MULTIPLE TASKS DERIVED FROM AN INTERACTIVE MODULE LINEAR TRANSFORMATION AND EIGENSPACE

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Introduction

Advent of new technologies now makes it possible to implement activities to facilitate learning. In fact, the current means of use of technology can be classified into those used to disseminate information, assess knowledge, provide feedback, use as high computing tools, and to provide platforms to reveal conceptual meaning of abstract ideas (Dogan, 2013, 2012, 2011; Dogan-Dunlap, 2010, 2006, 2004a, 2004b; Dogan, Carrizales and Beaven, 2011). Videos and social media such as YouTube are among the technologies to disseminate information. WebCT, blackboard, and webassign are some of these technologies that provide means to assess knowledge as well as provide feedback. Matlab and *Mathematica* are among the high computing technologies. Some of these technologies such as GSP and Geogebra can furthermore be used to provide means to reveal meaning behind abstract topics (Dogan, 2013, 2012, 2011; Dogan-Dunlap, 2010, 2006, 2004a, 2004b; Dogan, Carrizales and Beaven, 2011).

Focus in this paper will be on the technologies that provide means to reveal meaning, specifically on an activity in a GSP environment. This activity is developed to reveal meaning of abstract mathematical ideas of linear transformations, eigenvalues and their connections to other topics (Johnson, Riess and Arnold, 2002). As many of us are aware, mathematics is an abstract area, which makes it difficult for most novice learners to easily access meaning (Dogan, 2013, 2012, 2011; Dogan-Dunlap, 2010, 2006, 2004a, 2004b; Dogan, Carrizales and Beaven, 2011). For instance, understanding the ideas and accessing the meanings of linear map from R^n to R^m is not easily available for many students from its formal descriptions using symbolic expressions. Thus, to many learners, having T: R^3->R^2 where $T(\begin{bmatrix} a \\ b \end{bmatrix})=\begin{bmatrix} a-b \\ a+b \end{bmatrix}$ (Johnson, Riess and Arnold, 2002) may mean a bunch of symbols to manipulate. This aspect furthermore becomes a major obstacle toward understanding the formal description, and being able to connect this description to relevant topics such as image, range and dimension ideas (Dogan, 2013, 2012, 2011; Dogan-Dunlap, 2010, 2006, 2004a, 2004b; Dogan, Carrizales and Beaven, 2011). This in fact causes many of the observed classroom mistakes such as giving rank value for a transformation that is not the focus when asked for a basis of a vector space. Another observed mistake implying formalism as the source is the reference to a matrix as a set (Dogan, 2013, 2012, 2011; Dogan-Dunlap, 2010, 2006, 2004a, 2004b; Dogan, Carrizales and Beaven, 2011).

Ability to form a strong understanding and make relevant and accurate connections between various aspects of abstract objects require one to be able to uncover meaning behind symbolic descriptions. Our online module is formed precisely for this purpose of revealing meaning behind formal descriptions of various aspects of linear transformations, their eigenvalues and eigenvectors.

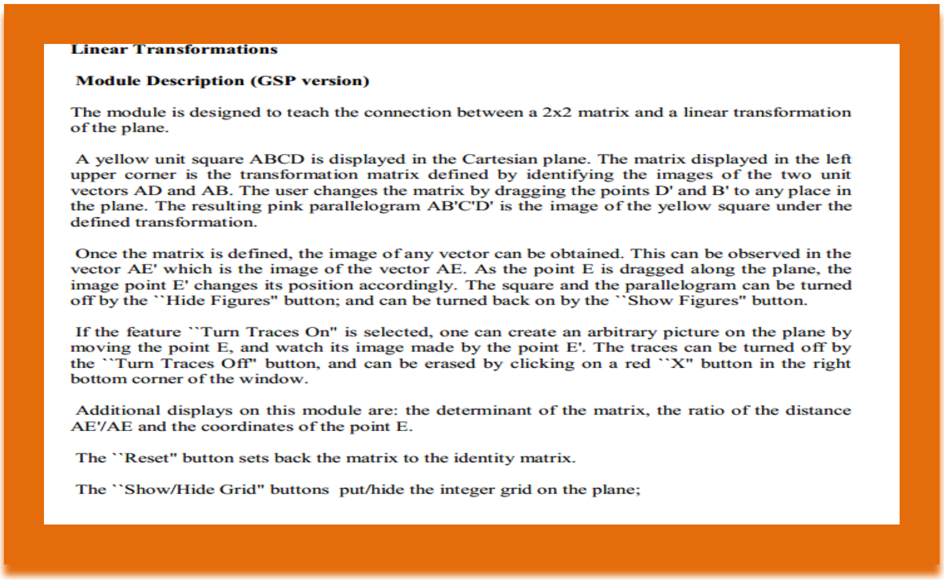
Now, we will turn our attention to the description of the module, and to a set of proposed topics along with the perspectives in which these topics can be covered to maximize meaning extraction.

Modules

Set of web-based modules developed with the support of an NSF grant (CCLI-0737485) is used in responding to basic abstract linear algebra concepts. Modules provide online interactive dynamic geometric features of linear algebra ideas (Johnson, Riess and Arnold, 2002), and this project furthermore provides a set of investigative questions both addressing geometric and abstract aspects that are integrated into the modules (See fig. 1).

Figure 1. Links and resources for the NSF project.

Modules are developed independently of the investigations to ensure that they can be utilized for multiple topics. Investigations included at
http://www.math.utep.edu:8080/jsp/
are the representative of many more instructional materials that can be designed utilizing the modules. Primary goal of these investigations is to provide the first-hand experience needed for an effective and accurate cognition of linear algebra concepts that are commonly introduced through abstract means. That is, these investigations are to provide concrete features preparing learners for a meaningful assimilation of abstract concepts that are mainly introduced formally. Even though our philosophy has been to utilize the modules for the beforehand introduction and investigation of basic features of abstract concepts, modules can also be utilized for assignments that focus on the reinforcement of information already introduced. See fig. 1 for a view of the website hosting the links to the modules and the relevant investigations.



Linear Transformations

Module Description (GSP version)

The module is designed to teach the connection between a 2x2 matrix and a linear transformation of the plane.

A yellow unit square ABCD is displayed in the Cartesian plane. The matrix displayed in the left upper corner is the transformation matrix defined by identifying the images of the two unit vectors AD and AB. The user changes the matrix by dragging the points D' and B' to any place in the plane. The resulting pink parallelogram AB'C'D' is the image of the yellow square under the defined transformation.

Once the matrix is defined, the image of any vector can be obtained. This can be observed in the vector AE' which is the image of the vector AE. As the point E is dragged along the plane, the image point E' changes its position accordingly. The square and the parallelogram can be turned off by the ``Hide Figures" button; and can be turned back on by the ``Show Figures" button.

If the feature ``Turn Traces On" is selected, one can create an arbitrary picture on the plane by moving the point E, and watch its image made by the point E'. The traces can be turned off by the ``Turn Traces Off" button, and can be erased by clicking on a red ``X" button in the right bottom corner of the window.

Additional displays on this module are: the determinant of the matrix, the ratio of the distance AE'/AE and the coordinates of the point E.

The ``Reset" button sets back the matrix to the identity matrix.

The ``Show/Hide Grid" buttons put/hide the integer grid on the plane;

Figure 2. Description of the module on linear transformation and eigenvalue.

Module seen in figures 2 and 3, focus of this paper, is constructed using GSP and run in a Java environment. Main tools provided in the module are: Matrix, Determinant, Ratio, numerical values of point E, boxes for Trace, Show and Hide Grid buttons, Show and Hide Figures, and a Reset button. As seen in fig.3, geometric figures are two parallelograms; one for the identity transformation and the other to set a second transformation. The parallelogram in yellow stands for the identity matrix, and the one in pink represents the second matrix. Furthermore, AE line represents input vectors and AE' line the image of AE under the transformation represented by the Pink parallelogram. Moreover, green dots represent input vectors and blue dots represent their images. See fig. 2 for the description of the module, and fig. 3 for a view of the module site and its tools.

To reiterate, module provides multiple tools intentionally to allow multiple activities to support multiple ideas. That is, it is not to provide a single idea or activity. One can design multiple activities for multiple subjects using the tools provided in this module. For instance, at the site,

http://www.math.utep.edu:8080/jsp/activities.html ,

one can find two separate investigations listed that are using the same module to support two separate sets of topics; eigenvalues and linear transformations (see fig. 1). One can also find in a dropdown menu, on the second column, a set of another 12 topics that can also be introduced using the same module. Of course, this module is not limited to only 12 topics, it can also be used to address many more ideas. In short, the main theme of the module is to address any idea relevant to linear transformations, eigenvalues and eigenvectors. Note that this same site provides a detailed description of the module. See middle column on figure 1 with the link "For more detail Click Here." Also, see figure 2 for the description of the module.

Let's now take a look at a couple of module tools through specific examples. One of the tools is to form matrices via the geometric representations of parallelograms. It should be noted here that transformations between Euclidean spaces can be represented as matrix multiplications (Johnson, Riess and Arnold, 2002). Each matrix entered into the module represents a linear transformation from R^2 to R^2. For instance, the matrix, $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ on fig. 3 represents a linear transformation that sends the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ via matrix multiplication: $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. These vectors would be represented on the module as the line segments AE and AE' respectively. Figure 3 shows both the algebraic and geometric descriptions of the matrix and the two vectors. Next, we will give some of the concepts among many more one can introduce using the very same module.

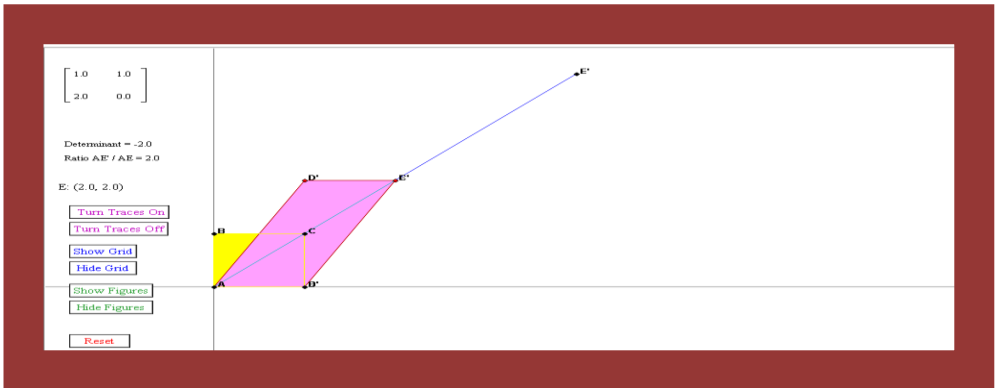


Figure 3. Module Tools and Geometric and algebraic representations of matrix, two vectors.

Linearity of Maps and Circle Vectors as Representative of Domain Vectors

One can begin with introducing the idea of maps being linear (Johnson, Riess and Arnold, 2002). To do this, first, learners may be directed to observing relations between AE vector, a scalar times AE vector and AE' line as their image vectors. If a specific AE vector and scalars are provided and asked to observe their geometric forms while taking note of the movement of AE', students may come to a conclusion that AE' vectors are originating from the same initial scalar used to obtain new vectors from AE. That is, AE' is also the very same scalar value (paired with AE) times the image of derived input vectors from AE.

This idea may be extended to considering all the vectors in the domain (in this case R^2) via a set of vectors forming a circle with center at the origin. From this idea, students can demonstrate the scalar multiples of vectors on the circle observing not only that vectors on the circle can be used as a spanning set for R^2, but also to observe that vectors on the circle would be enough to make accurate inferences about the images of all the vectors from the domain of the transformation (Johnson, Riess and Arnold, 2002). See figure 4 for a view of how this activity may reflect itself on the module. After this activity, students can continue into investigating (again using the module tools) the ideas of Range and Null Space of linear transformations. See figures 5a and 5b for a view of an activity that represents the range of a linear transformation.

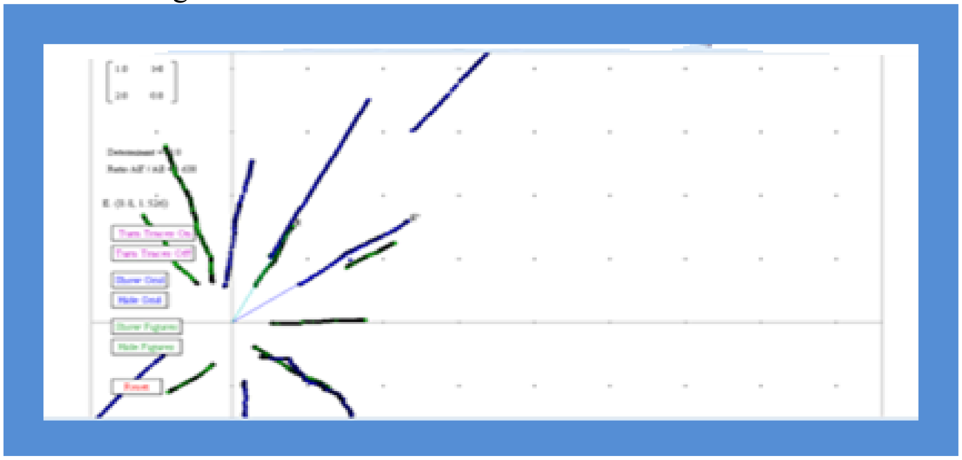


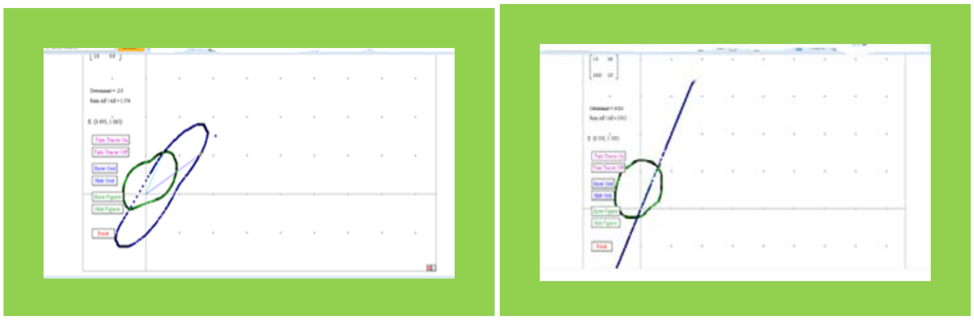
Figure 4. Demo on linearity and circle vector ideas.

Range of Linear Transformation

Let's now look at range ideas with an example. Consider the transformation T: R^2->R^2, where T(x)=Ax (*) with the matrix $A=\begin{bmatrix}2 & 0\\ 2 & 1\end{bmatrix}$ and the domain vectors $x=\begin{bmatrix}u\\ v\end{bmatrix}$. Formally, the range of T is defined using set notation and matrix multiplication : Range(T)={ $\begin{bmatrix}2 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}$ |for all $\begin{bmatrix}u\\ v\end{bmatrix}$ in R^2} (**) (Johnson, Riess and Arnold, 2002).

Clearly the description of the transformation in (*) is heavily arithmetic-oriented, and calls mainly for one to compute values. Moreover, the description of Range(T) in (**) similarly is very abstract that students (struggling with its formalism) may completely ignore this form (Dogan, 2013, 2012, 2011; Dogan-Dunlap, 2010, 2006, 2004a, 2004b; Dogan, Carrizales and Beaven, 2011). In fact, neither provides much hint on the overall behavior of, the visual aspects, and the structure of the Range of the transformation. On the other hand, the visual representations of input and output vectors as dots (green and blue respectively) provide an easily accessible overall behavior and the structure of the range. From figure 5a, one can infer easily that the range of T most probably will cover all vectors in R^2. That is, it will consider each vector in R^2 as its image vector. One furthermore may observe a peculiar behavior that the images of domain vectors are lined up favoring mainly one side (the right side) of the input vectors.

Considering that fig. 5a is depicting a non-singular matrix, students may furthermore be directed to make a conjecture about the range of all linear transformations represented by any non-singular matrix. In short, interactive tracing of the input and output vectors can provide details to the location of each range vectors. Another word, as many of the readers would agree, these observed behaviors may not be possible for many learners to gain solely based on algebraic and arithmetic approaches.



a b

Figure 5. Circle approach to identifying Range of a nonsingular and a singular matrix.

Now, let's also look at a singular transformation and observe both algebraic and geometric representations of its range vectors (See fig. 5b). Consider the transformation T: R^2->R^2, where T(x)=Ax with $A=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$, and $x=\begin{bmatrix}u\\ v\end{bmatrix}$. Formally, Range of T is again defined using set notation and matrix multiplication :

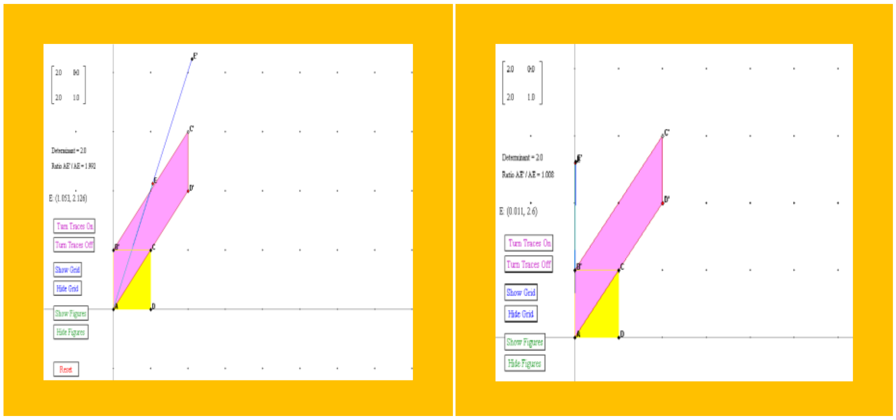
$$R(T)=\{\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} \mid \text{for all} \begin{bmatrix}u\\ v\end{bmatrix} \text{ in } R^2\}$$

Even though algebraic description may not reveal overall behavior of the function, the geometric representation of input and output vectors are seen easily to be clustering along the line spanned by the vector, $\begin{bmatrix}1\\ 2\end{bmatrix}$, which also forms the columns of the matrix A (See fig. 5b). In fact, this observation may lead to discussions on the column space of A and the range of T adding more activities for learners to compare range and column space of matrices via their geometric representations. For instance, one can go back to the transformation defined by $A=\begin{bmatrix}2 & 0\\ 2 & 1\end{bmatrix}$ earlier, and reconsider the observed peculiar behavior of range positioning on the right of inputs in green (see fig. 4 and fig. 5a). This behavior can be investigated and explained, this time in light of the connection between the range of the transformation and the column space of the matrix, directing attention to the span of the vectors $\begin{bmatrix}2\\ 2\end{bmatrix}$ and $\begin{bmatrix}0\\ 1\end{bmatrix}$ (Johnson, Riess and Arnold, 2002). Above example with the singular matrix $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$ and its behavior (fig. 5b) may indeed provide a transition to discussions on the eigenvalues and eigenvectors of linear transformations. With this example, asking appropriate questions, learners attention can be directed to input vectors along the line spanned by the vector, $\begin{bmatrix}1\\ 2\end{bmatrix}$, and their images as well as the value provided under the tool "Ratio AE'/AE" positioned on the center left (see fig. 3). Gradually, with more examples of transformations and their images, one can introduce the ideas of eigenvalues and eigenvectors of linear transformations.

Eigenvalues and Eigenvectors

For the transformation, $A=\begin{bmatrix}2 & 0\\ 2 & 1\end{bmatrix}$, mentioned above, let's go back to the domain vectors and their images depicted in Figures 6a and 6b. These figures show that the transformation defined by A appears to have two eigenvalues with ratios; AE'/AE≈2.0 and AE'/AE≈1.0 with two groups of eigenvectors forming two separate lines. Next, using the two observations, learners can form a conjecture addressing all eigenvalues and eigenvectors of the transformation. This may follow by formal mathematical arguments to prove their conjectures.

Now let's look at another example with a singular matrix (see fig. 7). We will consider the earlier example on fig. 5b of the transformation represented by $A=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$. In the earlier case, we looked at the behavior of the range of the transformation (fig. 5b). Now let's use the module to observe eigenvalues and eigenvectors (if any) of this transformation.



a b

Figure 6. Eigenvalues of a nonsingular transformation.

Comparing the figures 5b and 7 reveals a connection between the range and an eigenvalue of the transformation. In fact, AE'/AE ratio indicates that 3 might be an eigenvalue of the transformation and its eigenvectors are all the vectors in the range of the transformation (fig. 7). Learners can be further prompted to reconsider the nullspace of the transformation in the context of eigenvalues. This may aid them to begin thinking about the vectors in the nullspace and investigate whether these vectors may be considered for an eigenvalue. The earlier formal mathematical work on the nonsingular matrix may help learners to make connection to zero being an eigenvalue for the vectors of the nullspace. Since eigenvalue zero and its connection to vectors of the nullspace cannot be observed geometrically, formal algebraic discussion is necessary for the learners to make connections. One furthermore may continue with discussions on whether and in what situations zero would be an eigenvalue leading to singular matrices as the ones having zero as an eigenvalue.

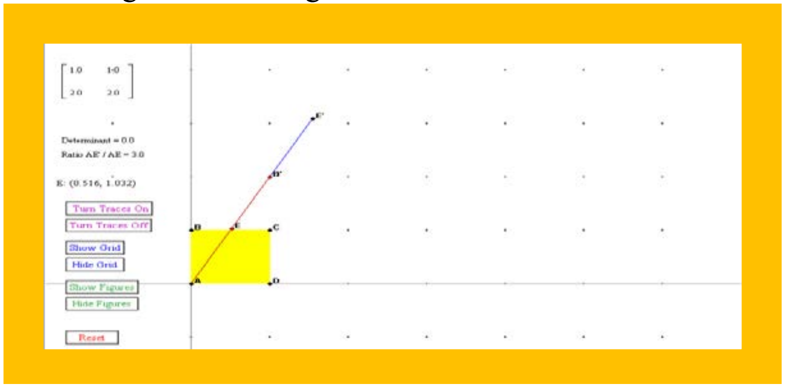


Figure 7. An eigenvalue of a singular transformation.

We can also give a transformation with no real eigenvalues by having learners to investigate the behavior of the matrix, $A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (see figure 8). This work may prompt learners to begin thinking about complex numbers as potential eigevalues leading to formal work on identifying complex eigenvalues.

Sign of Determinant and Orientation of Image Vectors

In fact in an actual implementation of the particular module, not only various aspects of linear transformations and eigenvalues were investigated, our students also observed an interesting behavior among the transformations whose geometric aspects observed. Many observed that there is a correlation between the sign of the determinant of the transformation and its direction in which image vectors located in comparison with the input vectors. It appears that if the determinant of a matrix is negative, then non-eigenvalue related image vectors are located to the left of input vectors and travel counterclockwise; and if the determinant is positive then they are located to the right of input vectors

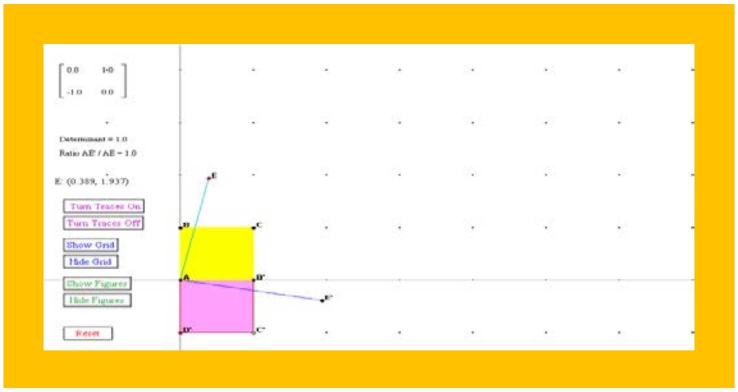
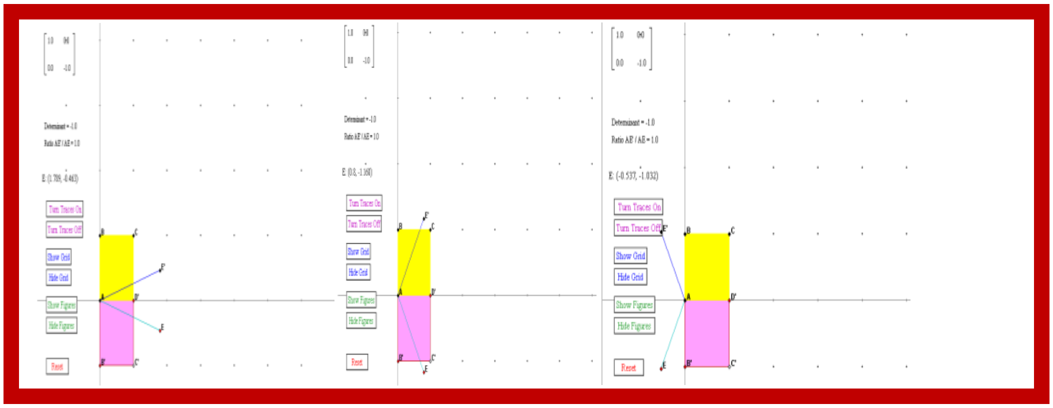


Figure 8. Transformation with no real eigenvalue.

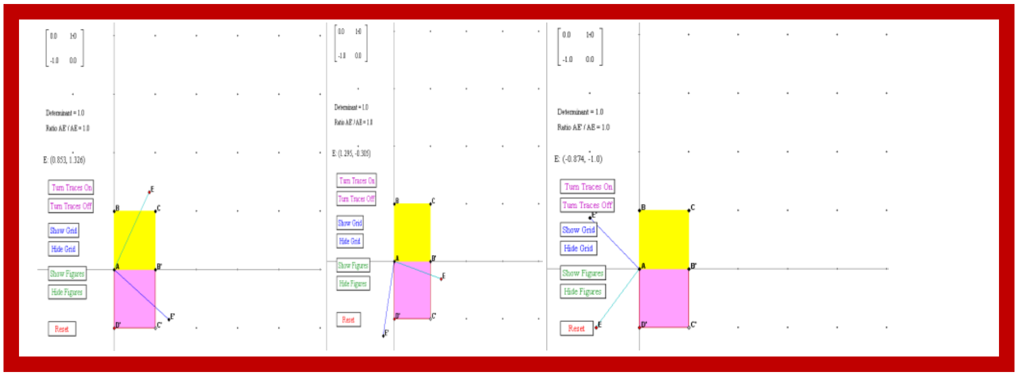
and travel clockwise. See figures 9a, 9b and 9c for an example of a matrix, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, with a negative determinant and the orientation of image vectors (in blue with E' point) compared to the input vectors (in green with E point).



a b c

Figure 9. Negative determinant and orientation of image vectors.

Clockwise orientation can be observed in the figures 10a, 10b and 10c for the matrix, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, with positive determinant.



a b c

Figure 10. Positive determinant and orientation of image vectors.

Conclusion

In this paper we discussed a single module, and how one can introduce multiple ideas using the very same module. From the examples we provided, readers may have sensed that it is possible to reveal abstract ideas via interactive dynamic online modules. In addition, our readers may have experienced the richness of a module that can handle the introduction of many aspects of linear transformations, eigenvalues and eigenvectors. In conclusion, we, the authors, strongly believe in the power of online interactive and dynamic modules that can reveal meaning behind abstract concepts handling various aspects of many topics single handedly.

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