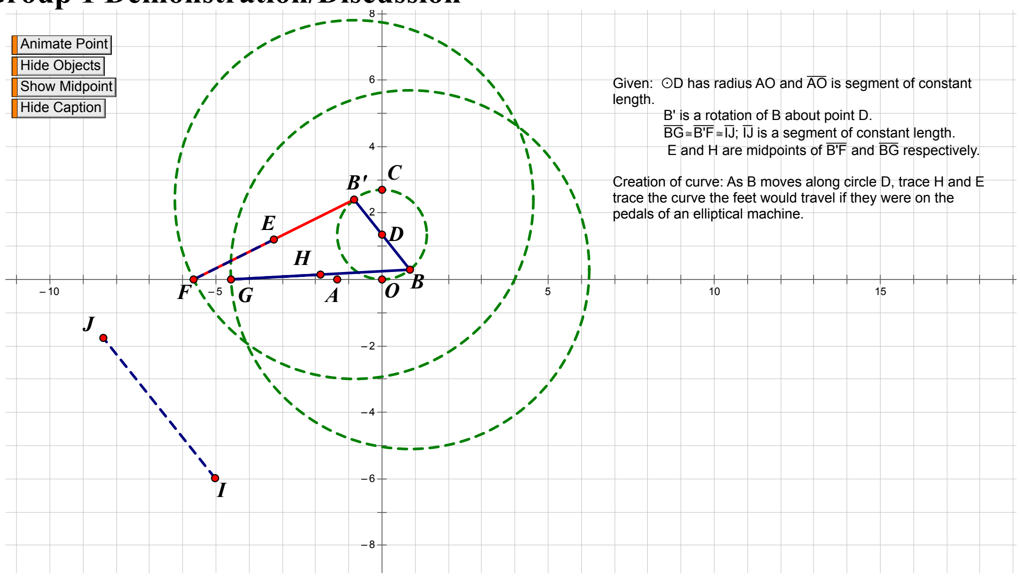

Figure 1—Group 1's model provided with all the geometry shown to create the curve

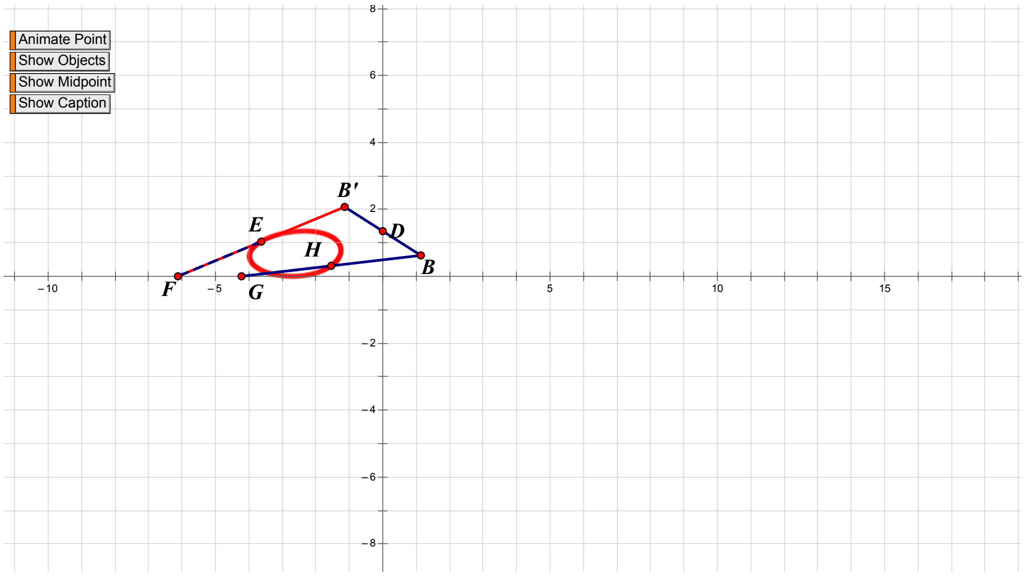

Figure 2—Group 1's simple version of the model

The members of group 1 were perplexed with the challenge of their curve not having a potential line of symmetry that would be parallel to the x or y-axis; therefore, they tried to make their problem simpler. They constructed a sketch where the base circle had center as the origin and radius of 1 unit. Then the segment with constant length of four units had one endpoint on the base circle and the other endpoint on the x-axis, left of the circle. This provided them with a curve in which they deemed their center to be the point with coordinates (-2,0). Figure 3 is a sketch of their new model and a discussion of whether or not their curve was an ellipse.

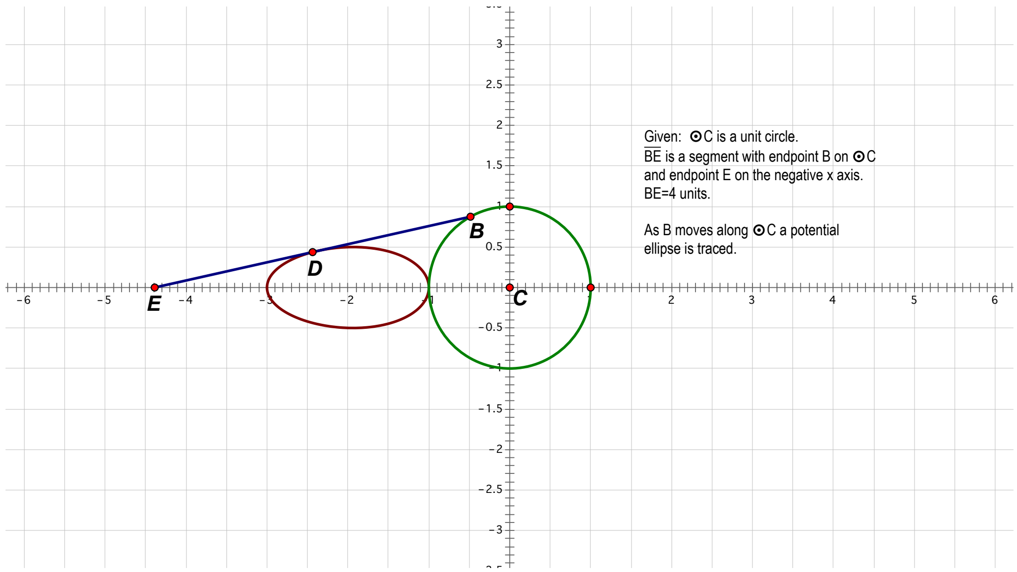

Figure 3—Group 1's updated model

Group 1 Discussion: "If we are to take a look at the supposed ellipse, we find that one side (-2.68, 0.35) and on the other (-1.26,0.35). If we are to check this to see if it is symmetrical, we find the midpoint to be in actuality (-1.97,0). This in itself proves itself not to be symmetrical as the midpoint of the Cartesian ellipse equation $\frac{(x+2)^2}{1}+\frac{y^2}{.5^2}=1$ is (-2,0). This shows the midpoint to be slightly off. It cannot be a classification of an ellipse due to this irregularity."

Group 2 Demonstration/Discussion

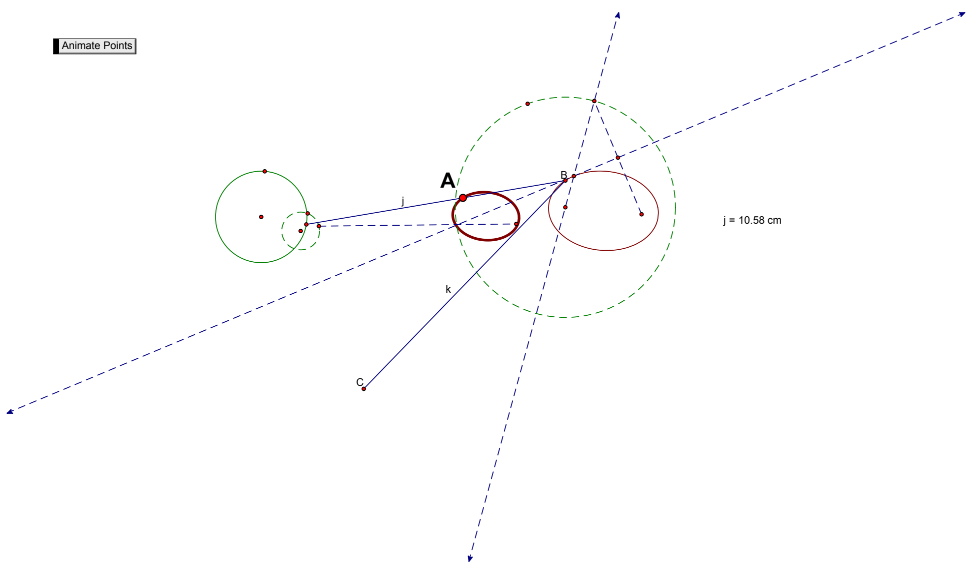

Figure 4— Group 2's model provided with all the geometry shown to create the curve

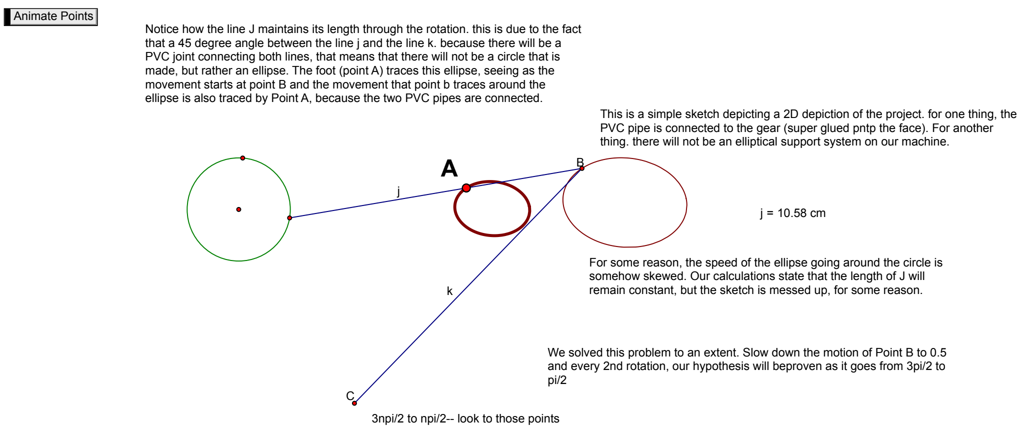

Figure 5—Group 2's simple version of the model

Group 2 Discussion:
Student Commentary 1: "Notice how the line j maintains its length through the rotation. This is due to the fact that a 45 degree angle between the line j and the line k. because there will be a PVC joint connecting both lines. That means that there will not be a circle that is made, but rather an ellipse. The foot (point A) traces this ellipse, seeing as the movement starts at point B and the movement that point B traces around the ellipse is also traced by Point A, because the two PVC pipes are connected."
Student Commentary 2: "This is a simple sketch depicting a 2D depiction of the project. for one thing, the PVC pipe is connected to the gear (super glued onto the face). For another thing, there will not be an elliptical support system on our machine."
Student Commentary 3: ''For some reason, the speed of the ellipse going around the circle is somehow skewed. Our calculations state that the length of J will remain constant, but the sketch is messed up, for some reason."
Student Commentary 4: "We solved this problem to an extent. Slow down the motion of Point B to 0.5 and every 2nd rotation, our hypothesis will be proven as it goes from 3pi/2 to pi/2"

Group 3 Demonstration/Discussion

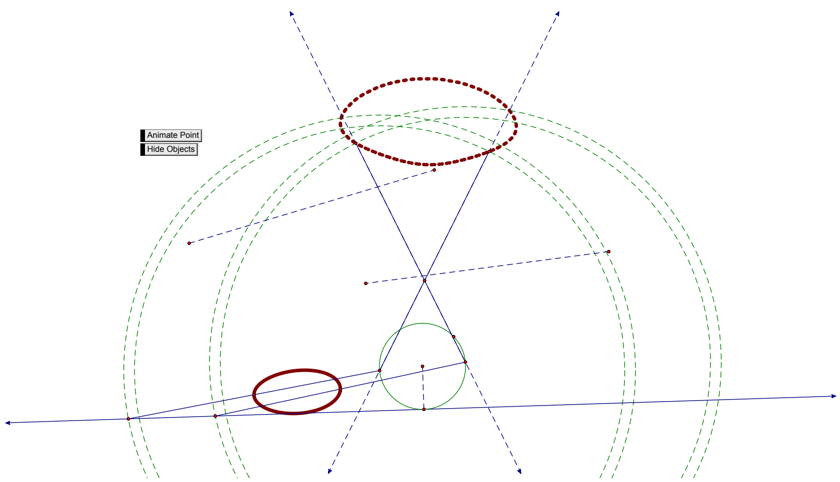

Figure 6—Group 3's model provided with all the geometry shown to create the solid curve

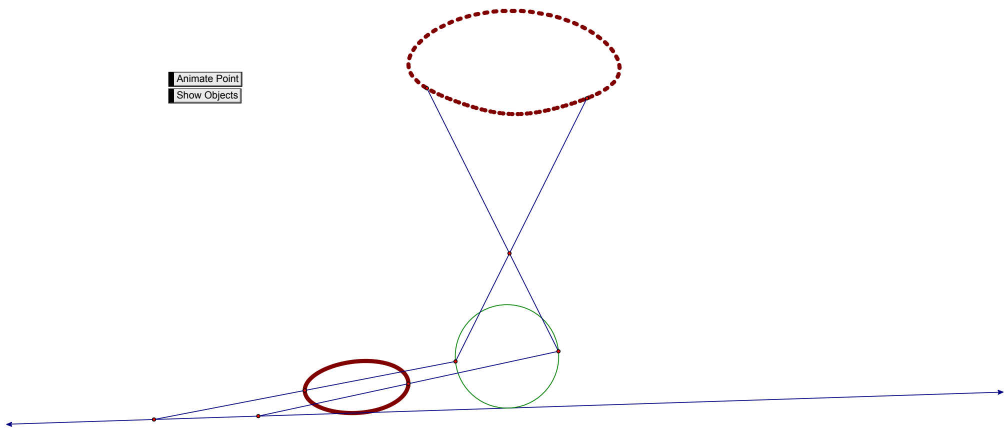

Figure 7—Group 3's simple version of the model

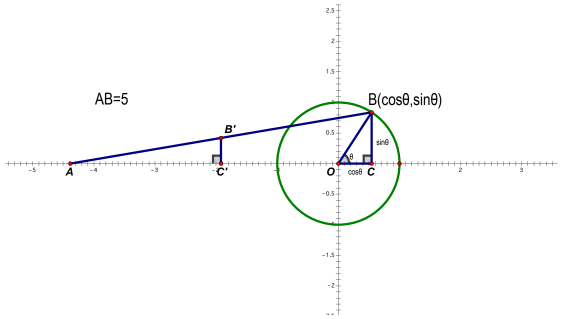

Figure 8—Group 3's diagram used in discussion

Group 3 Discussion: "Point A is a point on the x-axis, with an x-coordinate lesser than that of point B. Point B is on the unit circle, and point C, on the x-axis, has the same x-coordinate as B. B' is the midpoint of AB and C' the midpoint of AC. AB is equal to 5 for this proof. Let the value theta be the angle point B makes with the x-axis. Triangle AB'C' is similar to ABC by SAS, as AB' is .5AB, AC' is .5AC, and m<A is equal to itself. Because of this B'C'=.5BC. As the circle is a unit circle, point B has the coordinate of $(cos\theta, sin\theta)$. Point C, therefore has coordinates of $(cos\theta, 0)$. The distance between points B and C is $\sqrt{(cos\theta - cos\theta)^2 + (\sin\theta - 0)^2} = \sqrt{sin^2\theta} = sin\theta$. By the Pythagorean Theorem, as $\overline{AC}$ is horizontal and $\overline{BC}$ is vertical, forming a right angle, $AC = \sqrt{5^2 - sin^2\theta}$=$\sqrt{25 - sin^2\theta}$. AC' is therefore $\frac{\sqrt{25-sin^2\theta}}{2}$. Subtracting AC' from AC leads to CC', also $\frac{\sqrt{25-sin^2\theta}}{2}$. OC, where O is the origin is equal to $\sqrt{(cos\theta - 0)^2 + (0-0)^2} = \sqrt{cos^2\theta} = cos\theta$. As B'C'=.5BC and $BC = \sin\theta$ and $B'C' = .5\sin\theta$. C' has a y-coordinate of 0, and is directly below B', so the y-coordinate of B' is $\frac{\sin\theta}{2}$. To find the x-coordinate of B', the length of CC' is to be subtracted from the x-coordiante of C. This leads to $\cos\theta - \frac{\sqrt{25-sin^2\theta}}{2}$. Thus, the set of parametric

equation of B' which represents the pedal of an elliptical machine is the following for parameter θ: $x = \cos\theta - \frac{\sqrt{25-sin^2\theta}}{2}$ and $y = \frac{\sin\theta}{2}$.

A θ at 0 yields the point (-1.5,0). When $\theta = \frac{\pi}{2}$, the y-value is .5. When $\theta = \pi$, the point is (-3.5,0). When $\theta = \frac{3\pi}{2}$, the y-value is -.5. If the curve is an ellipse, the major, horizontal , axis of 2 and the minor axis is of length 1. The center of this ellipse is the midpoint of (-1.5,0) and (-3.5,0), or (-2.5,0). The parametric equation for this ellipse would be $x = cos\theta - 2.5$ and $y = \frac{\sin\theta}{2}$ for parameter θ. If graphed with the first set of parametric equations, the equation of the ellipse should generate identical points. It is especially apparent the y-values would be the same. The x-values, however, while close, still do diverge. When $\theta = .5$, the first set of parametric equations, which come from the elliptical, give an x-value of -1.611, while the second set from the actual ellipse gives an x-value of -1.622."

How compelling it was to compare the dynamic geometry versions of each group's elliptical machine. As well, the arguments had different levels of direction and depth. Group 1 offered a symmetry (or lack of symmetry) argument, as to why the curve traced was not an ellipse. Group 2 created a dynamic geometry model using an actual definition of an ellipse, and then had their segments trace along the ellipse. Group 3 created a dynamic model that was very similar to Group 1's and included the most analytic and detailed support of all three groups. Group 3 found the actual parametric equations for the curve and used those equations in the argument. After students turned in their conjectures with support, the class was most eager to discuss their solutions and use of evidence. Completing parts 1 through 4 of the project and addressing part 6 in the context of the dynamic geometry models gave students the opportunity to put their mathematical study into action and discuss this action. Up to this point, their projects had been explored using technology, groupwork, and analytical support. Then they moved to the truly physical portion of the project: part 5.

Part 5 required each group to create a scale model of a functioning elliptical exercise machine. Each group was given a budget of $20.00, and the group went to a local hardware store for their purchases. As well, they were given three days in the physics design lab where they had access to shop items such as bandsaws, drill presses, handsaws, hammers, drills, nuts, bolts, nails, and safety goggles. After safety training, each group started on this portion of the project. Each group successfully petitioned that their models be only of the portion of the elliptical machine that traces out the curve. Of the eleven students, four had never been in a workshop before. The students had completed their machine models by the end of the third day. Interestingly, their models were extremely similar to the models created with the dynamic geometry software. They did not have to create their physical scale model based on their dynamic geometry model, but they would have had to have come up with another argument about whether or not the curve traced by the physical elliptical machine model was an ellipse. Students were very efficient in their work.

Student understanding of the many connections between geometry and precalculus topics increased after the completion of the entire activity. The hands-on elements introduced before the actual project assignment had helped to put students in the mindset of looking for "how things work." This project satisfied many intentions of the *Maker Movement Manifesto,* which include making, sharing, learning, "tooling up," playing, participating, and supporting. Technology aided students because students were quickly able to see if their designs would indeed construct curves. Before students entered the workshop, they had tested different scenarios and entered the workshop better prepared due to the testing. As well, the exercise provided the students an opportunity to model an elliptical exercise machine and to evaluate and to see applications of linkages in real life.

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