

SOLAR ENERGY – “TAKE A SPHERICAL COW”

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As mathematicians, and, especially, as mathematicians teaching undergraduate mathematics courses, we have both an enormous responsibility and an enormous opportunity. Over the course of their lives our students will be making personal and public policy decisions that will shape the future of our planet and impact their wallets. As the complexity of such decisions increases, favorable outcomes demand the creativity, intellectual rigor, and intellectual integrity that are central to science and mathematics. As math and science educators we are well positioned to develop our students' decision-making skills by leveraging their interests in their planet and their wallets to motivate their study of mathematics and science and engage them as they apply mathematics and science to consequential and often controversial problems.

This paper and its companion paper – *Solar Energy – Can Parking Save Money and the Planet?* – use mathematics to study how solar energy can help us meet our energy needs[1]. Both papers use widely available inexpensive equipment to collect data. See Figure 1. The companion paper describes that equipment in detail. This paper uses three dimensional geometry, vectors, and linear algebra. It can be used in multivariable calculus courses, linear algebra courses, and especially courses in mathematical modeling. The companion paper uses much more elementary mathematics and can be used in college algebra and precalculus courses.



Figure 1: A backpackers' solar panel and rechargeable batteries.

Solar energy is one of the most promising sources of energy. It is renewable and, unlike fossil fuels, it doesn't fill our atmosphere with greenhouse gases like carbon dioxide. Solar energy is often used to generate electricity that in turn is used for many other things – for example, transportation, heating, and air conditioning. The basic problem with solar generated electric power is matching production with consumption. The Sun shines only during the day, for example, but we need to drive our cars and heat our homes at night as well as during the day. Batteries enable us to store electricity when production is higher than consumption and supply electricity when consumption is higher than production.

These two companion papers look at a simple example involving production, consumption, and storage [1][2]. Figure 1 shows a solar panel and a set of rechargeable AA batteries in a charging station. This gear is designed for use by backpackers on extended trips in the back country. Backpackers typically carry smart phones, GPS units, and other battery-powered equipment. Non-rechargeable batteries that power this equipment are both expensive and heavy. By using rechargeable batteries that can be recharged far away from house-current, backpackers can save money and weight.

The availability of solar energy is affected by the time of day, the time of the year, latitude, and weather. In this paper we build a model that considers only three of these factors – time of day, time of year, and latitude. We make a number of simplifying assumptions – we assume that the days are clear and cloudless and we also assume that the atmosphere has no effect. These are drastic assumptions – hence, our title “Take a Spherical Cow.” This paper is just the start of modeling the availability of solar energy. It is a great example to use in a mathematical modeling course because it illustrates the importance of assumptions and the importance of an iterative approach to modeling beginning with a simple, “spherical cow,” model to get some traction, then building ever more sophisticated and useful models. The interplay with experimentation is the key to success. We use the equipment shown in Figure 1 to collect data against-which we can compare our model. Models and their shortcomings lead to new experiments that, in turn, lead to new models.

The first step in building our first model is understanding the interplay of the seasons, latitude, and time of day. This is often a difficult concept as documented in the *A Private Universe Project*. Figure 2 is a screenshot from a simulation built using the DIY Modeling software produced by the NSF-funded DIY-Modeling project. The software and the simulation are free and can be downloaded from the DIY-Modeling web site[3]. This simulation can be used to help students understand the interplay of the seasons, latitude, and time of day. See also our Wolfram Demonstration[4].

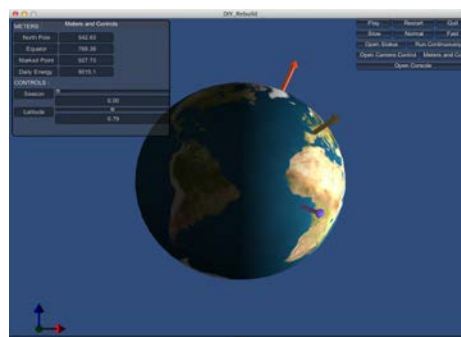


Figure 2: Simulation of the seasons, time of day, and latitude.

In our simplified model of an earth without atmosphere, the magnitude of solar energy intensity upon any point on the planet's surface can, if the sun is directly overhead reach a maximum of 1367.6 watts per square meter. Figure 3 shows the basic reason why the rate at which a particular point on the Earth receives energy per square meter from the Sun depends on the angle of elevation of the Sun. This angle depends on the time of day, the time of year, and the latitude of the point. We need a function $I(\phi, t, s)$ that tells us the rate at which a point whose latitude is ϕ receives energy from the Sun at the time of day t and time of year s . From Figure 3 we see that this function is

$$I(\phi, t, s) = 1367.6 \cos(\theta).$$

where the unit vector $\vec{N}$, called the normal vector, is perpendicular to the Earth's surface at our location and the unit vector $\vec{S}$ points from our location toward the Sun. The function $I(\phi, t, s)$ is almost given by $1367.6(\vec{N} \cdot \vec{S})$ but if the Sun were below the horizon then $\vec{N} \cdot \vec{S}$ would be negative, so this formula must be changed to

$$I(\phi, t, s) = 1367.6 \max(0, \vec{N} \cdot \vec{S}).$$

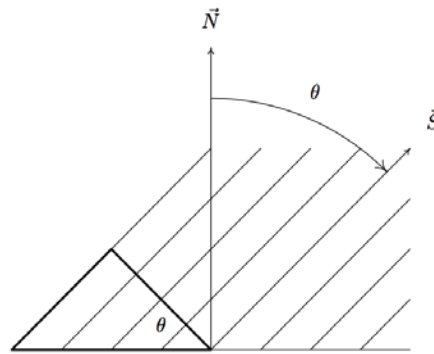


Figure 3: The Sun's angle and energy received.

To find the vectors $\vec{N}$ and $\vec{S}$ we use several coordinate systems. The first coordinate system is attached to the Earth with the origin at the center of the Earth and the z-axis running through the North and South Poles with the positive z-direction being north. The x- and y-axes and the equator are all in the xy-plane. The positive x-axis goes from the center of the Earth through the place where the prime meridian crosses the equator. The positive z-axis goes through the equator at longitude 90° E. See Figure 4. The point where the prime meridian crosses the equator is marked by a dot. The x-axis goes through this point. Notice that so far we are looking only at the Earth and this first coordinate system is attached to the Earth and revolves with the Earth.

We will use the variable ϕ to denote the latitude of a surface location, measured in radians. Thus, $\phi=0$ for points on the equator while $\phi=\pi/2$, refers to the North Pole, and $\phi=-\pi/2$ is the South pole respectively. Thus, in this coordinate system a point on the prime meridian with latitude ϕ has coordinates $\vec{u} = R_e(\cos(\phi), 0, \sin(\phi))$ where R_e denotes the radius of the Earth.

Now we need a second coordinate system. This second coordinate system is shown in

Figure 5. Its origin is at the center of the Earth and its z-axis goes through the two poles with the positive z-axis in the direction of the North Pole. The x- and y-axes, however, are fixed in space and do not revolve as the Earth revolves. If $\vec{u}$ denotes a position on the Earth in the first coordinate system and $\vec{v}(t)$ denotes the same point in the second coordinate system at time t in hours.

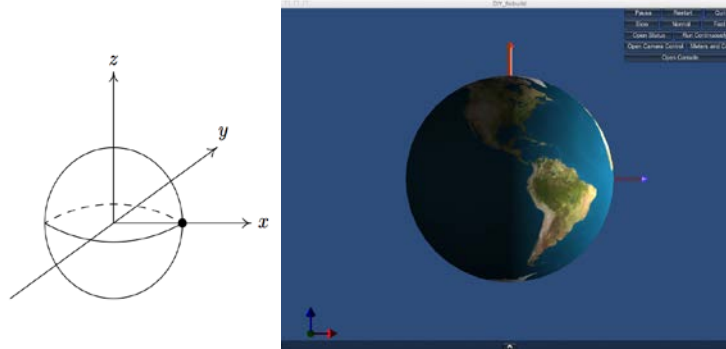


Figure 4: Revolving coordinate system for the Earth.

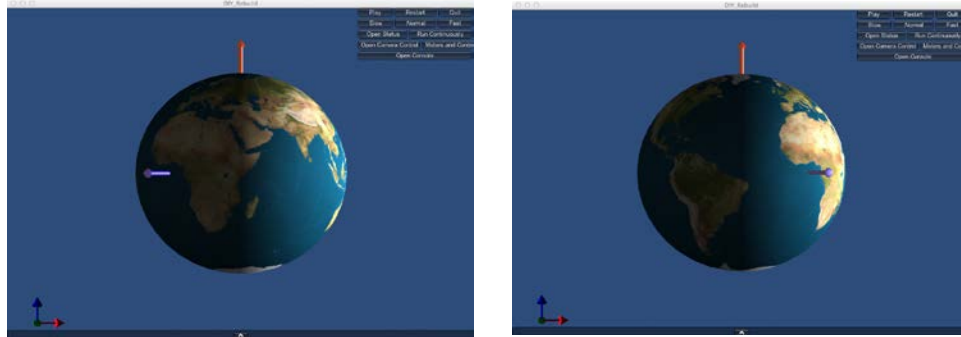


Figure 5: Two frames in a simulation in a fixed coordinate system for the Earth.

$$\vec{v}(t) = \begin{bmatrix} \cos(\frac{2\pi t}{24}) & -\sin(\frac{2\pi t}{24}) & 0 \\ \sin(\frac{2\pi t}{24}) & \cos(\frac{2\pi t}{24}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that in the second coordinate system points on the Earth are rotating around the Earth's axis once every 24 hours. In this coordinate system over the course of a day a point $\vec{u} = R_e \langle \cos(\phi), 0, \sin(\phi) \rangle$ on the prime meridian at latitude ϕ traces out a circle

$$\vec{v}(t) = \begin{bmatrix} \cos(\frac{2\pi t}{24}) & \sin(\frac{2\pi t}{24}) & 0 \\ -\sin(\frac{2\pi t}{24}) & \cos(\frac{2\pi t}{24}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_e \cos(\phi) \\ 0 \\ R_e \sin(\phi) \end{bmatrix} = \begin{bmatrix} R_e \cos(\frac{2\pi t}{24}) \cos(\phi) \\ -R_e \sin(\frac{2\pi t}{24}) \cos(\phi) \\ R_e \sin(\phi) \end{bmatrix}$$

or

$$\vec{v}(t) = R_e \left\langle \cos(\frac{2\pi t}{24}) \cos(\phi), -\sin(\frac{2\pi t}{24}) \cos(\phi), \sin(\phi) \right\rangle.$$

The upward pointing unit normal vector at this point is

$$\vec{N} = \left\langle \cos(\frac{2\pi t}{24}) \cos(\phi), -\sin(\frac{2\pi t}{24}) \cos(\phi), \sin(\phi) \right\rangle.$$

Now we need a third coordinate system. This coordinate system is shown in Figure 6 and is similar to the second second coordinate system in two ways. Its origin is at the center of the Earth and it is fixed in space. The xy-plane, however, is not the plane of the equator. Instead it is the plane of the ecliptic – that is, the plane containing the Earth's orbit around the Sun. The angle between these two planes is 23.4° , or 0.408 radians. If $\vec{v}(t)$ represents a point on the Earth in the second coordinate system then this same point is represented in the third coordinate system by the vector $\vec{w}(t)$ given by

$$\vec{w}(t) = \begin{bmatrix} \sin(0.408) & 0 & \cos(0.408) \\ 0 & 1 & 0 \\ -\cos(0.408) & 0 & \sin(0.408) \end{bmatrix} \vec{v}(t).$$

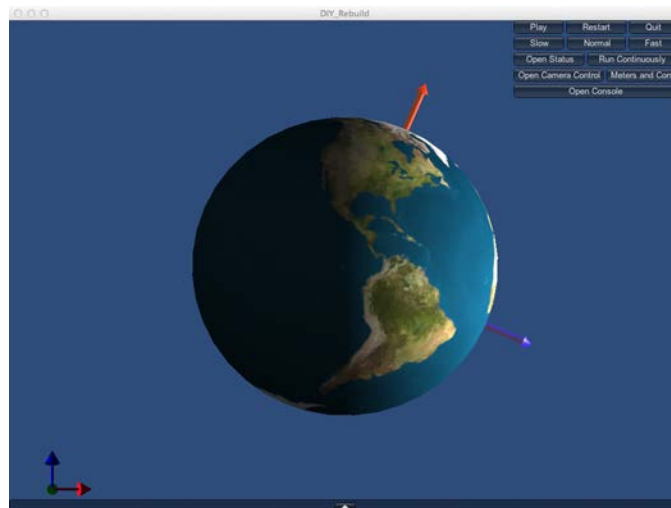


Figure 6: Tilting the Earth

To distinguish between the time of day, represented by variable t , we let the variable s

represent the normalized (i.e. $0 < s < 1$) location of the earth during the solar year. In this coordinate system the Sun appears to rotate about the Earth. If we measure time in years, then its position at time s is given by the vector

$$150,000,000 \langle \cos(2\pi s), \sin(2\pi s), 0 \rangle,$$

where we approximate the Earth's orbit around the Sun by a circle of radius 150,000,000 km. The unit vector pointing in this direction is.

$$\vec{S} = \langle \cos(2\pi s), \sin(2\pi s), 0 \rangle.$$

We put this all together to compute the function $I(\phi, t, s)$. The development of this formula with all its details makes a good set of exercises for students learning multivariable calculus, vectors, and linear algebra. Figure 7 shows a graph of this function at latitude 45 degrees, over the course of a day on the summer solstice, a day on the winter solstice and a day at either equinox.

Figures 8, 9, and 11 are three examples of actual data we collected using the equipment and procedures described in the companion paper. Figure 8 shows data collected on a very hot, humid, and hazy day with almost no clouds. Notice that the shape of the graph is almost an inverted V instead of the more rounded shape we see in Figure 7. This is one of the things we must explain in more sophisticated models. Notice the sharp dips. Figure 9 shows the data collected around one of these dips and Figure 10 shows the wind-blown cumulus clouds that caused this dip.

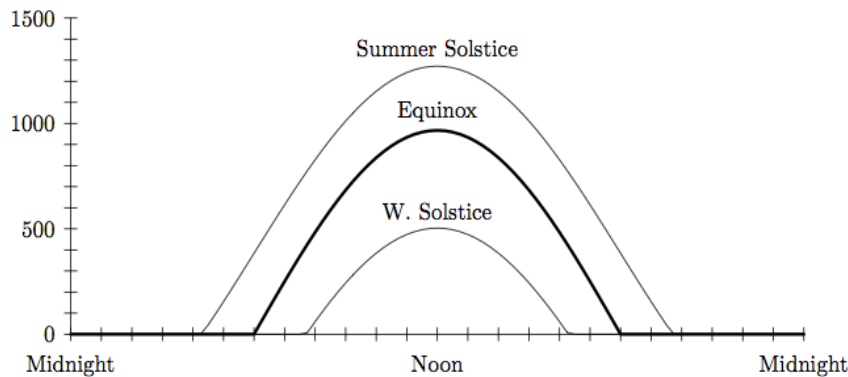


Figure 7: W/meter² at latitude 45 on the day of equinox and the two solstices.

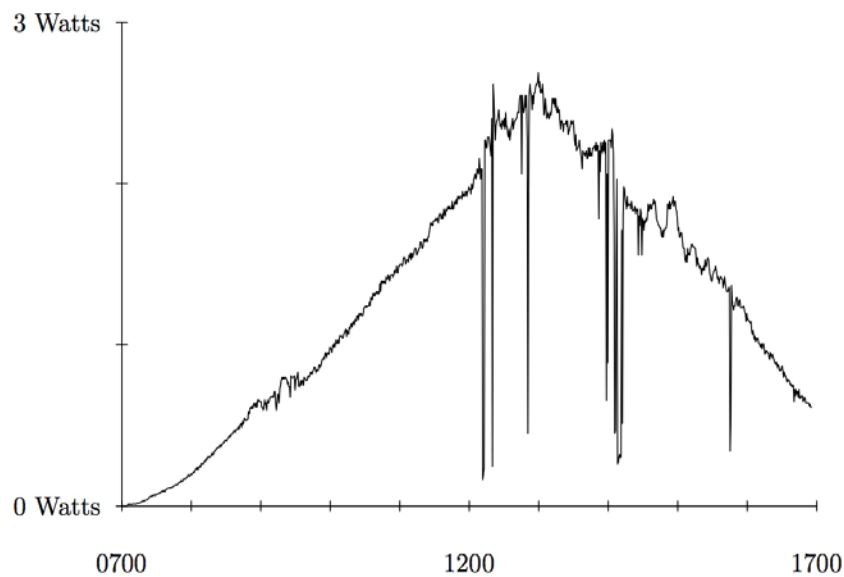


Figure 8: Data Recorded 21 August 2013.

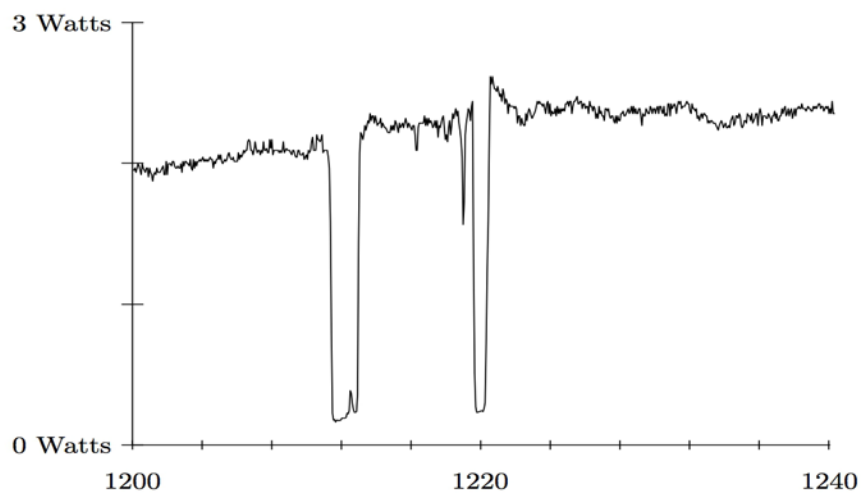


Figure 9: Data Recorded 21 August 2013 (1200 - 1240) .

We learned a tremendous amount exploring these phenomena – especially collecting data and comparing real data with our models. The two papers in this ICTCM are complements of one another. If our students learn half as much and have half as much fun collecting their own data and building their own models as we are having they will learn a lot of mathematics, a lot science, a lot about modeling and have a lot of fun.



Figure 10: Sun and Clouds at 1220.

References

- [1] Park, Russell, Wilkerson, Stephen, and Wattenberg, Frank – Can Parking Lots Replace Gasoline Stations?, submitted to ICTCM.
- [2] Wattenberg, Frank, Wilkerson, Drew and Park, Russell – Solar Energy – A Modeling Story, submitted to ICTCM.
- [3] DIY Modeling Appalachian State University, <http://diymodeling.appstate.edu/>
- [4] Wilkerson, Stephen, Wattenberg, Frank, Park, Russell, Wolfram Demonstration Projects, <http://demonstrations.wolfram.com/SolarEnergyIncidentOnEarthsSurface/>